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PARTIALLY ORDERED RINGS II

By

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Abstract. This paper is a continuation of [6]. We study partially ordered rings in terms of non-negative semi-cones and convex ideals, considering order-preserving homomorphisms, residue class rings, and certain product rings, etc.

1. Introduction

Partially ordered rings have been considered by several authors. Especially, the systematic foundation of lattice-ordered rings has been given by Birkhoff and Pierce [2]. Recently, an interesting result of a lattice-ordered skew field has been obtained in [10].

In this paper, we assume that all rings are non-zero commutative rings with identity. The symbol R means such a ring with the identity element denoted by 1, and I means an ideal of R (similar, for R' and I'), unless otherwise stated.

We shall consider commutative, partially ordered rings. As is well-known, for a ring R, there is a bijection between the set of partial orders of R which make it into a partially ordered ring and the set of those S of R having properties: $S \cap -S = \{0\}$; $S + S \subset S$ (S is closed under addition); $SS \subset S$ (S is closed under multiplication). In the previous paper [6], we call a subset S of R satisfying these three conditions a non-negative semi-cone as a generalization of "positive cones" of integral domains, as well as, "non-negative cones" of rings. For a partially ordered ring R, in order that the residue class ring R/I be a partially ordered ring with the canonical order induced from R, I is precisely a convex ideal, as is well-known ([4]). The concepts of "non-negative semi-cones" and the "convex ideals" play important roles in the theory of partially ordered rings (see [1], [2], [3] and

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[4], etc.). In view of these concepts, we shall study partially ordered rings, considering order-preserving homomorphisms, idempotents, residue class rings, and certain product rings, etc. Also, we give characterizations for typical subsets of the product rings to be non-negative semi-cones, and a characterization for non-negative cones of a certain product ring of a field.

2. Non-negative Semi-cones and Convex Ideals

Let A, B be subsets of R. Define $-A = \{-x \mid x \in A\}$, $A + B = \{x + y \mid x \in A, y \in B\}$, $AB = \{xy \mid x \in A, y \in B\}$, $aB = Ba = \{a\}B$ for $a \in R$, and $A \backslash \{0\} = \{x \mid x \in A, x \neq 0\}$. Also, for a subset C of R', define $A \times C = \{(x, y) \mid x \in A, y \in C\}$.

First, let us recall some basic definitions used in this paper. For other terminologies, see [4], [6], etc.

DEFINITION 2.1. A subset S of a ring R is a *non-negative semi-cone* ([6]) (resp. *non-negative cone* ([5])) of R if S satisfies the following (i), (ii), and (iii) (resp. (i), (ii), (iii), and (iv)):

(i) $S \cap (-S) = \{0\}$.

(ii) $S + S \subset S$.

(iii) $SS \subset S$.

(iv) $R = S \cup (-S)$.

A subset S of R is a *positive cone* ([5], [9]) of R if S satisfies the above (ii) and (iii), and $S \cup (-S) = R \backslash \{0\}$. For a positive cone S, $S \cup \{0\}$ is a non-negative cone.

We recall that $(R, \leq)$ is a *partially ordered ring* (resp. *ordered ring*) if $\leq$ is a partial order (resp. total order) on R such that $a \leq b$ implies $a + x \leq b + x$ for all x, and $a \leq b$ and $0 \leq x$ implies $ax \leq bx$. Also, $(R, \leq)$ is an *ordered integral domain* if it is an ordered ring which is an integral domain.

We note that for a non-negative semi-cone S of a ring R, we induce a canonical partial order $\leq_S$ in R by defining $x \leq_S y$ by $y - x \in S$, and $(R, \leq_S)$ is a partially ordered ring. Conversely, for a partially ordered ring $(R, \leq)$, we induce a canonical non-negative semi-cone $S = \{x \mid 0 \leq x\}$ of R with $\leq \, = \, \leq_S$. These are also valid for the relationship between "non-negative cones (resp. positive cones)" and "ordered rings (resp. ordered integral domains)". (A non-negative semi-cone S of a ring R is the set R^+ of all positive elements* of a po-ring (or partly ordered ring) $(R, \leq_S)$ in [2]).

* For a partially ordered ring $(R, \leq)$, elements x of R satisfying $x \geq 0$ are called *positive* in [2], [10], and other references.

DEFINITION 2.2. Let R, R' be rings. As is well-known, $R \times R'$ is a ring under component-wise addition and multiplication (i.e., for $(x, y), (z, w) \in R \times R'$, $(x, y) + (z, w) = (x + z, y + w)$, and $(x, y) \cdot (z, w) = (xz, yw)$). Let us call such a ring $R \times R'$ the *direct product ring*.

NOTATIONS. (1) The brief terminology "semi-cone" (resp. "cone") is used as an abbreviation of "non-negative semi-cone" (resp. "non-negative cone").

(2) The symbol $(R, \leq)$ (or simply, R) means a partially ordered ring, and S means the canonical semi-cone of R, and similar to the symbols $(R', \leq')$ (or simply, R') and S', unless otherwise stated.

(3) The symbol $R \times R'$ means the direct product ring, unless otherwise stated.

An element e of a ring R is called an *idempotent* if $e^2 = e$. For an idempotent e of R, $f = 1 - e$ is also an idempotent of R with $ef = 0$, $e + f = 1$.

The symbol $\mathbf{Z}$ denotes the ring of integers, and $\mathbf{Z}^*$ denotes the set of non-negative integers.

REMARK 2.3. Let S be a semi-cone of a ring R. Let $a, b \in S \cup \mathbf{Z}^*$. Then $aS + bS$ $(\subset S)$ is obviously a semi-cone of R (here, $aS + bS = S$ for $a = 1$ or $b = 1$). However, we have the following (1) and (2).

(1) $S + S'$ need not be a semi-cone of R for a semi-cone S' of R, and similar to Se for an idempotent e of R (indeed, let $R = \mathbf{Z} \times \mathbf{Z}$. Let $S = ((\mathbf{Z}^* \backslash \{0\}) \times \mathbf{Z}) \cup \{(0, 0)\}$, $S' = (\mathbf{Z} \times (\mathbf{Z}^* \backslash \{0\})) \cup \{(0, 0)\}$. Then S and S' are semi-cones of R. But, $S + S' = R$, and for an idempotent $e = (0, 1)$, $Se = \{0\} \times \mathbf{Z}$. Then neither $S + S'$ nor Se is a semi-cone of R).

(2) SS need not be a semi-cone of R (indeed, let $R = \mathbf{Z}$, and $S = 2\mathbf{Z}^* + 3\mathbf{Z}^*$. Then S is a semi-cone of R, but SS is not a semi-cone, because $4, 9 \in SS$, but $4 + 9 = 13 \notin SS$).

