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STRUCTURAL PROPERTIES OF IDEALS OVER $\mathscr{P}_\kappa\lambda$ I

By

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Abstract. We try to take a first step to a theory of the structural properties of ideals over $\mathscr{P}_\kappa\lambda$, that was studied in detail by Baumgartner, Taylor and Wagon [1] for κ. In defining the basic notions, P-points, Q-points, and selective ideals,we put importance on the behavior of the function on $\mathscr{P}_\kappa\lambda$ to the bounded ideal and Rudin-Keisler ordering.

Several facts hold similarly as on κ, for instance, the bounded ideal is a nowhere Q-point. However some differences exist such as the bounded ideal is isomorphic to another ideal. We state the sufficient condition for ideals to be Q-points and the weakly normal ideals selective.

1 Introduction

Throughout κ denotes a regular uncountable cardinal and λ a cardinal $\geq \kappa$. Let $\mathscr{P}_\kappa\lambda$ denote the set of the subsets of λ with the cardinality less than κ, that is, $\mathscr{P}_\kappa\lambda = \{x \subset \lambda : |x| < \kappa\}$.

Definition 1.1. Let $X \subset \mathscr{P}_\kappa\lambda$.

We say X is *unbounded* if for every $x \in \mathscr{P}_\kappa\lambda$ there exists $y \in X$ such that $x \subset y$. The set $\{X \subset \mathscr{P}_\kappa\lambda : X \text{ is not unbounded}\}$ denoted by $\mathrm{I}_{\kappa,\lambda}$ is called the *bounded ideal*. For each $a \in \mathscr{P}_\kappa\lambda$ let $\hat{a} = \{x \in \mathscr{P}_\kappa\lambda : a \subset x\}$. Thus $X \in \mathrm{I}_{\kappa,\lambda}$ if and only if $X \cap \hat{a} = \emptyset$ for some $a \in \mathscr{P}_\kappa\lambda$.

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X is said to be *closed* if $\bigcup C \in X$ for every $\subset$-chain $C \subset X$ with $|C| < \kappa$. X is a *club* if it is closed and unbounded. X is *stationary* if $X \cap C \neq \varnothing$ for any club C. The set $\{X \subset \mathscr{P}_\kappa\lambda : X$ is not stationary$\}$ denoted by $\mathrm{NS}_{\kappa,\lambda}$ is called the *non-stationary ideal*.

DEFINITION 1.2. We say I is an *ideal* over $\mathscr{P}_\kappa\lambda$ if the following hold:

(1) $I \subset \mathscr{P}(\mathscr{P}_\kappa\lambda)$,

(2) $\varnothing \in I$ and $\mathscr{P}_\kappa\lambda \notin I$,

(3) if $X \subset Y \in I$, then $X \in I$,

(4) $\bigcup D \in I$ for every $D \subset I$ with $|D| < \kappa$
(we say I is κ-*complete*),

(5) $\mathrm{I}_{\kappa,\lambda} \subset I$ (we say I is *fine*).

Let $I^+ = \mathscr{P}(\mathscr{P}_\kappa\lambda)\backslash I$ and $I^* = \{X \subset \mathscr{P}_\kappa\lambda : \mathscr{P}_\kappa\lambda\backslash X \in I\}$. For $X \in I^+$ $I \restriction X = \{Y \subset \mathscr{P}_\kappa\lambda : Y \cap X \in I\}$, which is an ideal extending I.

A function f is *regressive* if $f(x) \in x$ for every $x \in \mathrm{dom}(f)\backslash\{\varnothing\}$.

An ideal I over $\mathscr{P}_\kappa\lambda$ is *normal* if for any $X \in I^+$ and regressive function f on X there exists $Y \in \mathscr{P}(X) \cap I^+$ such that $f \restriction Y$ is constant.

I is *weakly normal* if for any $X \in I^+$ and regressive function f on X there exists $\gamma < \lambda$ such that $\{x \in X : f(x) \leq \gamma\} \in I^+$.

Note that $\mathrm{I}_{\kappa,\lambda}$ is the minimal, and $\mathrm{NS}_{\kappa,\lambda}$ the minimal normal ideal over $\mathscr{P}_\kappa\lambda$.

For a function f and $X \subset \mathrm{dom}(f)$ $f[X]$ denotes the set $\{f(x) : x \in X\}$ together with an abuse $f^{-1}[Y] = \{x : f(x) \in Y\}$ for a set Y.

The structural properties of ideals on κ was almost completely described in Baumgartner-Taylor-Wagon [1]. We state the basic notions.

DEFINITION 1.3. We say $I \subset \mathscr{P}(\kappa)$ is an *ideal* on κ if the following hold:

(a) $\varnothing \in I$ and $\kappa \notin I$,

(b) if $X \subset Y \in I$, then $X \in I$,

(c) I is κ-complete,

(d) $\mathrm{I}_\kappa \subset I$ ($\mathrm{I}_\kappa = \{x \subset \kappa : |x| < \kappa\}$, the bounded ideal on κ).

Suppose that I is an ideal on κ and $f : \kappa \to \kappa$.

(1) f is I-small if $\forall\alpha < \kappa f^{-1}[\{\alpha\}] \in I$.

(2) I is a P-point if for every I-small f there exists $X \in I^*$ such that $f \restriction X$ is I_κ-small.

(3) I is a Q-point if every I_κ-small f is injective on a set in I^*.

(4) I is selective if every I-small f is injetive on a set in I^*.

(5) $f_*(I) = \{X \subset \kappa : f^{-1}[X] \in I\}$.

Several notions for subsets of κ were first translated into $\mathscr{P}_\kappa\lambda$ by Jech [3]. Later Zwicker and some others tried to develop $\mathscr{P}_\kappa\lambda$ analogue of the theory for the structural properties unsccesfully [2], [7]. We have several choices and difficulties in defining basic notions in [1]. For instances,

(1) $X \in \mathrm{I}_\kappa$ if and only if $|X| < \kappa$. While some $X \in \mathrm{I}_{\kappa,\lambda}$ has the cardinality $|\mathscr{P}_\kappa\lambda|$ for $\lambda > \kappa$. Note that $\mathrm{I}_\kappa \neq \mathrm{I}_{\kappa,\kappa}$.
(2) For every $f : \kappa \to \kappa$ and $X \in \mathrm{I}_\kappa$, $f[X] \in \mathrm{I}_\kappa$. However, for some $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ and $X \in \mathrm{I}_{\kappa,\lambda}$, $f[X] \in \mathrm{I}^+_{\kappa,\lambda}$.

