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AN ASYMPTOTIC EXTENSION OF MORAN CONSTRUCTION IN METRIC MEASURE SPACES

By

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Abstract. In this paper, we define asymptotically generalized Cantor sets in metric measure spaces by generalizing the notion of λ-similarity maps. We define the notion of (λ, c, v)-similarity maps, and extend the Moran theorem about the generalized Cantor set in $\mathbf{R}^d$ to this general setting. As an example, we construct generalized Cantor sets in Riemannian manifolds by using (λ, c, v)-similarity maps.

1. Introduction

A bijective map $f : \mathbf{R}^d \to \mathbf{R}^d$ is called a *λ-similarity map* if there exists a real number $\lambda > 0$, such that

$$d(f(x), f(y)) = \lambda d(x, y)$$

for every $x, y \in \mathbf{R}^d$. Moran constructed general cantor sets in $\mathbf{R}^d$ by using the notion of *λ-similarity maps*, and determined the Hausdorff dimension of them as the similarity dimension (see [6], for instance). However, in general, it is difficult to construct a λ-similarity map in metric spaces. Actually, λ-similarity maps do not always exist on curved metric spaces. In the present paper, we generalize the notion of λ-similarity map to construct generalized Cantor sets in general metric measure spaces. This is done by introducing the notion of *(λ, c, v)-similarity maps*. Let X be a metric space and $A, B \subset X$. We call a bijective map $f : A \to B$ a *(λ, c, v)-similarity map* if there exist real number $c > 0$ and $0 < \lambda < 1$ such that for every $x, y \in A$,

$$\left| \frac{d(f(x), f(y))}{d(x, y)} - \lambda \right| \leq \lambda c |A| \tag{1}$$

$$B(f(x), \lambda r(1 - c|A|)) \subset f(B(x, r)) \tag{2}$$

$$|B| \leq v|A| \tag{3}$$

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whenever the ball $B(x,r) \subset A$, where $|A|$ is the diameter of A. The set B is called a (λ, c, v)-similar set of the set A.

An asymptotically generalized Cantor set in X is defined as follows. First assume the following:

(1) For a fixed integer $k > 1$, consider k subsets $\Delta_1, \dots, \Delta_k \subset X$, each of which is bounded and closed, satisfying $\overline{(\Delta_i)^0} = \Delta_i$, $\Delta_i \cap \Delta_j = \varnothing$ $(i \neq j)$, where Δ^0 and $\bar{\Delta}$ denote the interior and the closure of Δ respectively. These sets are called *basic sets*.

(2) Fix *ratio coefficients* $0 < \lambda_i < 1$ $(i = 1, 2, \dots, k)$ and a constant $c > 0$. For any $1 \leq i, j \leq k$, let Δ_{ij} be (λ_i, c, v)-similar sets of Δ_i such that:

(a) $\Delta_{ij} \subset \Delta_i$

(b) $\Delta_{ij} \cap \Delta_{ij'} = \varnothing$ $(j \neq j')$ $(j, j' \in \{1, 2, \dots, k\})$

(3) For any $n \geq 2$ and $\omega_1, \dots, \omega_n \in \{1, 2, \dots, k\}$, construct $(\lambda_{\omega_n}, c, v)$-similar sets $\Delta_{\omega_1 \cdots \omega_n}$ of $\Delta_{\omega_1 \cdots \omega_{n-1}}$ such that:

(a) $\Delta_{\omega_1 \cdots \omega_n} \subset \Delta_{\omega_1 \cdots \omega_{n-1}}$

(b) $\Delta_{\omega_1 \cdots \omega_n} \cap \Delta_{\omega_1 \cdots \omega_n'} = \varnothing$ $(\omega_n \neq \omega_n')$ $(\omega_n, \omega_n' \in \{1, 2, \dots, k\})$

Then, define a set C as

$$C := \bigcap_{n=1}^{\infty} \left(\bigcup_{\omega_1, \dots, \omega_n = 1}^{k} \Delta_{\omega_1 \cdots \omega_n} \right),$$

which is called an *asymptotically generalized Cantor set* in X.

Although it is hard to construct generalized Cantor sets in curved spaces via *λ-similarity maps*, it is easy to construct asymptotically generalized Cantor sets in curved spaces (see Example 3.2).

The main result in this paper is as follows.

Main Theorem. *Let X be a metric space with a regular Borel measure μ. Suppose that (X, μ) satisfies the following assumptions*:

(a) *Any closed ball $B(x,r)$ in X is compact.*

(b) *For any $x_0 \in X$ and $0 < r, \delta < 1$,*

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, \delta r))} \leq C(\delta)$$

where $C(\delta) > 0$ is a constant independent of x_0 and r.

Let C be an asymptotically generalized Cantor set in X with ratio coefficients $\lambda_1, \dots, \lambda_k$ defined above. Then the Hausdorff dimension of C is the same as the similarity dimension. Namely it is equal to t such that $\sum_{i=1}^{k} \lambda_i^t = 1$.