PROPOSITION 2.4. *Let S be a semi-cone of a ring R, and let e and $f = 1 - e$ be idempotents of R with $e, f \neq 0$. Then the following hold.*

(1) *Se is a semi-cone of Re iff $Se \cap (-Se) = \{0\}$. In particular, for $Se \subset S$, Se is a semi-cone of Re.*

(2) *If S_1 and S_2 are semi-cones of Re and Rf respectively, then $S_1 + S_2$ is a semi-cone of R.*

(3) *$S' = Se + Sf$ is a semi-cone of R iff so are Se of Re and Sf of Rf.*

Proof. (1) is obvious. (2) is routinely shown, noting $(Re)(Rf) = Re \cap Rf = \{0\}$. For (3), the if part holds by (2). For the only if part, noting $S'e = Se \subset S'$, Se is a semi-cone of Re by (1). Similarly, Sf is a semi-cone of Rf. $\square$

Let $h : R \to R'$ be a map with R and R' rings. We recall that h is an *epimorphism* if it is a ring homomorphism with $h(R) = R'$. For semi-cones S of R and S' of R', h is called *order-preserving* if $h(S) \subset S'$.

Let R be a ring. An ideal I of R is *proper* if $I \neq R$. For a proper ideal I of R, R/I denotes the *residue class ring* consisting of elements $[a] = I + a$ $(a \in R)$.

Definition 2.5 ([4]). For a proper ideal I of $(R, \leq)$, I is *convex* in R if whenever $0 \leq x \leq y$ and $y \in I$, then $x \in I$. We induce a canonical ordering relation on R/I as follows: For $a \in R$, define $[a] \geq 0$ if $[a] = [x]$ for some $x \geq 0$ in R (we use the same symbol $\leq$ in R/I without confusion).

We recall that a proper ideal I of $(R, \leq)$ is convex iff $(R/I, \leq)$ is a partially ordered ring; equivalently, $S' = \{[x] \mid [x] \geq 0\}$ is a semi-cone of R/I ([4], etc.).

We assume that R/I has the semi-cone S', unless otherwise stated.

Let $\varphi : (R, \leq) \to (R/I, \leq)$ be the natural map defined by $\varphi(x) = [x]$ for $x \in R$. Then φ is an order-preserving, epimorphism with $\varphi(S) = S'$.

The convexity of an ideal I of a partially ordered ring R is usually defined under I being proper in R. But, for convex ideals I and J of R, $I + J$ need not be proper in R (see Example 2.11(1)). Moreover, for some partially ordered ring $R \times R$, ideals $I_0 = 0 \times R$ and $I_0' = R \times 0$ are convex (see Remark 3.20 later), but $I_0 + I_0'$ is not proper in $R \times R$, and also for an idempotent $e = (0, 1)$, $I_0 e$ is not proper in Re.

In view of the above, let us introduce the following terminology.

Definition 2.6. Let J be an ideal of a ring R (including $J = R$), and S be a semi-cone of R. Let us say that J is *S-convex* in R if whenever $x \in S$, $y - x \in S$ and $y \in J$ imply $x \in J$. When $J \neq R$, we shall call such an S-convex ideal J *convex for S*. For $(R, \leq)$, obviously J is S-convex in R iff J is convex (for S), or $J = R$.

Proposition 2.7. *Let S be a semi-cone of a ring R, and let e and $f = 1 - e$ be idempotents of R with $e, f \neq 0$. Then the following hold.*

(1) *Let $Se \subset S$. If I is S-convex in R, then Ie is Se-convex in Re.*

(2) *Let $Se \subset S$ and $Sf \subset S$. I is S-convex iff Ie is Se-convex and If is Sf-convex.*

Proof. For (1), Se is a semi-cone of Re by Proposition 2.4(1). To see Ie is Se-convex in Re, let $xe, y - xe \in Se$ $(x \in S)$, and $y \in Ie$. Since $Se \subset S$ and $Ie \subset I$, $xe, y - xe \in S$, and $y \in I$. Since I is S-convex in R, $xe \in I$, hence $xe \in Ie$. For (2), the only if part holds by (1). For the if part, let $x, y - x \in S$, and $y \in I$. Then $xe, ye - xe \in Se$, and $ye \in Ie$. Since Ie is Se-convex in Re, $xe \in Ie$. Similarly, If is Sf-convex, so $xf \in If$. Hence $x = xe + xf \in Ie + If = I$. Thus I is S-convex. □

Remark 2.8. In (1) of Proposition 2.7, "$Se \subset S$" is essential. Also, it is impossible to replace "Se-convex" by "convex for Se" even if I is convex for S in R. We have similar mattes in (2) there. For these, see Example 2.20 later.

The following is a classical result, but let us give a proof for the readers.

Theorem 2.9. *Let $\sigma : (R, \leq) \to (R', \leq')$ be an epimorphism with $\sigma(S) = S'$, and let $J = Ker(\sigma)$. Then there exists a bijection Φ between the class of convex ideals I of R containing J and the class of convex ideals I' of R', defining by $\Phi(I) = \sigma(I)$ and $\Phi^{-1}(I') = \sigma^{-1}(I')$. Especially, J is a convex ideal of R.*

Proof. Let I be a convex ideal of R containing J. Evidently, $I = \sigma^{-1}(\sigma(I))$, hence $\sigma(I) \neq R'$. To see $\sigma(I)$ is convex in R', let $0 \leq' \sigma(x) \leq' \sigma(y)$ and $y \in I$. Since $0 \leq' \sigma(y - x)$ and $S' = \sigma(S)$, there exists $s \in S$ such that $\sigma(y - x) = \sigma(s)$. Thus $y - x - s = a$ for some $a \in J$. Since $0 \leq' \sigma(x)$, there exists similarly $t \in S$ such that $\sigma(x) = \sigma(t)$. Thus $x - t = b$ for some $b \in J$. Hence $s + t = y - (a + b)$. Since $J \subset I$, this implies that $s + t \in I$. Since I is convex in R, $s, t \in I$. Hence $\sigma(x) = \sigma(t) \in \sigma(I)$. Then $\sigma(I)$ is convex in R'. Conversely, let I' be a convex ideal of R'. Evidently, $\sigma(\sigma^{-1}(I')) = I'$ and $\sigma^{-1}(I') \supset J$. To see that $I = \sigma^{-1}(I')(\neq R)$ is convex in R containing J, let $0 \leq x \leq y$ and $y \in I$. Since $\sigma(S) \subset S'$, $0 \leq' \sigma(x) \leq' \sigma(y)$. Since $\sigma(y) \in I'$ and I' is convex in R', $\sigma(x) \in I'$, then $x \in I$. Thus I is convex in R. □

For convex ideals I and J of R, $I + J$ need not even be S-convex (see Example 2.20(4) later). While, for R being an ordered ring, the following holds.

Lemma 2.10. *Let $(R, \leq)$ be an ordered ring. If I_i $(i = 1, 2, \ldots, n)$ are convex ideals of R, then $I = I_1 + I_2 + \cdots + I_n$ is convex in R.*

Proof. It suffices to show that $I' = I_1 + I_2$ is convex. To see I' is proper, suppose not. Then $1 = a + b$ for some $a \in I_1$ and $b \in I_2$. We can assume $a \leq b$.