The follwing motivates our definition.

Fact 1.4. *For an ideal I on κ, $f_*(I)$ is an ideal if and only if f is I-small.*

Definition 1.5. Let I be an ideal over $\mathscr{P}_\kappa\lambda$ and $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$.

(1) f is *I-fine* if $\forall\alpha < \lambda\{x \in \mathscr{P}_\kappa\lambda : \alpha \notin f(x)\} \in I$.

(2) I is a *P-point* if for every I-fine f there exists $X \in I^*$ such that f is $\mathrm{I}_{\kappa,\lambda} \restriction X$-fine, that is, $\forall\alpha < \lambda\{x \in X : \alpha \notin f(x)\} \in \mathrm{I}_{\kappa,\lambda}$.

(3) I is a *Q-point* if every $\mathrm{I}_{\kappa,\lambda}$-fine f is injective on a set in I^*.

(4) I is *selective* if every I-fine f is injetive on a set in I^*.

(5) $f_*(I) = \{X \subset \mathscr{P}_\kappa\lambda : f^{-1}[X] \in I\}$.

By definition we have:

Fact 1.6. (1) *$f_*(I)$ is an ideal if and only if f is I-fine.*

(2) *Every normal ideal is a P-point.*

(3) *I is selective if it is both a P-point and a Q-point.*

(4) *Let I be a Q-point and $I \subset J$. Then, J is a Q-point. Furthermore, J is selective if J is a P-point.*

(5) *The following are equivalent for every $X \in \mathrm{I}^+_{\kappa,\lambda}$:*

(a) *f is $\mathrm{I}_{\kappa,\lambda} \restriction X$-fine,*
(b) $\mathrm{I}_{\kappa,\lambda} \restriction f[X] \subset f_*(\mathrm{I}_{\kappa,\lambda} \restriction X)$,
(c) $\forall Y \subset \mathscr{P}_\kappa\lambda \ (Y \cap f[X] \in \mathrm{I}_{\kappa,\lambda} \to f^{-1}[Y] \cap X \in \mathrm{I}_{\kappa,\lambda})$,
(d) $\forall Y \subset \mathscr{P}_\kappa\lambda(Y \cap X \in \mathrm{I}^+_{\kappa,\lambda} \to f[Y \cap X] \in \mathrm{I}^+_{\kappa,\lambda})$.

(6) *If f is I-fine and f is $\mathrm{I}_{\kappa,\lambda} \restriction X$-fine for some $X \in I^*$, there is an $\mathrm{I}_{\kappa,\lambda}$-fine g such that $f_*(I) = g_*(I)$; for instance, $g = f \restriction X \cup \mathrm{Id} \restriction (\mathscr{P}_\kappa\lambda \backslash X)$.*

Remark 1.7. The converse of (3) does not hold.

Fact 1.8 (Usuba). *Suppose that $\mathscr{P}_\kappa\lambda$ carries a selective ideal and the GCH holds. Then, there is a selective ideal which is not a P-point.*

2. The Bounded Ideal

Clearly the bounded ideal $\mathrm{I}_{\kappa,\lambda}$ is a P-point.

First we give some definitions and present known facts on I_κ.

Definition 2.1. Let J and K be ideals on a set S.

(1) J and K are *isomorphic* and denoted by $J \cong K$ if there is a bijection $F : S \to S$ such that $K = F_*(J)$.

(2) We write $J \cong^* K$ if $K = F_*(J)$ and $F \restriction X$ is injective for some $X \in J^*$.

Fact 2.2. (1) I_κ *has no isomorph except itself.*

(2) $f_*(\mathrm{I}_\kappa) = \mathrm{I}_\kappa \restriction f[\kappa]$ *for any* I_κ*-small* f.

(3) *For any* $A, B \in \mathrm{I}_\kappa^+ \backslash \mathrm{I}_\kappa^*$, $\mathrm{I}_\kappa \restriction A \cong \mathrm{I}_\kappa \restriction B$.

(4) *If* $f \restriction X$ *is injective and* $I \neq \mathrm{I}_\kappa \restriction X$ *for some* $X \in I^*$, *then* $I \cong f_*(I)$. *Thus, for two ideals on* κ, $\cong$ *and* $\cong^*$ *are equivalent.*

We show some of the above do not hold in $\mathscr{P}_\kappa\lambda$. Our first interest is $f_*(\mathrm{I}_{\kappa,\lambda})$. It holds that $\mathrm{I}_{\kappa,\lambda} \restriction f[\mathscr{P}_\kappa\lambda] \subset f_*(\mathrm{I}_{\kappa,\lambda})$ for every f. However we have:

Theorem 2.3. *For each* $X \in \mathrm{I}_{\kappa,\lambda}^+ \backslash \mathrm{I}_{\kappa,\lambda}^*$ *there is an* $\mathrm{I}_{\kappa,\lambda}$*-fine function* f *such that* $f_*(\mathrm{I}_{\kappa,\lambda}) = \mathrm{I}_{\kappa,\lambda} \restriction X \neq \mathrm{I}_{\kappa,\lambda} \restriction f[\mathscr{P}_\kappa\lambda]$.

Proof. Let $Y = \mathscr{P}_\kappa\lambda \backslash X \in \mathrm{I}_{\kappa,\lambda}^+$ and choose a non-empty $a \in \mathscr{P}_\kappa\lambda$. Define f as:

(1) $f(x) = x$ for $x \in X \cap \hat{a}$

(2) $x \subset f(x) \in X$ for $x \in \hat{a} \backslash X$

(3) $x \subset f(x) \in Y \cap \hat{a}$ for $x \in \mathscr{P}_\kappa\lambda \backslash \hat{a}$

Since $x \subset f(x)$ for every $x \in \mathscr{P}_\kappa\lambda$, f is $\mathrm{I}_{\kappa,\lambda}$-fine.