In general, generalized Cantor sets containing well-known examples were also defined by a family of similarity contractions $\{f_1, \dots, f_n\}$ on a metric space as the unique nonempty compact set K satisfying $K = \bigcup_{i=1}^n f_i(K)$. Hutchinson [4] (cf. Kigami [7], Schief [9]) introduced the notion of the open set condition and extended Moran's result for generalized Cantor sets in $\mathbf{R}^d$ satisfying the open set condition.

Balogh and Rohner extended Hutchinson's result to doubling metric spaces ([11]). The assumption (b) of our Main Theorem is essentially the same as considering doubling metric spaces. Since analysis on doubling metric measure spaces is now very active (see for instance Assouad [1], Heinonen [3]), it is meaningful to consider generalized Cantor sets in such metric measure spaces. Recently we have obtained an asymptotic extension of Balogh and Rohner's result. This will appear in a forthcoming paper.

The organization of the present paper is as follows. In section 2, we prove Main Theorem. In the proof of Main Theorem, the proof of $\dim_H C \geq t$ is the most essential part. In particular, the assumption (b) on the Borel measure is needed in this part, to obtain a uniform bound on the number of small ball contained in a larger ball. Therefore the assumption (b) can be replaced by doubling conditions on metric spaces.

In section 3, we give an example of generalized Cantor sets on non-flat Riemannian manifold.

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2. Proof of Main Theorem

Definition 2.1. Let X be a metric space, $Z \subset X$ and α be a nonnegative real number. An ε-cover $\{U_i\}$ of Z is a finite or countable collection of sets U_i with $|U_i| \leq \varepsilon$ covering Z.

For an $\varepsilon > 0$ define $m(Z, \alpha, \varepsilon)$ by

$$m(Z, \alpha, \varepsilon) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^{\alpha} \mid \{U_i\} : \varepsilon\text{-cover of } Z \right\}.$$

The *α-dimensional Hausdorff measure of Z* is defined by the formula

$$m(Z, \alpha) = \lim_{\varepsilon \to 0} m(Z, \alpha, \varepsilon).$$

The *Hausdorff dimension* $\dim_H Z$ *of* Z is defined as

$$\dim_H Z := \sup\{\alpha \geq 0 \mid m(Z, \alpha) = \infty\} = \inf\{\alpha \geq 0 \mid m(Z, \alpha) = 0\}.$$

To prove Main Theorem, first we show that $\dim_H C \leq t$.

We call n the *depth* of the basic set $\Delta_{\omega_1 \cdots \omega_n}$.

Let c be the constant in the definition of a (λ, c, v)-similarity map in Introduction. By the construction of C, we have $|\Delta_{\omega_1 \cdots \omega_n}| \leq |\Delta_{\omega_1 \cdots \omega_{n-1}}| v$. Obviously there exists a number n_0 $(n_0 \gg 1)$ such that

$$c|\Delta_{\omega_1 \cdots \omega_{n_0}}| < 1,$$

For any $\varepsilon > 0$, let n be sufficiently large such that $\mathscr{U} = \{\Delta_{\omega_1 \cdots \omega_n} \mid 1 \leq \omega_j \leq k, 1 \leq j \leq n\}$ is an ε-cover of C. By the definition of (λ, c, v)-similarity map $f : \Delta_{\omega_1 \cdots \omega_{n-1}} \to \Delta_{\omega_1 \cdots \omega_n}$, we have

$$|\Delta_{\omega_1 \cdots \omega_n}| \leq \lambda_{\omega_n}(1 + c|\Delta_{\omega_1 \cdots \omega_{n-1}}|)|\Delta_{\omega_1 \cdots \omega_{n-1}}|.$$

Let $n = n_0 + m$, then

$$c|\Delta_{\omega_1 \cdots \omega_{n-1}}| \leq c|\Delta_{\omega_1 \cdots \omega_{n_0}}| v^{m-1} \leq v^{m-1}.$$

Thus

$$\begin{aligned}
m(C, t, \varepsilon) &\leq \sum_{(\omega_1, \ldots, \omega_n)} |\Delta_{\omega_1 \cdots \omega_n}|^t \\
&= \sum_{(\omega_1, \ldots, \omega_{n-1})} (|\Delta_{\omega_1 \cdots \omega_{n-1} 1}|^t + \cdots + |\Delta_{\omega_1 \cdots \omega_{n-1} k}|^t) \\
&\leq \sum_{(\omega_1, \ldots, \omega_{n-1})} (1 + c|\Delta_{\omega_1 \cdots \omega_{n-1}}|)^t |\Delta_{\omega_1 \cdots \omega_{n-1}}|^t (\lambda_1^t + \cdots + \lambda_k^t) \\
&\leq \sum_{(\omega_1, \ldots, \omega_{n-1})} (1 + v^{m-1})^t |\Delta_{\omega_1 \cdots \omega_{n-1}}|^t \\
&= (1 + v^{m-1})^t \sum_{(\omega_1, \ldots, \omega_{n-1})} |\Delta_{\omega_1 \cdots \omega_{n-1}}|^t \\
&\leq \cdots < (1 + v^{m-1})^t \cdots (1 + v)^t 2^t \sum_{\omega_1, \ldots, \omega_{n_0}} |\Delta_{\omega_1 \cdots \omega_{n_0}}|^t.
\end{aligned}$$