Then $0 < 1 \le 2b \in I_2$. Thus, $1 \in I_2$ by the convexity of I_2, so $I_2 = R$, a contradiction. Thus, I' is proper. Similarly, the convexity of I' is shown. Hence, I' is convex. □

Example 2.11. (1) For a partially ordered ring $R = (\mathbf{Z}, \le_S)$ with $S = n\mathbf{Z}^*$ ($n > 1$), if I and J are convex ideals of R, then $I + J$ is S-convex (by means of [6, Proposition 3.4]). But, it is impossible to replace "S-convex" by "convex" (indeed, let $I = 2\mathbf{Z}$ and $J = 3\mathbf{Z}$, and let $S = 6\mathbf{Z}^*$ in $\mathbf{Z}$. Then I and J are convex ideals of a partially ordered ring $(\mathbf{Z}, \le_S)$, but $I + J = \mathbf{Z}$).

(2) For an ordered integral domain D, let $D[x]$ be the polynomial ring over D, and for $f = a_0 + a_1 x + \cdots + a_n x^n$ in $D[x]$, $0 <_2 f$ means the first nonzero coefficient a_k is positive in D. Then $R = (D[x], \le_2)$ is an ordered integral domain (see [6], etc.). Thus, for any convex ideals I and J of R, $I + J$ is convex in R by Lemma 2.10.

Corollary 2.12. *Let $\sigma : (R, \le) \to (R', \le')$ be an epimorphism with $\sigma(S) = S'$, and I be an ideal of R, and let $J = Ker(\sigma)$. Then $I + J$ is a convex ideal of R iff so is $\sigma(I)$. For $(R, \le)$ being an ordered ring, if I is convex, then so is $\sigma(I)$.*

Proof. This is shown in view of Theorem 2.9, noting that $\sigma(I + J) = \sigma(I)$ with $I + J \supset J$, and $\sigma^{-1}(\sigma(I)) = I + J$. The latter part holds by Lemma 2.10. □

Remark 2.13. In the first half of Corollary 2.12, for I being convex in R, $\sigma(I)$ need not be S'-convex in R'; see Example 2.20(6) later. Also, the converse of the latter part need not hold (indeed, let $R = (\mathbf{Z}[x], \le_2)$. Then every ideal $A = (x^n)$ (generated by x^n ($n > 0$)) is convex in R (by [6, Remark 3.8]). Let $I = (x^2 + x)$, $I' = (x)$, $J = (x^2)$. Then I' and J are convex, but I is not convex in R (actually, $0 \le_2 x^2 \le_2 x^2 + x \in I$, but $x^2 \notin I$). Let $\varphi : R \to R/J$ be the natural map. Then $\varphi(I)$ $(= \varphi(I'))$ is convex in R/J, but I is not convex).

Corollary 2.14. *Let J be a convex ideal of R. For the natural map $\varphi : (R, \le) \to (R/J, \le)$ and an ideal I of R, $I + J$ is a convex ideal of R iff so is $I' = \varphi(I)$ of R/J. For $(R, \le)$ being an ordered ring, if I is convex, then so is $\varphi(I)$.*

Let us show that the (direct product) ring $R \times R'$ is never an ordered ring. While, there exists a certain product ring which is an ordered ring; see [5, Example 1] (or Proposition 3.21 later).

Theorem 2.15. (1) *Every ordered ring R has the largest convex ideal.*

(2) *The following* (a) *and* (b) *hold. Moreover,* (a) *and* (b) *are equivalent.*

(a) *For any rings R, R', the ring $R \times R'$ can not be an ordered ring* (*i.e.*, *$R \times R'$ has no cones*).

(b) *Any ordered ring R has no idempotents except $e = 0$ or $e = 1$.*

Proof. For (1), let $\{I_\lambda \mid \lambda \in \Lambda\}$ be the collection of all convex ideals in R. Then the sum $L = \sum_{\lambda \in \Lambda} I_\lambda$ is the largest convex ideal of R. Indeed, to see L is proper, suppose not. Then, for some I_{λ_i} $(i = 1, 2, \dots, n)$, $1 \in \sum_{i=1}^n I_{\lambda_i}$, so $R = \sum_{i=1}^n I_{\lambda_i}$. But, $\sum_{i=1}^n I_{\lambda_i}$ is proper by Lemma 2.10, a contradiction. Hence L is proper. The convexity of L is obvious by Lemma 2.10.

For (2), to see (a), suppose $(R \times R', \leq)$ is an ordered ring. We will show that $I = R \times 0$ and $J = 0 \times R'$ are convex in $R \times R'$, which implies $I + J \neq R \times R'$ by Lemma 2.10, but $I + J = R \times R'$, a contradiction. To see I is convex, let $(0,0) \leq (x, y) \leq (r, 0) \in I$. For $f = (0, 1) \in R \times R'$, $(0,0) \leq f^2 = f$. Then $(0,0)f \leq (x, y)f \leq (r, 0)f$. Thus $(0,0) \leq (0, y) \leq (0,0)$, so $y = 0$. Then $(x, y) = (x, 0) \in I$. Hence I is convex. Similarly, J is convex, using $e = (1, 0)$. Next, to see (a) $\Rightarrow$ (b), suppose some ordered ring R has an idempotent e with $e \neq 0$ and $e \neq 1$. Then $\sigma : R \to Re \times R(1-e)$ defined by $\sigma(r) = (re, r(1-e))$ is a (ring) isomorphism (actually, if $y = (re, r'(1-e))$, then for $x = re + r'(1-e)$, $\sigma(x) = y$. Also, if $\sigma(r) = (0,0)$, then $re = r(1-e) = 0$, thus $r = re + r(1-e) = 0$). Then, $Re \times R(1-e)$ is an ordered ring by the cone $\sigma(S)$, a contradiction to (a). For (b) $\Rightarrow$ (a), $e = (1, 0)$ is an idempotent in $R \times R'$, but $e \neq (0,0)$ and $e \neq (1,1)$. Hence $R \times R'$ is not an ordered ring. $\square$

Lemma 2.16. *For S-convex ideas I, I' of a ring R, $S + I = S + I'$ iff $I = I'$.*

Proof. This is shown by the proof of [6, Lemma 4.14], replacing "convex" by "S-convex" (actually, $R = S + I'$ implies $I' = R$). $\square$

In [6], we obtain the following result by means of the above lemma: Let $\sigma : R \to R'$ be an epimorphism, and I (resp. I') be a convex ideal of R (resp. R'). Assume that (*) $\sigma(I)$ is convex in R' or R is an ordered ring. If $\sigma(S + I) = S' + I'$ and $\sigma(S) = S'$, then $\sigma(I) = I'$. The convexity of I (or I') is essential ([6, Remark 4.16]), but let us consider the question whether the assumption (*) is essential. Namely,

Question 2.17. Let $\sigma : R \to R'$ be an epimorphism, and I (resp. I') be a convex ideal of R (resp. R'). If $\sigma(S + I) = S' + I'$ and $\sigma(S) = S'$, then $\sigma(I) = I'$?

For this question, we have the following by Corollary 2.12 and Lemma 2.16.

Proposition 2.18. *Let* $\sigma : R \to R'$ *be an epimorphism with* $\sigma(S) = S'$ *Let* $J = Ker(\sigma)$. *For convex ideals* I *of* R *and* I' *of* R', $\sigma(I) = I'$ *iff* $I + J$ *is convex in* R *and* $\sigma(S + I) = S' + I'$.