For any $x \in \mathscr{P}_\kappa\lambda$, $x \backslash a \in \mathscr{P}_\kappa\lambda \backslash \hat{a}$ and $f(x \backslash a) \in Y \cap \hat{a}$. Hence $f(x \backslash a) \supset (x \backslash a) \cup a = x$. Thus $f[\mathscr{P}_\kappa\lambda \backslash \hat{a}] \in \mathrm{I}_{\kappa,\lambda}^+$ and $f[\mathscr{P}_\kappa\lambda \backslash \hat{a}] \notin \mathrm{I}_{\kappa,\lambda} \restriction f[\mathscr{P}_\kappa\lambda]$.

Since $f[\hat{a}] \subset X$ and $f[\mathscr{P}_\kappa\lambda \backslash \hat{a}] \subset Y$, $f^{-1}[f[\mathscr{P}_\kappa\lambda \backslash \hat{a}]] = \mathscr{P}_\kappa\lambda \backslash \hat{a} \in \mathrm{I}_{\kappa,\lambda}$. Hence $f[\mathscr{P}_\kappa\lambda \backslash \hat{a}] \in f_*(\mathrm{I}_{\kappa,\lambda})$, which says $f_*(\mathrm{I}_{\kappa,\lambda}) \neq \mathrm{I}_{\kappa,\lambda} \restriction f[\mathscr{P}_\kappa\lambda]$.

To show $f_*(\mathrm{I}_{\kappa,\lambda}) = \mathrm{I}_{\kappa,\lambda} \restriction X$, first suppose that $Z \in \mathrm{I}_{\kappa,\lambda} \restriction X$. Since f is $\mathrm{I}_{\kappa,\lambda}$-fine, $f^{-1}[Z \cap X] \in \mathrm{I}_{\kappa,\lambda}$. $f^{-1}[Z \cap Y] \subset \mathscr{P}_\kappa\lambda \backslash \hat{a} \in \mathrm{I}_{\kappa,\lambda}$. So, $f^{-1}[Z] = f^{-1}[Z \cap X] \cup f^{-1}[Z \cap Y] \in \mathrm{I}_{\kappa,\lambda}$ and $Z \in f_*(\mathrm{I}_{\kappa,\lambda})$.

Second let $Z \in f_*(\mathrm{I}_{\kappa,\lambda})$. Since $f \restriction X \cap \hat{a} = \mathrm{Id.}$, $Z \cap X \cap \hat{a} \subset f^{-1}[Z \cap X \cap \hat{a}] \subset f^{-1}[Z] \in \mathrm{I}_{\kappa,\lambda}$. Hence $Z \cap X \in \mathrm{I}_{\kappa,\lambda}$. □

Remark 2.4. $\forall Y \subset \mathscr{P}_\kappa\lambda(f^{-1}[Y] \cap X \in \mathrm{I}_{\kappa,\lambda} \to Y \cap f[X] \in \mathrm{I}_{\kappa,\lambda}) \Leftrightarrow f_*(\mathrm{I}_{\kappa,\lambda} \restriction X) \subset \mathrm{I}_{\kappa,\lambda} \restriction f[X]$.

So, $f_*(\mathrm{I}_{\kappa,\lambda} \restriction X) = \mathrm{I}_{\kappa,\lambda} \restriction f[X] \Leftrightarrow f$ is $\mathrm{I}_{\kappa,\lambda} \restriction X$-fine and $\forall Y \subset \mathscr{P}_\kappa\lambda(f^{-1}[Y] \cap X \in \mathrm{I}_{\kappa,\lambda} \to Y \cap f[X] \in \mathrm{I}_{\kappa,\lambda})$.

In particular, $f_*(\mathrm{I}_{\kappa,\lambda}) = \mathrm{I}_{\kappa,\lambda} \Leftrightarrow f$ is $\mathrm{I}_{\kappa,\lambda}$-fine and $\forall X \subset \mathscr{P}_\kappa\lambda(f^{-1}[X] \in \mathrm{I}_{\kappa,\lambda} \to X \in \mathrm{I}_{\kappa,\lambda})$.

For every $X \subset \kappa$ and $f : \kappa \to \kappa$, the following is clear:

(1) $X \in \mathrm{I}_\kappa \to f[X] \in \mathrm{I}_\kappa$.

(2) $f^{-1}[X] \in \mathrm{I}_\kappa \to X \in \mathrm{I}_\kappa$.

For $X \subset \mathscr{P}_\kappa\lambda$ we have the following:

Proposition 2.5. *Suppose that f is $\mathrm{I}_{\kappa,\lambda}$-fine and $\forall Y \in \mathrm{I}_{\kappa,\lambda}$ $f[Y] \in \mathrm{I}_{\kappa,\lambda}$. Then, $f_*(\mathrm{I}_{\kappa,\lambda} \restriction X) = \mathrm{I}_{\kappa,\lambda} \restriction f[X]$ for every $X \in \mathrm{I}^+_{\kappa,\lambda}$.*

Proof. We only have to show $f_*(\mathrm{I}_{\kappa,\lambda} \restriction X) \subset \mathrm{I}_{\kappa,\lambda} \restriction f[X]$.

For $Y \in f_*(\mathrm{I}_{\kappa,\lambda} \restriction X)$, $f^{-1}[Y] \cap X \in \mathrm{I}_{\kappa,\lambda}$ and $f[f^{-1}[Y] \cap X] = Y \cap f[X]$. By our assumption, we have $Y \cap f[X] \in \mathrm{I}_{\kappa,\lambda}$. □

Remark 2.6. (1) It may happen that $f^{-1}[Y] \cap X \subsetneq f^{-1}[Y \cap f[X]]$.

(2) The assumption that $\forall Y \in \mathrm{I}_{\kappa,\lambda} f[Y] \in \mathrm{I}_{\kappa,\lambda}$ is stronger than $\forall X(f^{-1}[X] \in \mathrm{I}_{\kappa,\lambda} \to X \in \mathrm{I}_{\kappa,\lambda})$: choose an $\alpha < \lambda$ and set $f \restriction \widehat{\{\alpha\}} = \mathrm{Id} \restriction \widehat{\{\alpha\}}$ and $f(x) = x \cup \{\alpha\}$ if $\alpha \notin x$.

Now we show that $\cong^*$ is equivalent to $\cong$.

The following is clear.