Here when $m \to \infty$ the sequence $a_m = (1 + v^{m-1})^t \cdots (1 + v)^t 2^t$ converges. Hence $m(C, t) \leq K$ for some constant K, and therefore $\dim_H C \leq t$. □

To prove $\dim_H C \geq t$, we show Lemmas 2.2 and 2.6.

Lemma 2.2. *There exists a constant K, chosen independently of any cover, such that if $\mathcal{U} = \{U_i\}$ is any cover of C such that each U_i is a basic set, then $\sum_i |U_i|^t \geq K > 0$ holds.*

Proof. Let $\mathcal{U} = \{U_i\}$ be any cover of C by basic sets. $\mathcal{U}$ is called minimal if no proper subcollection of $\mathcal{U}$ covers C. Because C is compact, it suffices to establish $\sum_i |U_i|^t \geq K > 0$ when $\mathcal{U}$ is finite and minimal.

Let n be the maximum of the depths of all basic sets in $\mathcal{U}$, and let $\Delta_{\omega_1\cdots\omega_n}$ be a basic set of maximal depth in $\mathcal{U}$. Since $\mathcal{U}$ is minimal, it does not contain the basic set $\Delta_{\omega_1\cdots\omega_{n-1}}$. It follows that each of the basic set $\Delta_{\omega_1\cdots\omega_{n-1}j}$ for $j = 1, \dots, k$ is contained in $\mathcal{U}$.

Thus the sum $\sum_i |U_i|^t$ contains the partial sum

$$|\Delta_{\omega_1\cdots\omega_{n-1}1}|^t + \cdots + |\Delta_{\omega_1\cdots\omega_{n-1}k}|^t.$$

By the definition of (λ, c, v)-similarity map and t, we see

$$\begin{aligned}
|\Delta_{\omega_1\cdots\omega_{n-1}1}|^t + \cdots + |\Delta_{\omega_1\cdots\omega_{n-1}k}|^t &\geq \lambda_1^t (1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t |\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&\quad + \cdots + \lambda_k^t (1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t |\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&= (\lambda_1^t + \cdots + \lambda_k^t)(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t |\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&= (1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)^t |\Delta_{\omega_1\cdots\omega_{n-1}}|^t \\
&\geq (1 - v^{m-1})^t |\Delta_{\omega_1\cdots\omega_{n-1}}|^t.
\end{aligned}$$

We replace $\{\Delta_{\omega_1\cdots\omega_{n-1}j}\}_{j=1}^k$ by $\Delta_{\omega_1\cdots\omega_{n-1}}$. In this way we replace all the basic sets in $\mathcal{U}$ of depth n by the corresponding sets of depth $n-1$, to obtain a new covering $\mathcal{U}'$ by basic sets. We may assume that $\mathcal{U}'$ is minimal. Then we can repeat the previous argument, and obtain

$$\sum_i |U_i|^t \geq (1 - v^{m-1})^t \cdots (1 - v)^t (1 - c|\Delta_{\omega_1\cdots\omega_{n_0}}|)^t |\Delta_{\omega_1\cdots\omega_{n_0}}|^t.$$

But in the last expression, $a_m = (1 - v^{m-1})^t \cdots (1 - v)^t$ converges to a positive number and $(1 - c|\Delta_{\omega_1\cdots\omega_{n_0}}|)^t |\Delta_{\omega_1\cdots\omega_{n_0}}|^t$ is uniformly bounded from below. Therefore $\sum_i |U_i|^t \geq K > 0$ for a uniform positive number K. □

Lemma 2.3. *Let* $\lambda_{\min} = \min\{\lambda_1, \dots, \lambda_k\}$. *For each* $r > 0$, *set*

$$V(r) = \left\{ \Delta_{\omega_1 \cdots \omega_n} \,\middle|\, r\lambda_{\min} \le |\Delta_{\omega_1 \cdots \omega_n}| \le \frac{r}{\lambda_{\min}} \right\},$$

and given $x \in X$, *define* $V_x(r) = \{V \in V(r) \mid x \in V\}$. *Let* N *be the number of elements of* $V_x(r)$. *Then* $N \le M$, *where* M *is independent of* x *and* r.