The following holds by Proposition 2.18.

Corollary 2.19. *Let* J *be a convex ideal of* R. *Let* $\varphi : (R, \leq) \to (R/J, \leq)$ *be the natural map. For convex ideals* I *of* R *and* I' *of* $R' = R/J$, $\varphi(I) = I'$ *iff* $I + J$ *is convex in* R *and* $\varphi(S + I) = S' + I'$.

The convexity of $I + J$ in Proposition 2.18 and Corollary 2.19 is essential, which shows that Question 2.17 is negative. Indeed, we have the following.

Example 2.20. Let $R = \mathbf{Z} \times \mathbf{Z}$. Then $e = (0, 1)$ and $f = (1, 0)$ are idempotents of R. Let $1 < n \in \mathbf{Z}$, and let $I = \{0\} \times 2n\mathbf{Z}$, $I' = \{0\} \times n\mathbf{Z}$, $J = \mathbf{Z} \times \{0\}$. For a semi-cone $A = n\mathbf{Z}^*$ of $\mathbf{Z}$, let $S = \{(k, m) \in R \mid 0 \leq_A m \leq_A k\}$. Then S is a semi-cone of R, and the following (1)$\sim$(6) hold.

(1) Se, Sf, and $Se + Sf = S \times S$ are semi-cones of the ring R, but $Se \not\subset S$, $Sf \subsetneqq S$, and $S \subsetneqq Se + Sf$.

(2) I, I', J are convex ideals of $(R, \leq_S)$, and $Jf = Rf$.

(3) I $(= Ie)$ is not Se-convex, and I' $(= I'e)$ is convex in $(Re, \leq_{Se})$.

(4) $I + J = \mathbf{Z} \times 2n\mathbf{Z}$ is not S-convex in $(R, \leq_S)$ (indeed, $(0, 0) \leq_S (n, n) \leq_S (2n, 2n) \in (I + J)$, but $(n, n) \notin (I + J)$).

(5) $(S + I)e = Se + I'$, but $Ie \neq I'$.

(6) Let $\varphi : (R, \leq_S) \to (R/J, \leq_{S'})$ be the natural map with $S' = \varphi(S)$.

(a) For the convex ideal I, $\varphi(I)$ is not S'-convex in R/J.

(b) There holds that $\varphi(S + I) = S' + \varphi(I')$, but $\varphi(I) \neq \varphi(I')$.

Indeed, note that $\psi : (R/J, \leq_{S'}) \to (Re, \leq_{Se})$ by $\psi([r]) = re$ is an isomorphism with $\psi(S') = Se$. Then (a) follows from (3), and (b) holds by (5), since $(\psi \circ \varphi)(S + I) = (\psi \circ \varphi)(S) + (\psi \circ \varphi)(I')$, but $(\psi \circ \varphi)(I) \neq (\psi \circ \varphi)(I')$.

3. Products of Partially Ordered Rings

Let R and R' be partially ordered rings, but assume $S \neq \{0\}$ and $S' \neq \{0\}$ in this section. Let $S_0 = S \backslash \{0\}$ and $S'_0 = S' \backslash \{0\}$.

In what follows, let us also use the symbol "0" instead of the set "{0}". We will consider the following typical subsets of the product set $R \times R'$.

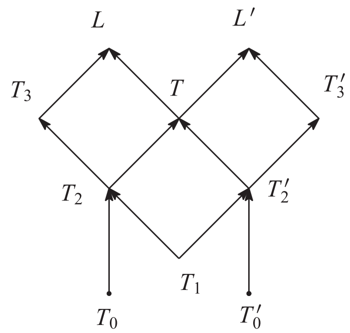
$T = S \times S'$.

$T_0 = S \times 0 \ (= (S_0 \times 0) \cup \{(0,0)\})$,
$T_1 = (S_0 \times S_0') \cup \{(0,0)\}$,
$T_2 = (S_0 \times S') \cup \{(0,0)\}$, and
$T_3 = (S_0 \times R') \cup \{(0,0)\} \ (= (S_0 \times R') \cup T_0)$.

$T_0' = 0 \times S' \ (= (0 \times S_0') \cup \{(0,0)\})$,
$T_1' = T_1$,
$T_2' = (S \times S_0') \cup \{(0,0)\}$, and
$T_3' = (R \times S_0') \cup \{(0,0)\} (= (R \times S_0') \cup T_0')$.

Remark 3.1. Obviously, the following (a), (b), and (c) hold, here we define the *lexicographic sets* L and L' in (c). Also, we have the diagram below.

(a) $T_0 \subset T_2$, $T_1 \subset T_2 \subset T_3$, $T_2 \subset T$; and $T_0' \subset T_2'$, $T_1' = T_1 \subset T_2' \subset T_3'$, $T_2' \subset T$.
(b) $T_2 = T_0 \cup T_1$, $T_2' = T_1 \cup T_0'$; and $T = T_0 \cup T_1 \cup T_0' = T_0 \cup T_2' = T_2 \cup T_0' = T_2 \cup T_2'$.
(c) $L = T_3 \cup T = T_3 \cup T_2' = T_3 \cup T_0'$; and $L' = T \cup T_3' = T_2 \cup T_3' = T_0 \cup T_3'$.



Remark 3.2. (1) Obviously, the sets T, T_i, T_i' $(i = 0,1,2,3)$, L, and L' satisfy (i) and (ii) in Definition 2.1 (with respect to these sets). But, neither $S \times R'$ nor $R \times S'$ satisfies (i) (cf. T_3 or T_3').

(2) None of sets $T_0 \cup T_0'$, $T_3 \cup T_3'$, L, and L' are semi-cones of $R \times R'$ (indeed, let $s \in S_0$ and $s' \in S_0'$. Then $(s,0) + (0,s') = (s,s') \notin T_0 \cup T_0'$; $(s,-1) \cdot (0,s') = (0,-s') \notin T_3 \cup T_3' \cup L$; and $(s,0) \cdot (-1,s') = (-s,0) \notin L'$).

(3) Let $\mathscr{C} = \{T_0, T_1, T_2, T_3, T_0', T_2', T_3', T\}$. Let $\mathscr{F}$ be the collection of finite unions of $\mathscr{C}$ (containing the sets in $\mathscr{C}$), but except unions which are never semi-cones of $R \times R'$. Then $\mathscr{F} = \mathscr{C}$, that is, $\mathscr{F} = \{T_0, T_1, T_2, T_3, T_0', T_2', T_3', T\}$. Indeed, let $\mathscr{C}(\leq 2)$ be the collection of unions of at most two sets in $\mathscr{C}$, and let $\mathscr{C}(> 2)$ be the collection of unions of more than two sets in $\mathscr{C}$. Using Remark 3.1, we show that $\mathscr{C}(\leq 2) = \mathscr{C} \cup \mathscr{C}^*$, where $\mathscr{C}^* = \{T_0 \cup T_0', T_3 \cup T_3', L, L'\}$, and $\mathscr{C}(> 2) \subset \mathscr{C}(\leq 2)$. But, any set in $\mathscr{C}^*$ is never a semi-cone of $R \times R'$ in view of (2). Thus $\mathscr{F} = \mathscr{C}$. (Actually, when R and R' are integral domains, all sets in $\mathscr{F}$ are semi-cones; see Corollary 3.4(1) below).