Lemma 2.7. *Suppose that I is an ideal over $\mathscr{P}_\kappa\lambda$, f is I-fine, and $f \restriction A$ is injective with $A \in I^*$. If one of the following holds, then $I \cong f_*(I)$:*

(1) $|\mathscr{P}_\kappa\lambda \backslash A| = |\mathscr{P}_\kappa\lambda \backslash f[A]|$

(2) $|\mathscr{P}_\kappa\lambda \backslash A| > |\mathscr{P}_\kappa\lambda \backslash f[A]|$ *and there is* $X \in \mathscr{P}(A) \cap I$ *such that* $|X| = |\mathscr{P}_\kappa\lambda \backslash A|$

(3) $|\mathscr{P}_\kappa\lambda \backslash A| < |\mathscr{P}_\kappa\lambda \backslash f[A]|$ *and there is* $X \in \mathscr{P}(A) \cap I$ *such that* $|X| = |\mathscr{P}_\kappa\lambda \backslash f[A]|$.

The proof of the next proposition is by the referee to whom the author is very grateful. She/He showed that the author's assumptions in Proposition 2.8, 2.9, and 2.10 of the original manuscript are not necessary.

Proposition 2.8. *Suppose that I is an ideal over $\mathscr{P}_\kappa\lambda$, $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ is I-fine, and $f \restriction A$ is injective for some $A \in I^*$. Then $I \cong f_*(I)$. (Hence $I \cong J$ and $I \cong^* J$ are equivalent.)*

Proof. If $|A| < \lambda^{<\kappa}$, (1) in Lemma 2.7 holds and the assertion follows. So, we assume $|A| = \lambda^{<\kappa}$.

Case 1: There exists $B \in \mathscr{P}(A) \cap I$ with $|B| = |A|$.

When $|\mathscr{P}_\kappa\lambda \backslash A| = |\mathscr{P}_\kappa\lambda \backslash f[A]|$, the assertion follows from (1) in Lemma 2.7. If $|\mathscr{P}_\kappa\lambda \backslash A| > |\mathscr{P}_\kappa\lambda \backslash f[A]|$, we can find $C \in \mathscr{P}(A) \cap I$ with $|C| = |\mathscr{P}_\kappa\lambda \backslash A|$ by our assumption and (2) in Lemma 2.7 works. In case that $|\mathscr{P}_\kappa\lambda \backslash A| < |\mathscr{P}_\kappa\lambda \backslash f[A]|$, we have $D \in \mathscr{P}(A) \cap I$ with $|D| = |\mathscr{P}_\kappa\lambda \backslash f[A]|$ and the assertion follows by (3) in Lemma 2.7.

Case 2: There is no $B \in \mathscr{P}(A) \cap I$ with $|B| = |A|$.

First note that $|\{x \in \mathscr{P}_\kappa\lambda : 0 \notin x\}| = \lambda^{<\kappa}$. If $|\mathscr{P}_\kappa\lambda \backslash A| < \lambda^{<\kappa}$, we have $|\{x \in \mathscr{P}_\kappa\lambda : 0 \notin x\} \backslash A| < \lambda^{<\kappa}$ hence $|\{x \in \mathscr{P}_\kappa\lambda : 0 \notin x\} \cap A| = \lambda^{<\kappa} = |A|$. However $\{x \in \mathscr{P}_\kappa\lambda : 0 \notin x\} \cap A \in I$, which contradicts our assumption. Hence $|\mathscr{P}_\kappa\lambda \backslash A| = \lambda^{<\kappa}$.

Suppose that $|\mathscr{P}_\kappa\lambda \backslash f[A]| < \lambda^{<\kappa}$. Since f is I-fine, we have that $|\{x \in A : 0 \notin f(x)\}| < \lambda^{<\kappa}$ by our assumption. Hence $|\{x \in f[A] : 0 \notin x\}| < \lambda^{<\kappa}$. Since $|\mathscr{P}_\kappa\lambda \backslash f[A]| < \lambda^{<\kappa}$, we know that $|\{x \in \mathscr{P}_\kappa\lambda : 0 \notin x\}| < \lambda^{<\kappa}$. This contradiction tells us that $|\mathscr{P}_\kappa\lambda \backslash f[A]| = \lambda^{<\kappa}$. Now we have that $|\mathscr{P}_\kappa\lambda \backslash A| = |\mathscr{P}_\kappa\lambda \backslash f[A]| = \lambda^{<\kappa}$ and the assertion holds by (1) in Lemma 2.7. □

It is possible to have an unbounded set of $\mathscr{P}_\kappa\lambda$ with the cardinality $< \lambda^{<\kappa}$. An ideal which satisfies the following might exist:

> there is $A \in I^*$ with the cardinality $\lambda^{<\kappa}$ such that every $X \in \mathscr{P}(A) \cap I$ has the cardinality $< \lambda^{<\kappa}$.

We show this is not the case.

Proposition 2.9. *Suppose $\lambda > \kappa$. Then every $X \subset \mathscr{P}_\kappa\lambda$ has a subset $Y \in \mathrm{I}_{\kappa,\lambda}$ with $|Y| = |X|$.*

Proof. We may assume $X \in \mathrm{I}^+_{\kappa,\lambda}$. Since $\lambda > \kappa$, we have that $|X| > \kappa$.

Case 1: $\mathrm{cof}(|X|) > \kappa$.

For $\alpha < \kappa$, let $B_\alpha = \{x \in X : \alpha \notin x\}$. Then $B_\alpha \in \mathscr{P}(X) \cap \mathrm{I}_{\kappa,\lambda}$ and $X = \bigcup\{B_\alpha : \alpha < \kappa\}$. Since $\mathrm{cof}(|X|) > \kappa$, there is $\alpha < \kappa$ with $|B_\alpha| = |X|$. Then B_α is as desired.

Case 2: $\mathrm{cof}(|X|) \leq \kappa$.

Since $|X| \geq \lambda \geq \kappa^+$, it holds that $|X| > \kappa^+$.

For $\alpha < \kappa^+$, let $C_\alpha = \{x \in X : x \cap \kappa^+ \subset \alpha\}$. Then $C_\alpha \in \mathscr{P}(X) \cap \mathrm{I}_{\kappa,\lambda}$ and $X = \bigcup\{C_\alpha : \alpha < \kappa^+\}$. Note that for $\alpha < \beta < \kappa^+$, we have $C_\alpha \subset C_\beta$. Then for every cardinal $\mu < |X|$, the set $\{\alpha < \kappa^+ : |C_\alpha| \leq \mu\}$ is bounded in κ^+; Otherwise, there is a cardinal $\mu < |X|$ such that $|C_\alpha| \leq \mu$ for every $\alpha < \kappa^+$. Then $|X| = |\bigcup\{C_\alpha : \alpha < \kappa^+\}| \leq \kappa^+ \times \mu < |X|$. Contradiction.