Proof. First we consider the case $x \in C$. We can write given $x \in C$ as

$$\{x\} = \bigcap_{n \ge 1} \Delta_{\omega_1 \cdots \omega_n}.$$

For the infinite sequence $\omega_1, \omega_2, \dots, \omega_n, \dots$, define the set E as

$$E = \left\{ n \,\middle|\, r\lambda_{\min} \le |\Delta_{\omega_1 \cdots \omega_n}| \le \frac{r}{\lambda_{\min}} \right\}.$$

Then the number of elements of E is equal to the number N of elements of $V_x(r)$. Now let $n' = \min E$, $n'' = \max E$, and let $n'' = n' + m$, $n'' \ge n_0$, $n' \ge n_0$. Beacause

$$|\Delta_{\omega_1 \cdots \omega_{n''}}| = |\Delta_{\omega_1 \cdots \omega_{n'+m}}| \le |\Delta_{\omega_1 \cdots \omega_{n'}}| v^m,$$

by the definition of n', n'', we have

$$r\lambda_{\min} \le |\Delta_{\omega_1 \cdots \omega_{n''}}| \le |\Delta_{\omega_1 \cdots \omega_{n'}}| v^m \le \frac{r}{\lambda_{\min}} v^m.$$

Therefore, $r\lambda_{\min} \le \frac{r}{\lambda_{\min}} v^m$. Hence, $m \le 2 \frac{\log \lambda_{\min}}{\log v} = M$. i.e., $N \le M$.

Next, we consider the general case $x \in X$. For any $x \in X$, define E as $E = \{\Delta_{\omega_1 \cdots \omega_n} \mid x \in \Delta_{\omega_1 \cdots \omega_n}\}$. If $n = 1$, there exits unique ω_1 such that $x \in \Delta_{\omega_1}$; if $n = 2$, there exits unique ω_2 such that $x \in \Delta_{\omega_1 \omega_2}$; similarly there exits unique ω_n such that $x \in \Delta_{\omega_1 \cdots \omega_n}$. If E is an infinite set, then $x \in C$. Because there exits unique infinite sequence $\omega_1, \omega_2, \dots, \omega_n, \dots$ such that $x \in \Delta_{\omega_1 \cdots \omega_n}$ and $\{x\} = \bigcap_{n \ge 1} \Delta_{\omega_1 \cdots \omega_n}$ for any n $(n \ge 1)$. Therefore, $x \in C$. If E is a finite set, i.e., $E = \{\Delta_{\omega_1}, \Delta_{\omega_1 \omega_2}, \dots, \Delta_{\omega_1 \cdots \omega_n}\}$, then $V_x(r) = \{\Delta_{\omega_1 \cdots \omega_{n_0}}, \dots, \Delta_{\omega_1 \cdots \omega_{n_0+m}}\}$ for suitable n_0 and m. Thus by an argument similar to Lemma 2.3, the number of elements of $V_x(r)$ is bounded above by a constant M (which is independent of x and r). □

Lemma 2.4. *If* $b_{\omega_1 \cdots \omega_n} = \max\{r \mid B(x, r) \subset \Delta_{\omega_1 \cdots \omega_n}\}$, *then*

$$b_{\omega_1 \cdots \omega_n} \ge \lambda_{\omega_n} b_{\omega_1 \cdots \omega_{n-1}} (1 - c|\Delta_{\omega_1 \cdots \omega_{n-1}}|).$$

Proof. Let x be the center point of a largest ball included in $\Delta_{\omega_1\cdots\omega_{n-1}}$. By the definition of $(\lambda_{\omega_n}, c, v)$-similarity map $f : \Delta_{\omega_1\cdots\omega_{n-1}} \to \Delta_{\omega_1\cdots\omega_n}$, we have

$$B(f(x), \lambda_{\omega_n} b_{\omega_1\cdots\omega_{n-1}}(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)) \subset f(B(x, b_{\omega_1\cdots\omega_{n-1}}))$$

Thus, $B(f(x), \lambda_{\omega_n} b_{\omega_1\cdots\omega_{n-1}}(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)) \subset \Delta_{\omega_1\cdots\omega_n}$, therefore

$$b_{\omega_1\cdots\omega_n} \geq \lambda_{\omega_n} b_{\omega_1\cdots\omega_{n-1}}(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|). \qquad \square$$

Lemma 2.5. *If* $b_{\omega_1\cdots\omega_n} = \max\{r \mid B(x,r) \subset \Delta_{\omega_1\cdots\omega_n}\}$, *then there exists a constant* k_0 *such that*

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \leq k_0,$$

for any n *and any* $\omega_1, \omega_2, \ldots, \omega_n$.