We give characterizations for the sets in the collection $\mathscr{F}$ to be semi-cones of the ring $R \times R'$.

Theorem 3.3. *Let R and R' be partially ordered rings. Then the following hold.*

(1) *T, T_0, and T_0' are semi-cones of $R \times R'$.*

(2) *T_1 is a semi-cone of $R \times R'$ iff* (i) *$S_0S_0 \subset S_0$ and $S_0'S_0' \subset S_0'$, otherwise* (ii) *$S_0S_0 = 0$ and $S_0'S_0' = 0$.*

(3) *T_2 is a semi-cone of $R \times R'$ iff $S_0S_0 \subset S_0$ or $S_0'S_0' = 0$.*

(4) *T_2' is a semi-cone of $R \times R'$ iff $S_0S_0 = 0$ or $S_0'S_0' \subset S_0'$.*

(5) *T_3 is a semi-cone of $R \times R'$ iff $S_0S_0 \subset S_0$.*

(6) *T_3' is a semi-cone of $R \times R'$ iff $S_0'S_0' \subset S_0'$.*

Proof. (1) is obvious. It is routine to see the if parts in (2)$\sim$(6). So, we will see their "only if" parts. For (2), assume $S_0S_0 \neq 0$. Then $S_0'S_0' \subset S_0'$. Indeed, take some $a, b \in S_0$ with $ab \neq 0$. Suppose $S_0'S_0' \not\subset S_0'$. Take $c, d \in S_0'$ with $cd = 0$. Thus $(a, c) \cdot (b, d) = (ab, 0) \notin T_1$, a contradiction. Hence $S_0'S_0' \subset S_0'$. Similarly, $S_0'S_0' \neq 0$ implies $S_0S_0 \subset S_0$. Therefore, the following are equivalent: $S_0S_0 \subset S_0$; $S_0S_0 \neq 0$; $S_0'S_0' \subset S_0'$; and $S_0'S_0' \neq 0$. Hence (i) or (ii) holds. For (3) and (4), assume $S_0'S_0' \neq 0$. Take $c, d \in S_0'$ with $cd \neq 0$. Suppose $S_0S_0 \not\subset S_0$. Take $a, b \in S_0$ with $ab = 0$. Thus $(a, c) \cdot (b, d) = (0, cd) \notin T_2$, a contradiction. Thus $S_0S_0 \subset S_0$. (4) is similarly shown. For (5) and (6), $(x, 1) \cdot (y, 1) = (xy, 1) \in T_3$ implies $S_0S_0 \subset S_0$. (6) is similarly shown. □

Corollary 3.4. *Let R and R' be partially ordered rings. Then the following hold.*

(1) *T, T_0, and T_0' are semi-cones of $R \times R'$. For R and R' being integral domains, the other sets in $\mathscr{F}$ are also semi-cones of $R \times R'$.*

(2) *Let* $1 \in S$ *and* $1 \in S'$. *Then the following hold.*
(a) T_1 *is a semi-cone of* $R \times R'$ *iff* $S_0 S_0 \subset S_0$ *and* $S_0' S_0' \subset S_0'$.
(b) T_2 *is a semi-cone of* $R \times R'$ *iff* $S_0 S_0 \subset S_0$.
(c) T_2' *is a semi-cone of* $R \times R'$ *iff* $S_0' S_0' \subset S_0'$.

Remark 3.5. For a partially ordered ring R, (a) $S_0 S_0 \subset S_0$ need not hold, and (b) $S_0 S_0 \subset S_0$ need not imply that R is an integral domain. Indeed, for (a), take a semi-cone $S'' = T$ of $R'' = R \times R'$, and for (b), take a semi-cone $S'' = T_1$ of R'' with R and R' integral domains, in view of Corollary 3.4(1). Then the partially ordered ring $(R'', \leq_{S''})$ is a desired one.

For convenience, henceforth let us assume that all sets in $\mathscr{F}$ are semi-cones of the ring $R \times R'$ (cf. Corollary 3.4), unless otherwise stated.

Let $P_R : R \times R' \to R$ and $P_{R'} : R \times R' \to R'$ be the projections (i.e., $P_R(x, y) = x$, $P_{R'}(x, y) = y$). These projections are obviously epimorphisms.

Remark 3.6. P_R or $P_{R'}$ need not be order-preserving, and also need not preserve the convexity of an ideal. Indeed, let us see these for P_R (similar for $P_{R'}$). Evidently, $P_R(T_3') = R$, hence P_R is not order-preserving. Let $R = R' = \mathbf{Z}$, and $1 < n \in \mathbf{Z}$. Let $S' = n\mathbf{Z}^*$, and $S'' = \{(k, m) \in \mathbf{Z} \times \mathbf{Z} \mid 0 \leq_{S'} k \leq_{S'} m\}$. Then, S'' is a semi-cone, and $I'' = 2n\mathbf{Z} \times 0$ is a convex ideal, but $P_R(I'') = 2n\mathbf{Z}$ is not convex in $(\mathbf{Z}, \leq_{S'})$ (cf. Example 2.20).

The following is well-known or routinely shown.

Lemma 3.7. *For a subset* A *of the ring* $R \times R'$, A *is an ideal of* $R \times R'$ *iff* $A = P_R(A) \times P_{R'}(A)$, $P_R(A)$ *is an ideal of* R, *and so is* $P_{R'}(A)$ *of* R'.

Proposition 3.8. *For* $T = S \times S'$, *and an ideal* J *of* $R \times R'$, J *is* T*-convex iff* $P_R(J)$ *is* S*-convex in* R *and* $P_{R'}(J)$ *is* S'*-convex in* R'.

Proof. For the if part, to see J is T-convex, let $(0,0) \leq_T (x, y) \leq_T (a, b) \in J$. Then $0 \leq_S x \leq_S a \in P_R(J)$, so $x \in P_R(J)$. Similarly, $y \in P_{R'}(J)$. Then $(x, y) \in J$ by Lemma 3.7. Hence J is T-convex. For the only if part, to see $P_R(J)$ is S-convex in R, let $0 \leq_S x \leq_S a \in P_R(J)$. Then $(0,0) \leq_T (x,0) \leq_T (a,0)$, and $(a,0) \in J$ by Lemma 3.7. Since J is T-convex, $(x,0) \in J$, so $x \in P_R(J)$. Hence $P_R(J)$ is S-convex. Similarly, $P_{R'}(J)$ is S'-convex in R'. □

The following holds by Proposition 3.8 and Lemma 2.16, related to Question 2.17.

PROPOSITION 3.9. *Let J be a T-convex ideal of $R \times R'$, and let J' be an S-convex ideal of R. Then $P_R(T+J) = S+J'$ iff $P_R(J) = J'$.*

REMARK 3.10. Let us give analogues to Propositions 3.8 and 3.9 for the sets in $\mathscr{F}$. For an ideal I of $R \times R'$, let us consider conditions (p_1) $P_R(I) \cap S_0 \neq \emptyset$, and (p_2) $P_{R'}(I) \cap S_0' \neq \emptyset$. We note that (p_1) (resp. (p_2)) holds if R (resp. R') is an ordered ring. Then the following hold for ideals I and J of $R \times R'$.