Now we know that for every $\mu < |X|$, the set $\{\alpha < \kappa^+ : |C_\alpha| \leq \mu\}$ is bounded in κ^+. Since $\mathrm{cof}(|X|) \leq \kappa$, there is $\alpha < \kappa^+$ such that $|C_\alpha| > \mu$ for every $\mu < |X|$. Now C_α is as reqired. □

Proposition 2.10. *There is a bijection* $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ *such that* $f_*(\mathrm{I}_{\kappa,\lambda}) \supset \mathrm{I}_{\kappa,\lambda} \restriction A$ *for some* $A \in \mathrm{I}^+_{\kappa,\lambda} \backslash \mathrm{I}^*_{\kappa,\lambda}$.

Proof. Let $\{s_\alpha : \alpha < \lambda^{<\kappa}\}$ be an enumeration of $\mathscr{P}_\kappa\lambda$. By induction on $\alpha < \lambda^{<\kappa}$ we define $\langle (x_\alpha, y_\alpha) \mid \alpha < \lambda^{<\kappa}\rangle$ such that

(a) $s_\alpha \subset x_\alpha, y_\alpha$
(b) $x_\alpha \neq y_\alpha$
(c) $x_\alpha, y_\alpha \notin \{x_\beta, y_\beta : \beta < \alpha\}$.

This is possible since $|\hat{a}| = \lambda^{<\kappa}$ for every $a \in \mathscr{P}_\kappa\lambda$; there is an injection from $\mathscr{P}_\kappa(\lambda \backslash a)$ into $\hat{a}$.

Define $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ by $f(s_\alpha) = x_\alpha$. Since $f(s_\alpha) \supset s_\alpha$ for every $\alpha < \lambda^{<\kappa}$, f is $\mathrm{I}_{\kappa,\lambda}$-fine. We know that $\{x_\alpha : \alpha < \lambda^{<\kappa}\}$ and $\{y_\alpha : \alpha < \lambda^{<\kappa}\}$ are two disjoint unbounded sets. Hence $\{x_\alpha : \alpha < \lambda^{<\kappa}\} = f[\mathscr{P}_\kappa\lambda] \in \mathrm{I}^+_{\kappa,\lambda} \backslash \mathrm{I}^*_{\kappa,\lambda}$. Cleary f is injective. Hence $f_*(\mathrm{I}_{\kappa,\lambda}) \cong \mathrm{I}_{\kappa,\lambda}$ by Proposition 2.8, and we have $f_*(\mathrm{I}_{\kappa,\lambda}) \supset \mathrm{I}_{\kappa,\lambda} \restriction f[\mathscr{P}_\kappa\lambda]$. □

Corollary 2.11. $\mathrm{I}_{\kappa,\lambda}$ *has an isomorph other than itself.*

Moreover the referee kindly pointed out the following:

Proposition 2.12. $\mathrm{I}_{\kappa,\lambda}$ *is isomorphic to* $\mathrm{I}_{\kappa,\lambda} \restriction A$ *for some* $A \in \mathrm{I}^+_{\kappa,\lambda} \backslash \mathrm{I}^*_{\kappa,\lambda}$.

Proof. Define $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ by

$$f(x) = \begin{cases} x \cup \{\sup(x\cap\kappa)\} & \text{if } \sup(x\cap\kappa) \notin x \\ x \cup \{\sup(x\cap\kappa)+1\} & \text{otherwise} \end{cases}$$

It is easily seen that f is an $\mathrm{I}_{\kappa,\lambda}$-fine injection and $f[\mathscr{P}_\kappa\lambda] \in \mathrm{I}^+_{\kappa,\lambda} \backslash \mathrm{I}^*_{\kappa,\lambda}$. By 2.8 it holds that $f_*(\mathrm{I}_{\kappa,\lambda}) \cong \mathrm{I}_{\kappa,\lambda}$. We have $f_*(\mathrm{I}_{\kappa,\lambda}) = \mathrm{I}_{\kappa,\lambda} \restriction f[\mathscr{P}_\kappa\lambda]$ if we show $f_*(\mathrm{I}_{\kappa,\lambda}) \subset \mathrm{I}_{\kappa,\lambda} \restriction f[\mathscr{P}_\kappa\lambda]$.

Let $X \in f_*(\mathrm{I}_{\kappa,\lambda})$. Then we have $a \in \mathscr{P}_\kappa\lambda$ such that for any $x \in \hat{a}$, $f(x) \notin X$. Let $b = a \cup \{\sup(a\cap\kappa)+1\}$. We show $X \cap f[\mathscr{P}_\kappa\lambda] \cap \hat{b} = \emptyset$. Choose any $x \in \mathscr{P}_\kappa\lambda$ such that $b \subset f(x) \in X$. In case that $\sup(x\cap\kappa) \notin x$, we have $a \cup \{\sup(a\cap\kappa)+1\} \subset x \cup \{\sup(x\cap\kappa)\}$ and $\sup(x\cap\kappa)$ is a limit ordinal. Hence $a \subset x \in f^{-1}[X]$, which contradicts to the choice of a. When $\sup(x) \in x$, it holds that $a \cup \{\sup(a\cap\kappa)+1\} \subset x \cup \{\sup(x\cap\kappa)+1\}$. Again we have $a \subset x$. □

Definition 2.13. An ideal I is a *weak Q-point* (*weakly selective*) if for any $\mathrm{I}_{\kappa,\lambda}$-fine ($I$-fine) f and $X \in I^+$ there is $Y \in \mathscr{P}(X) \cap I^+$ such that $f \restriction Y$ is injective. I is said to be a *nowhere Q-point* if for any $X \in I^+$ $I \restriction X$ is not a Q-point.

Theorem 2.14. (1) *The bounded ideal* $\mathrm{I}_{\kappa,\lambda}$ *is a weak Q-point* (*hence weakly selective*).