Proof. By the definition of $(\lambda_{\omega_n}, c, v)$-similarity map $f : \Delta_{\omega_1\cdots\omega_{n-1}} \to \Delta_{\omega_1\cdots\omega_n}$, we have

$$|\Delta_{\omega_1\cdots\omega_n}| \leq \lambda_{\omega_n}(1 + c|\Delta_{\omega_1\cdots\omega_{n-1}}|)|\Delta_{\omega_1\cdots\omega_{n-1}}|,$$

Therefore we obtain

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \leq \frac{(1 + c|\Delta_{\omega_1\cdots\omega_{n-1}}|)|\Delta_{\omega_1\cdots\omega_{n-1}}|}{(1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|)b_{\omega_1\cdots\omega_{n-1}}}.$$

There exists n_0 such that for any $n \geq n_0$

$$\frac{1 + c|\Delta_{\omega_1\cdots\omega_{n-1}}|}{1 - c|\Delta_{\omega_1\cdots\omega_{n-1}}|} \leq 1 + 3c|\Delta_{\omega_1\cdots\omega_{n-1}}|.$$

Thus we have

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \leq \frac{|\Delta_{\omega_1\cdots\omega_{n-1}}|}{b_{\omega_1\cdots\omega_{n-1}}}(1 + 3c|\Delta_{\omega_1\cdots\omega_{n-1}}|).$$

By the construction of C, we have $|\Delta_{\omega_1\cdots\omega_n}| \leq |\Delta_{\omega_1\cdots\omega_{n-1}}|v$. Hence, there exists $n_1 \geq n_0$ such that

$$3c|\Delta_{\omega_1\cdots\omega_{n_1}}| < 1.$$

Now let $n = n_1 + m$, then we get

$$|\Delta_{\omega_1\cdots\omega_n}| \leq |\Delta_{\omega_1\cdots\omega_{n_1}}|v^m.$$

Therefore we obtain

$$3c|\Delta_{\omega_1\cdots\omega_{n-1}}| \le 3c|\Delta_{\omega_1\cdots\omega_{n_1}}|v^{m-1} \le v^{m-1},$$

and hence

$$1 + 3c|\Delta_{\omega_1\cdots\omega_{n-1}}| \le 1 + v^{m-1}.$$

Thus we have

$$\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \le \frac{|\Delta_{\omega_1\cdots\omega_{n-1}}|}{b_{\omega_1\cdots\omega_{n-1}}}(1 + v^{m-1}).$$

Therefore we obtain

$$\begin{aligned}\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} &\le \frac{|\Delta_{\omega_1\cdots\omega_{n-1}}|}{b_{\omega_1\cdots\omega_{n-1}}}(1 + v^{m-1}) \\ &\le \frac{|\Delta_{\omega_1\cdots\omega_{n-2}}|}{b_{\omega_1\cdots\omega_{n-2}}}(1 + v^{m-2})(1 + v^{m-1}) \\ &\le \cdots \le \frac{|\Delta_{\omega_1\cdots\omega_{n_1}}|}{b_{\omega_1\cdots\omega_{n_1}}}2(1 + v)\cdots(1 + v^{m-2})(1 + v^{m-1}).\end{aligned}$$

Here when $m \to \infty$ the sequence $a_m = 2(1 + v)\cdots(1 + v^{m-1})$ converges. Thus there exists a constant k_1 such that $\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \le k_1$ for any $n \ge n_1$. Let $k_2 = \max\left\{\frac{|\Delta_{\omega_1}|}{b_{\omega_1}}, \frac{|\Delta_{\omega_1\omega_2}|}{b_{\omega_1\omega_2}}, \ldots, \frac{|\Delta_{\omega_1\cdots\omega_{n_1}}|}{b_{\omega_1\cdots\omega_{n_1}}}\right\}$, and $k_0 = \max\{k_1, k_2\}$. Then $\frac{|\Delta_{\omega_1\cdots\omega_n}|}{b_{\omega_1\cdots\omega_n}} \le k_0$ for any n and any $\omega_1, \omega_2, \ldots, \omega_n$. □

LEMMA 2.6. *Let U be a bounded subset of X, and write $r = |U|$. Then U intersects at most $M' = C(\delta)M$ elements of $V(r)$, where M is the constant given in Lemma 2.3 and $\delta = \frac{\lambda_{\min}^2}{4k_0 + 4k_0\lambda_{\min} + \lambda_{\min}^2}$.*

PROOF. Fix an arbitrary point $x_0 \in U$, and consider the ball $B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r\right) \subset X$. Then we have $U \subset B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r\right)$, and choose maximal points $\{x_i\}_{i=1}^N \subset B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r\right)$ such that $d(x_i, x_j) \ge \frac{r\lambda_{\min}}{k_0}$ for any $i \ne j$, where k_0 is a constant defined in Lemma 2.5. We show that $N \le C(\delta)$, where $\delta = \frac{\lambda_{\min}^2}{4k_0 + 4k_0\lambda_{\min} + \lambda_{\min}^2}$ and $C(\delta)$ is the constant given in the condition (b) in Main Theorem. Consider the ball $B\left(x_i, \frac{r\lambda_{\min}}{2k_0}\right)$, and the ball $B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r + \frac{r\lambda_{\min}}{2k_0}\right)$. Then we have

$$\bigcup_{i=1}^N B\left(x_i, \frac{r\lambda_{\min}}{2k_0}\right) \subset B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r + \frac{r\lambda_{\min}}{2k_0}\right).$$