(1) (a) $P_R(I)$ is S-convex in R if I is A-convex for $A = T_i$ $(i = 0, 2, 3)$, T_i' $(i = 1, 2, 3)$, but assume (p_2) for T_i' $(i = 1, 2, 3)$. Also, $P_{R'}(I)$ is S'-convex in R if I is A-convex for $A = T_i$ $(i = 1, 2, 3)$, T_i' $(i = 0, 2, 3)$, but assume (p_1) for T_i $(i = 1, 2, 3)$. Conversely,

(b) J is A-convex in $R \times R'$ for $A = T_1, T_2$, or T_2' if $P_R(J)$ is S-convex and $P_{R'}(J)$ is S'-convex. Also, J is T_0-convex if $P_R(J)$ is S-convex. Similarly, J is T_0'-convex if $P_{R'}(J))$ is S'-convex.

(2) Proposition 3.8 remains true for T_1, T_2, and T_2', but for T_1 (resp. T_2; T_2'), assume (p_1) and (p_2) (resp. (p_1); (p_2)). Also, J is T_0-convex iff $P_R(J)$ is S-convex. Similarly, J is T_0'-convex iff $P_R'(J)$ is S'-convex.

(3) Proposition 3.9 remains true for T_i $(i = 0, 2, 3)$, T_1', T_2', but assume (p_2) for T_1', T_2'. Also, for $P_{R'}$, the similar result holds for T, T_1, T_2, T_i' $(i = 0, 2, 3)$, but assume (p_1) for T_1, T_2. While, Proposition 3.9 need not hold for $A = T_0'$ or T_3'.

Indeed, (1) is shown as in the proof of Proposition 3.8. (For example, for (a), to see $P_R(I)$ is S-convex in R for $T_3' = (R \times S_0') \cup \{(0,0)\}$, let $0 \leq_S x \leq_S a \in P_R(I)$, and take $p \in P_{R'}(I) \cap S_0'$ by (p_2). Then $(0,0) \leq_{T_3'} (x, p) \leq_{T_3'} (a, 2p) \in P_R(I) \times P_R'(I) = I$. Thus, $(x, p) \in I$, so $x \in P_R(I)$). (2) and (3) hold in view of (1). For the last part of (3), let J' be a convex ideal and $S \ni 1$ in R. Then $J = J' \times 0$ is convex for A, and $P_R(J) = J'$. But, $P_R(A+J) \neq S+J'$. To see this, suppose $P_R(A+J) = S+J'$. Then, for $A = T_0'$, $J' = S+J' \ni 1$, so $J' = R$, a contradiction. For $A = T_3'$, $R = S+J'$, so $R = J'$ by Lemma 2.16, a contradiction).

EXAMPLE 3.11. In Proposition 3.9, the convexity of the ideals J and J' is essential for J and J' being proper. Indeed, let $R = (\mathbf{Z}[x], \leq_2)$. Let $I = (x)$, and $A = (2x)$. Then I is convex in R. But, A is not convex in R (indeed, $0 \leq_2 x \leq_2$

$2x \in A$, but $x \notin A$). Also, (*) $S + I = S + A$ holds. For $R' = R$, $I^* = I \times I$ and $A^* = A \times A$, the following hold.

(1) I^* is convex in $R \times R'$ for T, but A is not convex in R. While, $P_R(T + I^*) = S + A$ by (*), but $P_R(I^*) \neq A$.

(2) A^* is not convex in $R \times R'$ for T, but I is convex in R. While, $P_R(T + A^*) = S + I$ by (*), but $P_R(A^*) \neq I$.

Let us recall the following ring on the product set $P = R \times R$ of ring R with itself.

Definition 3.12. Let R be a ring. For $(a,b) \in P$, let $P(a,b) = (P, +, *)$ be the commutative ring defined by the following addition (i) and multiplication (ii):

For $(x, y), (z, w) \in P$, let

(i) $(x, y) + (z, w) = (x + z, y + w)$.

(ii) $(x, y) * (z, w) = (xz + ayw, xw + yz + byw)$.

Then $e = (1,0)$ is the identity element, and for $u = (0,1)$, $u * u = (a,b)$, and $(x, y) = (x, 0) * e + (y, 0) * u$ in $P(a,b)$.

The ring $P(0,0)$ is an algebra over R which has a basis $\{e, u\}$ with $u * u = (0,0)$, and it is called the *trivial extension* of R by itself (see [8], etc.). This ring gives useful examples related to ring structures and order structures, or extensions. We investigate order structures of the ring $P(0,0)$ in terms of semi-cones or cones. (We consider $P(a,b)$ in [7] in terms of ring structures).

Notation. For a ring R, the symbol $R \ltimes R$ denotes the ring $P(0,0)$.

Remark 3.13. (1) Let $R[x]$ be the polynomial ring over a ring R, and $I = (x^2)$. Then $R \ltimes R$ is (ring) isomorphic to $R[x]/I$ by a map $(a,b) \mapsto [a + bx]$.

(2) For a subset A of $R \ltimes R$, let $A^* = \{(x, -y) \mid (x, y) \in A\}$. Then A is a semi-cone of $R \ltimes R$ iff so is A^*, and also for a semi-cone A of $R \ltimes R$, I is a convex ideal of $R \ltimes R$ for A iff so is I^* for A^*, by a (ring) isomorphism $(x, y) \mapsto (x, -y)$.

Let us consider the sets T, T_i, T'_j $(i, j = 0, 1, 2, 3)$ in $R \ltimes R$, putting $R' = R$ and $S' = S$.

Remark 3.14. (1) $T_0 \cup T'_0$ is not a semi-cone of $R \ltimes R$ by Remark 3.2(2). Also, T'_3 is not a semi-cone of $R \ltimes R$, and any union of T, T_i, T'_j $(i, j = 0, 1, 2, 3)$

containing T_3' is not a semi-cone of $R \ltimes R$ (indeed, for $s \in S_0$, $(-1, s) * (0, s) = (0, -s) \notin T_3'$. The latter part holds by $(0, -s) \notin T \cup T_i \cup T_j'$).

(2) Let $\mathscr{C} = \{T_0, T_1, T_2, T_3, T_0', T_2', T_3', T\}$. Let $\mathscr{F}_P$ be the collection of finite unions of $\mathscr{C}$ (containing the sets in $\mathscr{C}$), but except unions which are never semi-cones of $R \ltimes R$. Then, $\mathscr{F}_P = \{T_0, T_1, T_2, T_3, T_0', T_2', T, L\}$ by (1), reviewing the poof of Remark 3.2(3). (Actually, when R is an integral domain, all sets in $\mathscr{F}_P$ are semi-cones of $R \ltimes R$; see Corollary 3.17(1) later).

For a semi-cone S of R, let us consider the following conditions around condition (*) $S_0 S_0 \subset S_0$ on R.