(2) $\mathrm{I}_{\kappa,\lambda}$ *is a nowhere Q-point.*

Proof. Choose any $A \in \mathrm{I}^+_{\kappa,\lambda}$ and set $\gamma = \min\{|Y| : Y \in (\mathrm{I}_{\kappa,\lambda} \restriction A)^+\}$.

(1) Let $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ be $\mathrm{I}_{\kappa,\lambda}$-fine. We will find a $B \in \mathscr{P}(A) \cap \mathrm{I}^+_{\kappa,\lambda}$ such that $f \restriction B$ is injective.

Let $X \in \mathscr{P}(A) \cap \mathrm{I}^+_{\kappa,\lambda}$ with $|X| = \gamma$ and $\{x_\alpha : \alpha < \gamma\}$ be an enumeration of X. By induction we define $\langle s_\alpha \mid \alpha < \gamma\rangle$ such that

(a) $x_\alpha \subset s_\alpha \in X$

(b) $f(s_\alpha) \neq f(s_\beta)$ for any $\beta < \alpha$.

Suppose $\langle s_\beta \mid \beta < \alpha\rangle$ is defined. There is $a \in \mathscr{P}_\kappa\lambda$ such that $\{f(s_\beta) : \beta < \alpha\} \cap \hat{a} = \emptyset$. Since f is $\mathrm{I}_{\kappa,\lambda}$-fine, $\{x \in \mathscr{P}_\kappa\lambda : a \subset f(x)\} \in \mathrm{I}^*_{\kappa,\lambda}$. Choose s_α from $\widehat{x_\alpha} \cap X \cap \{x \in \mathscr{P}_\kappa\lambda : a \subset f(x)\} \neq \emptyset$.

Now $\{s_\alpha : \alpha < \gamma\}$ is a desired set.

(2) We define an $\mathrm{I}_{\kappa,\lambda}$-fine $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ such that for any $X \in (\mathrm{I}_{\kappa,\lambda} \restriction A)^*$ $f \restriction X$ is not injective. Pick a $B = \{b_\xi : \xi < \gamma\} \in \mathrm{I}^+_{\kappa,\lambda} \cap \mathscr{P}(A)$.

By induction on $\xi < \gamma$ we define $\langle (x_\xi, y_\xi) \mid \xi < \gamma \rangle$ such that

(a) $x_\xi \neq y_\xi \in \widehat{b_\xi} \cap A$
(b) $x_\xi, y_\xi \notin \{x_\zeta : \zeta < \xi\} \cup \{y_\zeta : \zeta < \xi\}$

Suppose $\langle (x_\zeta, y_\zeta) \mid \zeta < \xi \rangle$ is defined. Since $\{x_\zeta : \zeta < \xi\} \cup \{y_\zeta : \zeta < \xi\} \in \mathrm{I}_{\kappa,\lambda}$ and $\widehat{b_\xi} \cap A \in \mathrm{I}^+_{\kappa,\lambda}$, we can find $x_\xi \neq y_\xi \in \widehat{b_\xi} \cap A \backslash \{x_\zeta : \zeta < \xi\} \cup \{y_\zeta : \zeta < \xi\}$.

Set $C = \{x_\xi : \xi < \gamma\} \cup \{y_\xi : \xi < \gamma\}$. Then $C \in \mathscr{P}(A) \cap \mathrm{I}^+_{\kappa,\lambda}$.

Let $f \upharpoonright (\mathscr{P}_\kappa\lambda \backslash C) = \mathrm{Id} \upharpoonright (\mathscr{P}_\kappa\lambda \backslash C)$ and for each $\xi < \gamma$ $f(x_\xi) = f(y_\xi) \supset x_\xi \cup y_\xi$. f is $\mathrm{I}_{\kappa,\lambda}$-fine since $f(x) \supset x$ for every $x \in \mathscr{P}_\kappa\lambda$.

Suppose that $X \in (\mathrm{I}_{\kappa,\lambda} \upharpoonright A)^*$. Since $\mathscr{P}_\kappa\lambda \backslash X \in \mathrm{I}_{\kappa,\lambda} \upharpoonright A$, we have an $a \in \mathscr{P}_\kappa\lambda$ such that $A \cap \hat{a} \cap (\mathscr{P}_\kappa\lambda \backslash X) = \emptyset$. For some $\xi < \gamma$, $a \subset b_\xi$. $\widehat{b_\xi} \cap A \cap (\mathscr{P}_\kappa\lambda \backslash X) = \emptyset$ hence $\widehat{b_\xi} \cap A \subset X$. Now $x_\xi, y_\xi \in X$ and $f(x_\xi) = f(y_\xi)$. □

Remark 2.15. Usuba proved more general fact that I is a nowhere Q-point if $\mathrm{non}(I) = \mathrm{cof}(I)$, where $\mathrm{non}(I) = \min\{|X| : X \in I^+\}$ and $\mathrm{cof}(I) = \min\{|W| : W \subset I \wedge \forall X \in I \exists Y \in W \ X \subset Y\}$. It is easily seen that $\mathrm{non}(\mathrm{I}_{\kappa,\lambda}) = \mathrm{cof}(\mathrm{I}_{\kappa,\lambda})$.

3. Weakly Normal Ideals and Selectivity

For an ideal I on κ the weak normality coincides with the normality, and the sup-function is injective on κ. This implies the selectivity of normal ideals and the fact an ideal extending the non-stationary ideal is a Q-point.

In the following we state a $\mathscr{P}_\kappa\lambda$ version. The proof is essentially by Menas' [4] for fine ultrafilters over $\mathscr{P}_\kappa\lambda$.

Definition 3.1. Let $J_w = \{X \subset \mathscr{P}_\kappa\lambda : \exists f : X \to \lambda$, regressive, $\forall \gamma < \lambda \{x \in X : f(x) \leq \gamma\} \in \mathrm{I}_{\kappa,\lambda}\}$.

Shioya [6] proved (3) and (4) in the follwoing.

Fact 3.2. (1) *J_w is an ideal and* $\{x : \sup(x) \in x\} \in J_w$.

(2) *$J_w = \mathrm{I}_{\kappa,\lambda}$ if and only if* $\mathrm{cof}(\lambda) < \kappa$.

(3) *J_w is the minimal weakly normal ideal over $\mathscr{P}_\kappa\lambda$ if* $\mathrm{cof}(\lambda) = \kappa$.