Since $B\big(x_i, \frac{r\lambda_{\min}}{2k_0}\big) \cap B\big(x_j, \frac{r\lambda_{\min}}{2k_0}\big) = \varnothing$ $(i \neq j)$, we get

$$\sum_{i=1}^{N} \mu\left(B\left(x_i, \frac{r\lambda_{\min}}{2k_0}\right)\right) \leq \mu\left(B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r + \frac{r\lambda_{\min}}{2k_0}\right)\right),$$

Take i_0 such that $\min_{1 \leq i \leq N} \mu\big(B\big(x_i, \frac{r\lambda_{\min}}{2k_0}\big)\big) = \mu\big(B\big(x_{i_0}, \frac{r\lambda_{\min}}{2k_0}\big)\big)$. Then we get

$$N\mu\left(B\left(x_{i_0}, \frac{r\lambda_{\min}}{2k_0}\right)\right) \leq \mu\left(B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r + \frac{r\lambda_{\min}}{2k_0}\right)\right).$$

Thus we have

$$N \leq \frac{\mu\big(B\big(x_0, \big(1 + \frac{1}{\lambda_{\min}}\big)r + \frac{r\lambda_{\min}}{2k_0}\big)\big)}{\mu\big(B\big(x_{i_0}, \frac{r\lambda_{\min}}{2k_0}\big)\big)}.$$

Because $B\big(x_0, \big(1 + \frac{1}{\lambda_{\min}}\big)r + \frac{r\lambda_{\min}}{2k_0}\big) \subset B\big(x_{i_0}, 2\big(1 + \frac{1}{\lambda_{\min}}\big)r + \frac{r\lambda_{\min}}{2k_0}\big)$, we obtain

$$N \leq \frac{\mu\big(B\big(x_{i_0}, 2\big(1 + \frac{1}{\lambda_{\min}}\big)r + \frac{r\lambda_{\min}}{2k_0}\big)\big)}{\mu\big(B\big(x_{i_0}, \frac{r\lambda_{\min}}{2k_0}\big)\big)} \leq C(\delta),$$

where $\delta = \frac{\lambda_{\min}^2}{4k_0 + 4k_0\lambda_{\min} + \lambda_{\min}^2}$.

Next we show that if $V \in V(r)$ intersects U, it must contain one of $\{x_i\}$. Take a point $y \in V \cap U$. Let x be the center point of a largest ball included in V. Then we have

$$\begin{aligned} d(x, x_0) &\leq d(x, y) + d(y, x_0) \\ &\leq |V| + |U| \leq \left(\frac{1}{\lambda_{\min}} + 1\right)r. \end{aligned}$$

Therefore, we obtain $x \in B\big(x_0, \big(1 + \frac{1}{\lambda_{\min}}\big)r\big)$. Furthermore, we have

$$B\left(x_0, \left(1 + \frac{1}{\lambda_{\min}}\right)r\right) \subset \bigcup_{i=1}^{N} B\left(x_i, \frac{r\lambda_{\min}}{k_0}\right).$$

Thus there exists a point x_i $(i = 1, 2, \ldots, N)$ such that $x \in B\big(x_i, \frac{r\lambda_{\min}}{k_0}\big)$. Hence $x_i \in B\big(x, \frac{r\lambda_{\min}}{k_0}\big)$. By Lemma 2.5, $\frac{|V|}{b(V)} \leq k_0$. Therefore, $b(V) \geq \frac{|V|}{k_0} \geq \frac{r\lambda_{\min}}{k_0}$. i.e., $V \supset B(x, b(V)) \supset B\big(x, \frac{r\lambda_{\min}}{k_0}\big)$. Hence, $x_i \in V$. Because each of $\{x_i\}$ is contained in at most M such sets V, it follows that the total number of elements V of $V(r)$ which intersect U is bounded above by $M' = C(\delta)M$. □

The proof of $\dim_H C \geq t$. Let $\mathscr{U} = \{U_i\}$ be any ε-cover of C. For each U_i, write $r_i = |U_i|$, and let $U_{i,1}, \dots, U_{i,m(i)}$ be the basic sets in $V(r_i)$ which intersect U_i. It follows from the above Lemma 2.6 that $m(i) \leq M'$. Furthermore, since $U_{i,j} \in V(r_i)$, we have

$$|U_{i,j}| \leq \frac{|U_i|}{\lambda_{\min}},$$

and

$$\sum_{j=1}^{m(i)} |U_{i,j}|^t \leq m(i) \frac{|U_i|^t}{\lambda_{\min}^t} \leq \frac{M'}{\lambda_{\min}^t} |U_i|^t.$$

i.e.

$$|U_i|^t \geq \frac{\lambda_{\min}^t}{M'} \sum_{j=1}^{m(i)} |U_{i,j}|^t.$$

Summing over all the elements of $\mathscr{U}$ yields

$$\sum_i |U_i|^t \geq \frac{\lambda_{\min}^t}{M'} \sum_i \sum_{j=1}^{m(i)} |U_{i,j}|^t.$$

Since $\{U_{i,j}\}$ is a cover of C by basic sets, we may apply Lemma 2.2 to obtain

$$\sum_i |U_i|^t \geq \frac{\lambda_{\min}^t}{M'} K > 0,$$

where K is the constant in Lemma 2.2. Hence $\dim_H C \geq t$, and Main Theorem follows. □

3. Generalized Cantor Sets in Riemannian Manifold

Finally we construct a generalized Cantor set in a complete Riemannian manifold.