(c_1) For $x, z \in S_0$, if $xz \in S_0$ (i.e., $xz \neq 0$), then $xS_0 \subset S_0$ or $zS_0 \subset S_0$.

(c_2) For $x, z \in S_0$, if $xz \notin S_0$ (i.e. $xz = 0$), then $xS_0 = 0$ and $zS_0 = 0$.

We can replace "$xS_0 \subset S_0$ or $zS_0 \subset S_0$" by "$xS_0 + zS_0 \neq 0$" in (c_1). Also, we can replace "$xS_0 = 0$ and $zS_0 = 0$" by "$xS_0 + zS_0 = 0$ (or $xS + zS = 0$)" in (c_2).

REMARK 3.15. (1) None of (*) (i.e., $S_0 S_0 \subset S_0$), (c_1), and (c_2) hold for some partially ordered ring R.

(2) Obviously, (*) implies (c_1) and (c_2). But, (c_1) and (c_2) need not imply (*) by the following (3) and (4).

(3) (c_2) implies (c_1). But, the converse does not hold for some ordered ring R.

(4) For $S_0 \ni 1$, (c_2) implies (*). But, (c_2) need not imply (*) without $S_0 \ni 1$.

Indeed, (1) is shown by the proof of Remark 3.5(a), but assume $SS \neq 0$ in T. For (3), assume (c_2) holds. If $xz \in S_0$ for $x, z \in S_0$, then $xS_0 \subset S_0$ and $zS_0 \subset S_0$. To see this, suppose $xS_0 \not\subset S_0$, then $xy = 0$ for some $y \in S_0$. Thus $xS_0 = 0$ by (c_2), hence $xz = 0$, a contradiction. Thus $xS_0 \subset S_0$ (similarly, $zS_0 \subset S_0$). Then (c_1) holds. For the latter part, let $(R, \leq)$ be the ordered ring in [5, Example 1]. Then we may consider the ordered ring $(R, \leq)$ as the ring $R' = K \ltimes K = \{(a, b) \mid a, b \in K\}$ with K an ordered field, where R' has a cone $S' = L$ (cf. Corollary 3.17(1) below). Then $u = (0, 1) \in S_0'$ and $u * u = (0, 0)$. But, $e = (1, 0) \in S_0'$, then $uS_0' \neq \{(0, 0)\}$. Hence (c_2) does not hold (also, $S_0' S_0' \not\subset S_0'$). For $x = (a, b)$, $z = (c, d) \in S_0'$ with $x * z \neq (0, 0)$. Then $a \neq 0$ or $c \neq 0$, so $a > 0$ or $c > 0$ in K. Hence $xS_0' \subset S_0'$ or $zS_0' \subset S_0'$. Then (c_1) holds. Hence, R' is a desired one (for S'). For (4), suppose $S_0 S_0 \not\subset S_0$, then $xz = 0$ for some $x, z \in S_0$. But, $x = x1 \in xS_0 = 0$ by (c_2), so $x = 0$, a contradiction. Hence, $S_0 S_0 \subset S_0$. For the latter part in (4), let $R' = R \ltimes R$, $A = T_0'$ $(= 0 \times S)$, and $A_0 = 0 \times S_0$. Then $AA = \{(0, 0)\}$. Thus, (c_2) holds, but $A_0 A_0 \not\subset A_0$. Then R' is a desired one (for A).

We give characterizations for the sets in the collection $\mathscr{F}_P$ to be semi-cones of $R \ltimes R$, in comparison with Theorem 3.3 for $R \times R'$.

Theorem 3.16. *Let R be a partially ordered ring. Then the following hold.*
(1) *T, T_0, and T_0' are semi-cones of $R \ltimes R$.*
(2) *T_1 is a semi-cone of $R \ltimes R$ iff (c_2) holds.*
(3) *T_2 is a semi-cone of $R \ltimes R$ iff (c_2) holds.*
(4) *T_2' is a semi-cone of $R \ltimes R$ iff (c_1) holds.*
(5) *T_3 is a semi-cone of $R \ltimes R$ iff $S_0S_0 \subset S_0$.*
(6) *$T_3 \cup T_0'$ $(= L)$ is a semi-cone of $R \ltimes R$ iff $S_0S_0 \subset S_0$.*

Proof. For (1), the result is obviously shown.

For (2), to see the if part, let $(x, y), (z, w) \in S_0 \times S_0$. By (c_2) with Remark 3.15(3), for $xz = 0$, $xw + yz = 0$, and for $xz \neq 0$, $xw + yz \in S_0$. Thus $(x, y) * (z, w) = (xz, xw + yz) \in T_1$. For the only if part, suppose (c_2) does not hold. Then we assume that for some $x, z, w \in S_0$, $xz = 0$, but $xw \neq 0$. Let $y \in S_0$, then $xw + yz \neq 0$. Thus $(x, y), (z, w) \in S_0 \times S_0$, but $(x, y) * (z, w) = (xz, xw + yz) = (0, xw + yz) \notin T_1$, a contradiction. Then (c_2) holds.

For (3), the result is shown as in the proof of (2).

For (4), to see the if part, let $(x, y), (z, w) \in S \times S_0$. If $xz = 0$, $(x, y) * (z, w) = (0, xw + yz) \in T_2'$, so assume $xz \neq 0$. Then $xw + yz \in S_0$ by (c_1), hence $(x, y) * (z, w) = (xz, xw + yz) \in T_2'$. For the only if part, suppose that (c_1) doesn't hold. Then for some $x, z, y, w \in S_0$, $xz \neq 0$, $xw = 0$, and $yz = 0$. Thus, $(x, y) * (z, w) = (xz, xw + yz) = (xz, 0) \notin T_1$, a contradiction. Then (c_1) holds.

For (5) and (6), their if parts are routine. For their only if parts, suppose $S_0S_0 \not\subset S_0$, and take $x, y \in S_0$ with $xy = 0$. Then $(x, -1), (y, 0) \in T_3$, but $(x, -1) * (y, 0) = (0, -y) \notin T_3 \cup T_0'$, a contradiction. Hence, $S_0S_0 \subset S_0$. □

The following holds by Theorem 3.16 and Remark 3.15.

Corollary 3.17. *Let R be a partially ordered ring. Then the following hold.*
(1) *T, T_0, and T_0' are semi-cones of $R \ltimes R$. For R being an integral domain, the other sets in $\mathscr{F}_P$ are also semi-cones of $R \ltimes R$.*
(2) *For $S \ni 1$, T_1 (or T_2) is a semi-cone of $R \ltimes R$ iff $S_0S_0 \subset S_0$.*

In view of the previous corollary, for an ordered integral domain R, the lexicographic set L is a cone of $R \ltimes R$, though L is not even a semi-cone of the ring $R \times R$ (by Remark 3.2(2)).

It is well-known (or routinely shown) that for a field K, any non-zero, proper ideal of $K \ltimes K$ (resp. $K \times K$) is $0 \times K$ (resp. $0 \times K$ or $K \times 0$). We note that $I_0 = 0 \times R$ is an ideal in $R \ltimes R$, but $I_0' = R \times 0$ is not an ideal (I_0 and I_0' are ideals in $R \times R$). Let us consider the convexity of I_0 in $R \ltimes R$ (or I_0, I_0' in $R \times R$).