(4) *The minimal weakly normal ideal over $\mathscr{P}_\kappa\lambda$ is a proper extension of J_w if* $\mathrm{cof}(\lambda) > \kappa$.

Lemma 3.3. *Let* $\mathrm{cof}(\lambda) \geq \kappa$, *$J_w \subset I$ be an ideal over $\mathscr{P}_\kappa\lambda$, f $\mathrm{I}_{\kappa,\lambda}$-fine, and $f[\mathscr{P}_\kappa\gamma] \in \mathrm{I}_{\kappa,\lambda}$ for all $\gamma < \lambda$. Then there is $S \in I^*$ such that* $\sup(x) = \sup(y)$ *whenever $f(x) = f(y)$ and $\{x, y\} \subset S$.*

Proof. Otherwise, $X = \{x \in \mathscr{P}_\kappa\lambda : \exists y_x(\sup(y_x) < \sup(x) \wedge f(x) = f(y_x))\} \in I^+$. Since $X \notin J_w$, we have a $\gamma < \lambda$ such that $Y = \{x \in X : \sup(y_x) < \gamma\} \in \mathrm{I}^+_{\kappa,\lambda}$. Let $Z = \{y_x : x \in Y\}$. Since f is $\mathrm{I}_{\kappa,\lambda}$-fine, $f[Y] \in \mathrm{I}^+_{\kappa,\lambda}$. However $f[Y] = f[Z] \subset f[\mathscr{P}_\kappa\gamma] \in \mathrm{I}_{\kappa,\lambda}$. Contradiction. □

It might be better to assume $f[X] \in \mathrm{I}_{\kappa,\lambda}$ for all $X \in \mathrm{I}_{\kappa,\lambda}$, which implies $f[\mathscr{P}_\kappa\gamma] \in \mathrm{I}_{\kappa,\lambda}$ for all $\gamma < \lambda$. Other choices are $\gamma^{<\kappa} < \lambda$ or the sup function is $\leq \eta$ to one for some $\eta < \lambda$.

By the same argument we have the following theorem, which seems the most natural $\mathscr{P}_\kappa\lambda$ version of the fact that an ideal extension of NS_κ is a Q-point:

Theorem 3.4. *Let* $\mathrm{cof}(\lambda) \geq \kappa$.

(1) *Suppose that* $J_w \subset I$ *and* $\sup \restriction X$ *is injective for some* $X \in I^*$. *Then* I *is a Q-point.*

(2) *If* I *is a weakly normal ideal over* $\mathscr{P}_\kappa\lambda$ *and* $\sup \restriction X$ *is injective for some* $X \in I^*$, *then* I *is selective.*

Remark 3.5. Usuba proved several facts for weak normality and selectivity. For instances:

(1) Suppose that λ is regular. Then, weakly normal prime ideal I is selective if and only if $\sup \restriction X$ is injective for some $X \in I^*$.

(2) For $n \in \omega$, $\mathscr{P}_\kappa\kappa^{+n}$ carries a normal selective ideal I such that $\sup \restriction X$ is not injective for any $X \in I^*$.

(3) $\mathrm{NS}_{\kappa,\kappa^{+n}}$ is not a Q-point for any $n \in \omega$.

Definition 3.6. Two ideals I and J are *coherent* if there is an ideal K such that $I \cup J \subset K$. This is equivalent to $I^* \cap J = \varnothing$.

In [1] the coherence with the non-stationary ideal NS_κ is mentioned. We investigate that with J_w.

Definition 3.7. For an ideal I, let $R(I) = \{f : X \to \mathscr{P}_\kappa\lambda : f$ is I-fine and $X \in I^*\}$. For $f, g \in R(I)$, $f \lessdot g$ if $\{x : \sup(f(x)) < \sup(g(x))\} \in I^*$.

Clearly $\lessdot$ is well-founded.

Lemma 3.8. *Suppose that* $\mathrm{cof}(\lambda) \geq \kappa$, f *is* $\lessdot$*-minimal in* $R(I)$, *and* $\sup(f(x)) \notin f(x)$ *for any* $x \in \mathrm{dom}(f)$. *For every* $A \in f_*(I)^*$ *and regressive function* g *on* A, *there is a* $\gamma < \lambda$ *such that* $\{x \in A : g(x) \leq \gamma\} \in f_*(I)^+$.

Proof. Let $A \in f_*(I)^*$, g be regressive on A, and $X = f^{-1}[A]$. Since $\sup(y) \notin y$ for any $y \in f[X]$, $g(y) < \sup(y)$ for every $y \in f[X]$. Define $h : X \to \mathscr{P}_\kappa\lambda$ by $h(x) = f(x) \cap g(f(x))$. Then $\sup(h(x)) < \sup(f(x))$ for all $x \in X$. Hence h is not I-fine, and we have a $\gamma < \lambda$ such that $Y = \{x \in X : \gamma \notin h(x)\} \in I^+$. Since f is I-fine, we may assume $\gamma \in f(x)$ for all $x \in Y$. Thus, $g(f(x)) \leq \gamma$ for every $x \in Y$. Now $\{x : g(x) \leq \gamma\} \in f_*(I)^+$. □

Definition 3.9. A function $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ is said to be *incompressible* for I if $f \in R(I)$ and $\lessdot$-minimal in $R(I \restriction A)$ for all $A \in I^+$.

Proposition 3.10. *If $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ is incompressible for I and $\sup(f(x)) \notin f(x)$ for any $x \in \mathscr{P}_\kappa\lambda$, then $f_*(I \restriction A)$ is weakly normal for every $A \in I^+$.*

Proof. Suppose that $A \in I^+$ and h is regressive on $X \in f_*(I \restriction A)^+$. Set $Y = f^{-1}[X] \cap A$. Then $Y \in I^+$ and $X \in f_*(I \restriction Y)^*$. Since f is $\lessdot$-minimal in $R(I \restriction Y)$, $\{x \in X : h(x) \leq \gamma\} \in f_*(I \restriction Y)^+$ for some $\gamma < \lambda$ by the previous lemma. Clearly $f_*(I \restriction Y)^+ \subset f_*(I \restriction A)^+$. □

Definition 3.11. For an ideal I over $\mathscr{P}_\kappa\lambda$, let $R'(I) = \{f \in R(I) : \forall\gamma < \lambda(\{x : \sup(f(x)) \leq \gamma\} \in \mathrm{I}_{\kappa,\lambda})\}$.