Let M be a complete Riemannian manifold. For a point $p \in M$, let $B(0,r) = \{v \in T_pM \mid \|v\| \leq r\}$. If r is sufficiently small, then the exponential map $exp_p : B(0,r) \to M$ is a diffeomorphism onto $B(p,r) = \{q \in M \mid d(p,q) \leq r\}$. For any $v \in B(0,r)$, let γ_v be a geodesic such that $\gamma_v(0) = p$, $\dot{\gamma}_v(0) = v$. Then by definition, $exp_p(v) = \gamma_v(1)$.

Let K_M be the sectional curvature of M. Take a positive number Λ such that $-\Lambda^2 \leq K_M \leq \Lambda^2$ on $B(p, r)$. By Rauch Comparison Theorem (cf. [CHE]), for any $u, v \in B(0, r)$,

$$\frac{\sin \Lambda r}{\Lambda r} \leq \frac{d(exp_p(u), exp_p(v))}{\|u - v\|} \leq \frac{\sinh \Lambda r}{\Lambda r}.$$

Proposition 3.1. *For a constant λ with $0 < \lambda < 1$, let $p_1 \in B(p, r) \subset M$ with $d(p_1, p) \leq (1 - \lambda)r$. Let $\tilde{f}_1 : T_{p_1}M \to T_{p_1}M$ be the λ-similarity map given by $v \mapsto \lambda v$. Let $I_0 : T_pM \to T_{p_1}M$ be a linear isometry. Let $A_0 := B(p, r)$, $\tilde{A}_0 := exp_p^{-1}(A_0) = B(0_p, r) \subset T_pM$, $\tilde{A}_1 := I_0(\tilde{A}_0) = B(0_{p_1}, r) \subset T_{p_1}M$, $\tilde{B}_1 := \tilde{f}_1(\tilde{A}_1) = B(0_{p_1}, \lambda r)$, $A_1 := exp_{p_1}(\tilde{B}_1) = B(p_1, \lambda r)$. Then $f_0 := exp_{p_1} \circ \tilde{f}_1 \circ I_0 \circ exp_p^{-1} : A_0 \to A_1$ is a (λ, c, v)-similarity map, where $c = \frac{\Lambda^2}{16}(\lambda^2 + 1)$.*

Proof. For any $x, y \in A_0$, by Rauch Comparison Theorem, we have

$$\frac{d(e^{-1}(x), e^{-1}(y))}{d(x, y)} \leq \frac{\Lambda r}{\sin \Lambda r}, \quad \frac{d(f_0(x), f_0(y))}{d(\tilde{f}_1(I_0(e^{-1}(x))), \tilde{f}_1(I_0(e^{-1}(y))))} \leq \frac{\sinh \Lambda\lambda r}{\Lambda\lambda r},$$

and therefore

$$\frac{d(f_0(x), f_0(y))}{d(x, y)} \leq \lambda \frac{\Lambda r}{\sin \Lambda r} \frac{\sinh \Lambda\lambda r}{\Lambda\lambda r},$$

where $e^{-1} = exp_p^{-1}$.

When $r \ll 1$, by Taylor expansion we get

$$\frac{\Lambda r}{\sin \Lambda r} \leq 1 + \frac{1}{7}\Lambda^2 r^2, \quad \frac{\sinh \Lambda\lambda r}{\Lambda\lambda r} \leq 1 + \frac{1}{7}\Lambda^2\lambda^2 r^2.$$

Thus, we have

$$\begin{aligned}
\frac{d(f_0(x), f_0(y))}{d(x, y)} &\leq \lambda \frac{\Lambda r}{\sin \Lambda r} \frac{\sinh \Lambda\lambda r}{\Lambda\lambda r} \\
&\leq \lambda\left(1 + \frac{1}{7}\Lambda^2 r^2\right)\left(1 + \frac{1}{7}\Lambda^2\lambda^2 r^2\right) \\
&\leq \lambda\left(1 + \frac{1}{7}\Lambda^2\lambda^2 r^2 + \frac{1}{7}\Lambda^2 r^2 + \frac{1}{49}\Lambda^4\lambda^2 r^4\right) \\
&\leq \lambda\left(1 + \frac{1}{8}\Lambda^2\lambda^2 r^2 + \frac{1}{8}\Lambda^2 r^2\right) \quad (r \ll 1) \\
&\leq \lambda + \lambda\left(\frac{1}{8}\Lambda^2\lambda^2 r^2 + \frac{1}{8}\Lambda^2 r^2\right).
\end{aligned}$$