Let $pr : R \ltimes R$ (or $R \times R$) $\to R$ be the projection defined by $pr(x, y) = x$. Then pr is an epimorphism.

LEMMA 3.18. *Let A be a semi-cone of $R \ltimes R$ (or $R \times R$). Then $I_0 = 0 \times R$ is a convex ideal for A iff $pr(A) \cap -pr(A) = 0$.*

PROOF. Let $\leq = \leq_A$. For the if part, let $(0,0) \leq (x, y) \leq (0,b) \in I_0$. Then $(0,0) \leq (-x, b - y)$, hence $x \in pr(A) \cap -pr(A)$, so $x = 0$. Thus, $(x, y) = (0, y) \in I_0$. For the only if part, let $x \in pr(A) \cap -pr(A)$. Then for some $y, y' \in R$, $(0,0) \leq (x, y)$ and $(0,0) \leq (-x, y')$. Then, $(0,0) \leq (x, y) \leq (x, y) + (-x, y') = (0, y + y') \in I_0$. Since I_0 is convex in $R \ltimes R$ (or $R \times R$), $(x, y) \in I_0$, hence $x = 0$. □

Obviously, $I_0 = 0 \times R$ is convex in $R \ltimes R$ for the semi-cones in $\mathscr{F}_P$. Also, the following holds (hence, for R being a field, I_0 is the only non-zero, convex ideal).

PROPOSITION 3.19. *For an integral domain R, I_0 is convex for any semi-cone A of $R \ltimes R$.*

PROOF. To see $pr(A) \cap -pr(A) = 0$, let $x \in pr(A) \cap -pr(A)$. Then $x = pr(x, y) = -pr(z, w)$ for some $(x, y), (z, w) \in A$. Then $x = -z$, and hence $(x, y) + (z, w) = (x + z, y + w) = (0, y + w) \in A$. Thus $(x, y) * (0, y + w) = (0, x(y + w)) \in A$, and similarly, $(0, z(y + w)) \in A$. Hence $(0, x(y + w)) = -(0, z(y + w)) \in A \cap -A$. Thus $x(y + w) = 0$. Since R is an integral domain, $x = 0$ or $y + w = 0$. If $y + w = 0$, then $y = -w$, so $(x, y) = (-z, -w) = -(z, w) \in A \cap -A$, thus $x = 0$. Then $pr(A) \cap -pr(A) = 0$, which implies that I_0 is convex in $R \ltimes R$ by Lemma 3.18. □

REMARK 3.20. For the ring $R \times R$, $I_0 = 0 \times R$ is obviously a convex ideal of $R \times R$ for the semi-cones in $\mathscr{F}$, but remove T_3' even if R is an integral domain. Also, for an integral domain R, I_0 is convex for a semi-cone A of $R \times R$ if $A \ni (a, 0)$ for some $a \neq 0$ (indeed, let $x \in pr(A) \cap -pr(A)$, and $(a, 0) \in A$ with $a \neq 0$. Then $(ax, 0), (-ax, 0) \in A$. Thus $(ax, 0) \in A \cap -A$, hence $x = 0$. Thus, I_0 is convex for A by Lemma 3.18). Also, for $I_0' = R \times 0$, similarly the analogous results hold.

Proposition 3.21. (1) *For an ordered* (*resp. partially ordered*) *integral domain* R, L *and* $L^* = \{(x, -y) \mid (x, y) \in L\}$ *are cones* (*resp. semi-cones*) *of* $R \ltimes R$.

(2) *For an ordered field* K *and a cone* A *of* $K \ltimes K$, *the following are equivalent.*

(a) $A \supset T_0$ $(= S \times 0)$.

(b) $pr(A) \supset S$.

(c) $pr(A) = S$.

(d) $A = L$ *or* $A = L^*$.

Proof. (1) holds in view of Corollary 3.17(1) and Remark 3.13(2).

For (2), obviously, the implication (d) $\Rightarrow$ (c) $\Rightarrow$ (b) holds. (a) $\Rightarrow$ (d) holds by putting $a = (a, 0)$, $b = (b, 0)$, and $e = (1, 0)$, $u = (0, 1)$ in the proof of Example 1 in [5]. Indeed, A is a cone, so $u \in A$ or $u \in -A$. In case of $u \in A$, let $(a, b) \in L$. If $a = 0$, then $b \in S$, so $(b, 0) \in A$ by (a), thus $(a, b) = (0, b) = (b, 0) * u \in A$. If $a \neq 0$, then $a \in S$, and $(a, 0) \in A$ by (a), thus $(a, b) = (a, 0) * (1, b/2a)^2 \in A$. Hence, $L \subset A$, so $A = L$. In case of $u \in -A$ (i.e., $-u = (0, -1) \in A$), let $(a, -b) \in L^*$. Then, similarly $(a, -b) \in A$. Thus, $A = L^*$. For (b) $\Rightarrow$ (a), let $s \in S$. Then $(s, s') \in A$ for some $s' \in K$ by (b). Thus, for $s \neq 0$, $(s, 0) = (s, s') * (1, -s'/2s)^2 \in A$. Hence, $T_0 \subset A$. □

Corollary 3.22. *Let* K *be an ordered field such that* (*) *for each* $a \in S$, *there exists* $b \in K$ *with* $a = b^2$ (*in particular*, K *is the field of real numbers*, *or the field of algebraic real numbers over the rational number field*). *Then for a cone* A *of* $K \ltimes K$, $A = L$ *or* $A = L^*$.

Proof. To see $A \supset T_0$, let $(a, 0) \in T_0$. Then for some $b \in K$, $(a, 0) = (b^2, 0) = (b, 0)^2 \in A$. Then $A = L$ or L^* by Proposition 3.21. The parenthetic part implies (*), as is well-known. □

For a field K, we will give a characterization for cones of $K \ltimes K$. The following lemma is obvious.

Lemma 3.23. *For a subring* R' *and cone of* A *of* R, $A \cap R'$ *is a cone of* R'.

Theorem 3.24. *For a field* K, *let* $\mathscr{S}$ *be the collection of all cones of* K, *and let* $\tilde{\mathscr{S}}$ *be the collection of all cones of* $K \ltimes K$. *Then* $\tilde{\mathscr{S}} = \{L(S), L(S)^* \mid S \in \mathscr{S}\}$, *where* $L(S) = (S_0 \times K) \cup (0 \times S)$.

PROOF. For a cone S of K, $L(S)$ and $L(S)^*$ are cones of $K \ltimes K$ in view of Proposition 3.21(1). Conversely, let A be a cone of $K \ltimes K$, and let $K' = K \ltimes 0$. Since K' is a subring of $K \ltimes K$, $S = A \cap K'$ is a cone of K' by Lemma 3.23. But, we can consider S as a cone in K by a (ring) isomorphism $K' \to K$, $(x, 0) \mapsto x$. Since $A \supset S \times 0$, $A = L(S)$ or $A = L(S)^*$ by Proposition 3.21(2). Thus, $\tilde{\mathscr{S}} = \{L(S), L(S)^* \mid S \in \mathscr{S}\}$. □

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