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THE BAIRE PROPERTY OF CERTAIN HYPO-GRAPH SPACES

By

Katsuhisa Koshino

Abstract. Let X be a compact metrizable space and Y be a non-degenerate dendrite with an end point $\mathbf{0}$. For each continuous function $f : X \to Y$, we define the hypo-graph $\downarrow f = \bigcup_{x \in X} \{x\} \times [\mathbf{0}, f(x)]$ of f, where $[\mathbf{0}, f(x)]$ is the unique path from $\mathbf{0}$ to $f(x)$ in Y. Then we can regard $\downarrow \mathrm{C}(X, Y) = \{\downarrow f \mid f : X \to Y \text{ is continuous}\}$ as a subspace of the hyperspace consisting of non-empty closed sets in $X \times Y$ equipped with the Vietoris topology. In this paper, we prove that $\downarrow \mathrm{C}(X, Y)$ is a Baire space if and only if the set of isolated points of X is dense.

1. Introduction

The study on topological properties of function spaces plays a significant role in geometric functional analysis. It is one of the most interesting problems for many researchers when a function space is a Baire space. In this paper, we define a hypo-graph of each continuous function from a compact metrizable space to a non-degenerate dendrite and endow the set of hypo-graphs with certain topology. We will discuss the Baire property of the hypo-graph space.

Throughout the paper, we assume that all maps are continuous, but functions are not necessarily continuous. Moreover, let X be a compact metrizable space and Y be a non-degenerate dendrite with an end point $\mathbf{0}$. Recall that a *dendrite* is a Peano continuum, namely a connected, locally connected, compact metrizable space, containing no simple closed curves. A point of a space is called an *end point* provided that it has an arbitrarily small open neighborhood whose boundary is a

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singleton. Every non-degenerate dendrite contains at least two end points, see [9, Chapter III, (6.1) and Chapter V, (1.1)]. It is well-known that any two distinct points of a dendrite are joined by a unique arc [9, Chapter V, (1.2)]. For any two points $x, y \in Y$, the symbol $[x, y]$ means the unique arc between x and y if $x \neq y$, or the singleton $\{x\} = \{y\}$ if $x = y$.

For each function $f : X \to Y$, we define the *hypo-graph* $\downarrow f$ of f as follows:

$$\downarrow f = \bigcup_{x \in X} \{x\} \times [\mathbf{0}, f(x)] \subset X \times Y.$$

Observe that if f is continuous, then the hypo-graph $\downarrow f$ is closed in $X \times Y$. Let $\mathrm{Cld}(X \times Y)$ be the hyperspace of non-empty closed sets in $X \times Y$ endowed with the Vietoris topology. Then we can regard the set

$$\downarrow \mathrm{C}(X, Y) = \{\downarrow f \mid f : X \to Y \text{ is continuous}\}$$

of hypo-graphs of continuous functions from X to Y as a subset of $\mathrm{Cld}(X \times Y)$. We shall equip $\downarrow \mathrm{C}(X, Y)$ with the relative topology of $\mathrm{Cld}(X \times Y)$.

A closed subset A of a metric space $W = (W, d)$ is a Z-*set* in W if for each map $\varepsilon : W \to (0, \infty)$, there exists a map $f : W \to W$ such that $d(f(x), x) < \varepsilon(x)$ for all $x \in W$ and $f(W) \cap A = \emptyset$. This notion plays a central role in the theory of infinite-dimensional topology. A countable union of Z-sets is said to be a Z_σ-*set*. Note that every Z-set is nowhere dense, and hence every space that is a Z_σ-set in itself is not a Baire space. In this paper, we will give necessary and sufficient conditions for $\downarrow \mathrm{C}(X, Y)$ to be a Baire space as follows (Z. Yang [7] proved the case that Y is the closed unit interval $\mathbf{I} = [0, 1]$ and $\mathbf{0} = 0$):

Main Theorem. *The following are equivalent*:

(1) $\downarrow \mathrm{C}(X, Y)$ *is a Baire space*;
(2) $\downarrow \mathrm{C}(X, Y)$ *is not a* Z_σ-*set in itself*;
(3) *The set of isolated points of* X *is dense.*

2. Preliminaries

In this section, we introduce some notation and lemmas used later. For a metric space $W = (W, d)$ and $\varepsilon > 0$, let $B_d(x, \varepsilon) = \{y \in W \mid d(x, y) < \varepsilon\}$. The metric d is *convex* if any two points x and y in W have a mid point z, that is, $d(x, z) = d(y, z) = d(x, y)/2$. It is easy to verify that when d is convex and complete, there is a path between x and y isometric to the interval $[0, d(x, y)]$. Every Peano continuum admits a convex metric, see [1] and [5, 6]. From now on, we use an admissible metric d_X on X and an admissible convex metric d_Y on Y.