Fact 3.12. *If f is $\mathrm{I}_{\kappa,\lambda}$-fine, then $f \in R'(I)$.*

Fact 3.13. *For every ideal I over $\mathscr{P}_\kappa\lambda$, the following are equivalent.*

(1) *I and J_w are coherent.*

(2) *For every $A \in I^*$ and regressive h on A, there is $\gamma < \lambda$ such that $\{x \in A : h(x) \leq \gamma\} \in \mathrm{I}^+_{\kappa,\lambda}$.*

Lemma 3.14. *Let $\mathrm{cof}(\lambda) \geq \kappa$ and $f \in R'(I)$.*

(1) *If f is $\mathrm{I}_{\kappa,\lambda}$-fine, $\lessdot$-minimal in $R'(I)$, and $\{x : \sup(f(x)) \notin f(x)\} \in I^*$, then $f_*(I)$ and J_w are coherent.*

(2) *If $f_*(I)$ and J_w are coherent and $f[X] \in \mathrm{I}_{\kappa,\lambda}$ for all $X \in \mathrm{I}_{\kappa,\lambda}$, then f is $\lessdot$-minimal in $R'(I)$.*

Proof. (1) Suppose that $X \in f_*(I)^*$ and g is regressive on X. For $x \in f^{-1}[X]$, let $h(x) = f(x) \cap g(f(x))$. We may assume $\sup(f(x)) \notin f(x)$ for any $x \in f^{-1}[X]$. Hence $\{x : \sup(h(x)) < \sup(f(x))\} \in I^*$. Thus, for some $\gamma < \lambda$, $Y = \{x : \sup(h(x)) \leq \gamma\} \in \mathrm{I}^+_{\kappa,\lambda}$. For any $x \in Y$, $\gamma + 1 \notin f(x) \cap g(f(x))$. Since f is $\mathrm{I}_{\kappa,\lambda}$-

fine, $\{x : \gamma + 1 \notin f(x)\} \in \mathrm{I}_{\kappa,\lambda}$. So, $Z = \{x \in Y : g(f(x)) \leq \gamma\} \in \mathrm{I}^+_{\kappa,\lambda}$. Now $f[Z] \in \mathrm{I}^+_{\kappa,\lambda}$ since f is $\mathrm{I}_{\kappa,\lambda}$-fine. For every $x \in f[Z]$, $g(x) \leq \gamma$.

(2) Suppose that $A \in I^*$, $g \in R(I)$, and $\sup(g(x)) < \sup(f(x))$ for all $x \in A$. For each $y \in f[A]$, choose $x_y \in A$ such that $y = f(x_y)$ and set $h(y) = \sup(g(x_y))$. Then $h(y) < \sup(y)$ for all $y \in f[A]$. Since $f_*(I)$ is coherent with J_w and $f[A] \in f_*(I)^*$, $\{y \in f[A] : h(y) \leq \gamma\} \in \mathrm{I}^+_{\kappa,\lambda}$ for some $\gamma < \lambda$. By our assumption, we have $\{x_y : \sup(g(x_y)) \leq \gamma\} \in \mathrm{I}^+_{\kappa,\lambda}$. Hence $g \notin R'(I)$. □

Remark 3.15. (1) If J_w and $f_*(I)$ are coherent, then $\{x : \sup(f(x)) \notin f(x)\} \in I^+$.

(2) If $J_w \subset f_*(I)$, then $\{x : \sup(f(x)) \notin f(x)\} \in I^*$.

Definition 3.16. A function $f : \mathscr{P}_\kappa\lambda \to \mathscr{P}_\kappa\lambda$ is said to be *weakly incompressible* for I if $f \in R'(I)$ and $\lessdot$-minimal in $R'(I \restriction A)$ for all $A \in I^+$.

We get a result analogous to the previous lemma.

Proposition 3.17. *Let* $\mathrm{cof}(\lambda) \geq \kappa$ *and* $f \in R'(I)$.

(1) *If* f *is* $\mathrm{I}_{\kappa,\lambda}$*-fine, weakly incompressible, and* $\{x : \sup(f(x)) \notin f(x)\} \in I^*$, *then* $J_w \subset f_*(I)$.

(2) *If* $J_w \subset f_*(I)$ *and* $f[X] \in \mathrm{I}_{\kappa,\lambda}$ *for all* $X \in \mathrm{I}_{\kappa,\lambda}$, *then* f *is weakly incompressible for* I.

Proof. (1) Let $X \in f_*(I)^+$. Then, $f^{-1}[X] \in I^+$ and f is $\lessdot$-minimal in $R'(I \restriction f^{-1}[X])$. Since $X \in f_*(I \restriction f^{-1}[X])^*$, $X \notin J_w$ by the above lemma.

(2) Clear by the above lemma. □

Definition 3.18. An ideal I over $\mathscr{P}_\kappa\lambda$ is a *weak P-point* if for every $X \in I^+$ and $I \restriction X$-fine f there is $Y \in \mathscr{P}(X) \cap I^+$ such that f is $\mathrm{I}_{\kappa,\lambda} \restriction Y$-fine.

Proposition 3.19. *Suppose that* $\mathrm{cof}(\lambda) \geq \kappa$, I *is a weak P-point*, $f[X] \in \mathrm{I}_{\kappa,\lambda}$ *for every* $X \in \mathrm{I}_{\kappa,\lambda}$, *and* $\{x : \sup(f(x)) \notin f(x)\} \in I^*$. *Then the following are equivalent.*

(1) f *is incompressible for* I.

(2) $f_*(I)$ *is weakly normal.*

(3) $J_w \subset f_*(I)$.

(4) f *is weakly incompressible for* I.

PROOF. By the previous argument we only have to show (4) $\to$ (1). Let $A \in I^+$, $g \in R(I \restriction A)$, and $\sup(g(x)) < \sup(f(x))$ for all $x \in A$. Since I is a weak P-point, we can find a $B \in \mathscr{P}(A) \cap I^+$ such that for every $\gamma < \lambda$, $\{x \in B : \gamma \notin g(x)\} \in \mathrm{I}_{\kappa,\lambda}$. Now $g \in R'(I \restriction B)$ and $g \lessdot f$ in $R'(I \restriction B)$, which contradicts to f is weakly incompressible. □

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