Furthermore, since $|A_0| = 2r$, we obtain

$$\frac{d(f_0(x), f_0(y))}{d(x,y)} - \lambda \le \frac{\lambda\Lambda^2}{8}(\lambda^2+1)r^2 \quad (r \ll 1)$$

$$\le \frac{\lambda\Lambda^2}{16}(\lambda^2+1)2r$$

$$= \frac{\lambda\Lambda^2}{16}(\lambda^2+1)|A_0|.$$

Similarly, we have $\frac{d(f_0(x), f_0(y))}{d(x,y)} - \lambda \ge -\frac{\lambda\Lambda^2}{16}(\lambda^2+1)|A_0|$. Letting $c = \frac{\Lambda^2}{16}(\lambda^2+1)$, we obtain $\left|\frac{d(f_0(x), f_0(y))}{d(x,y)} - \lambda\right| \le \lambda c|A_0|$, and hence f_0 is a (λ, c, v)-similarity map. □

Example 3.2. For $0 < \lambda \le \frac{1}{2}$, let k_0 be a maximal number of disjoint closed balls of radius λ which is contained in the unit ball of $\mathbf{R}^n$. Let M be an n-dimensional complete Riemannian manifold of Ricci curvature $\ge (n-1)\kappa$ and $p \in M$ for a constant κ. If r is sufficiently small, then $B(p,r)$ is almost isometric to $B(0,r) \subset T_pM$. Let $1 < k \le k_0$ and $r_1 = \lambda r$. Then we can take k disjoint balls $\{B(p_i, r_1)\}_{i=1}^k$ in $B(p,r)$. By Proposition 3.1, $B(p_i, r_1)$ is a (λ, c, v)-similar set of $B(p,r)$ for some uniform constant c. let $r_2 = \lambda r_1$, then we can take k disjoint balls $\{B(p_{ij}, r_2)\}_{j=1}^k$ in each ball $B(p_i, r_1)$, and $B(p_{ij}, r_2)$ is a (λ, c, v)-similar set of $B(p_i, r_1)$. Repeating this procedure, we can construct basic sets $B(p_{i_1\cdots i_n}, r_n)$ $(r_n = \lambda^n r, i_1, \dots, i_n = 1, 2, \dots k)$, and we can define an asymptotically generalized cantor set C in M as

$$C := \bigcap_{n=1}^{\infty}\left(\bigcup_{i_1,\dots,i_n=1}^{k} B(p_{i_1\cdots i_n}, r_n)\right).$$

Let μ be the Riemannian measure of M. We denote by $V_\kappa^n(r)$ the volume of a r-ball in the n-dimensional space form M_κ^n of constant curvature κ. By Gromov-Bishop Comparison Theorem, we have

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, \delta r))} \le \frac{V_\kappa^n(r)}{V_\kappa^n(\delta r)} = \frac{\int_0^r \sinh\sqrt{|\kappa|}t\,dt}{\int_0^{\delta r} \sinh\sqrt{|\kappa|}t\,dt} \le C_{n,\kappa}(\delta),$$

for any $x_0 \in M$ and $0 < \delta, r < 1$, where $C_{n,\kappa}(\delta)$ is a positive constant depending only on n, κ and δ.

Hence by Main Theorem we have $\dim_H C = -\frac{\log k}{\log \lambda}$.

References

[1] Assouad, P. Plongements lipschitziens dans R^n. Bull. Soc. Math. France 111, **4** (1983), 429–448.
[2] D. Burago, Yu. Burago, and S. Ivanov, A Course in Metric Geometry. American Mathematical Society, 2001.
[3] Heinonen, J. Lectures on analysis on metric spaces. Universitext. Springer-Verlag, New York, 2001.
[4] Hutchinson, J. E. Fractals and self-similarity. Indiana Univ. Math. J. 30, **5** (1981), 713–747.
[5] J. Cheeger and D. Ebin, Comparison theorems in Riemannian Geometry. North-Holland, 1975.
[6] Yakov Pesin and Vaughn Climehaga, Lectures on Fractal Geometry and Dyamical systems. American Mathematical Society, 2009.
[7] Kigami, J. Hausdorff dimensions of self-similar sets and shortest path metrics. J. Math. Soc. Japan 47, **3** (1995), 381–404.
[8] Moran, P. A. P. Additive functions of intervals and Hausdorff measure. Proc. Cambridge Philos. Soc. **42** (1946), 15–23.
[9] Schief, A. Separation properties for self-similar sets. Proc. Amer. Math. Soc. 122, **1** (1994), 111–115.
[10] Schief, A. Self-similar sets in complete metric spaces. Proc. Amer. Math. Soc. 124, **2** (1996), 481–490.
[11] Z. M. Balogh and H. Rohner, Self-similar sets in doubling spaces, Illinois J. Math. **51(4)** (2007), 1275–1297.
[12] Daruhan Wu, The Hausdorff dimension of generalized Cantor sets. Master's Thesis, University of Tsukuba, 2013.

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