Arcs in a dendrite have the following good property with respect to an admissible convex metric [2].

Lemma 2.1. *There exists a map* $\gamma : Y^2 \times \mathbf{I} \to Y$ *such that for any distinct points* $x, y \in Y$, *the map* $\gamma(x, y, *) : \mathbf{I} \ni t \mapsto \gamma(x, y, t) \in Y$ *is an arc from* x *to* y *and the following holds*:

- *For each* $x_i, y_i \in Y$, $i = 1, 2$, $d_Y(\gamma(x_1, y_1, t), \gamma(x_2, y_2, t)) \le \max\{d_Y(x_1, x_2), d_Y(y_1, y_2)\}$ *for all* $t \in \mathbf{I}$.

We also use an admissible metric ρ on $X \times Y$ defined by

$$\rho((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}.$$

Since X and Y are compact, the topology of $\mathrm{Cld}(X \times Y)$ is induced by the *Hausdorff metric* ρ_H defined as follows:

$$\rho_H(A, B) = \inf\left\{ r > 0 \;\middle|\; A \subset \bigcup_{(x,y) \in B} B_\rho((x, y), r), B \subset \bigcup_{(x,y) \in A} B_\rho((x, y), r) \right\}.$$

For each $A \in \mathrm{Cld}(X \times Y)$, we define a set-valued function $A : X \to \mathrm{Cld}^*(Y)$ as follows:

$$A(x) = \{y \in Y \mid (x, y) \in A\} \subset Y,$$

where $\mathrm{Cld}^*(Y)$ is the set of closed subsets in Y. Here we set $A(B) = \{A(x) \mid x \in B\}$ for $B \subset X$. Moreover, for each subset $B \subset X$, let

$$A|_B = \{(x, y) \in A \mid x \in B\} \subset X \times Y.$$

The following lemma, that has been proved in [3], is a key lemma of this paper.

Lemma 2.2 (Digging Lemma). *Let* Z *be a metrizable space and* $\phi : Z \to {\downarrow}\mathrm{C}(X, Y)$ *be a map. Suppose that* X *contains a non-isolated point* a. *Then for each map* $\varepsilon : Z \to (0, 1)$, *there exist maps* $\psi : Z \to {\downarrow}\mathrm{C}(X, Y)$ *and* $\delta : Z \to (0, 1)$ *such that for each* $x \in Z$,

(a) $\rho_H(\psi(x), \phi(x)) < \varepsilon(x)$,
(b) $\psi(x)(B_{d_X}(a, \delta(x))) = \{\{\mathbf{0}\}\}$.

3. Proof of Main Theorem

This section is devoted to proving the main theorem. For the sake of convenience, we denote the set of isolated points of X by X_0. Let $\overline{{\downarrow}\mathrm{C}(X, Y)}$ be the closure of $\downarrow\mathrm{C}(X, Y)$ in $\mathrm{Cld}(X \times Y)$. Since X and Y are compact, $\mathrm{Cld}(X \times Y)$ is also compact, and hence $\overline{\downarrow\mathrm{C}(X, Y)}$ is a compactification of $\downarrow\mathrm{C}(X, Y)$.

Lemma 3.1. *The following holds*:

$$\overline{\downarrow\mathrm{C}(X, Y)} = \{A \in \mathrm{Cld}(X \times Y) \mid (*)\},$$

where

$(*)$ *for each* $x \in X$, (i) $A(x) \neq \emptyset$, (ii) $[\mathbf{0}, y] \subset A(x)$ *if* $y \in A(x)$, *and* (iii) $A(x)$ *is an arc or the singleton* $\{\mathbf{0}\}$ *if* $x \in X_0$.

Proof. For simplicity, let $\mathscr{A} = \{A \in \mathrm{Cld}(X \times Y) \mid (*)\}$. Obviously, $\downarrow\mathrm{C}(X, Y) \subset \mathscr{A}$. First, we shall show that $\mathscr{A}$ is a closed set in $\mathrm{Cld}(X \times Y)$. To this end, take any sequence $\{A_n\}_{n\in\mathbf{N}}$ in $\mathscr{A}$ that converges to $A \in \mathrm{Cld}(X \times Y)$. According to [4, Lemma 1.11.2],

$$(\star)\quad A = \left\{ (x, y) \in X \times Y \;\middle|\; \begin{array}{l} \text{for each } n \in \mathbf{N}, \text{ there is } (x_n, y_n) \in A_n \\ \text{such that } \lim_{n\to\infty}(x_n, y_n) = (x, y) \end{array} \right\}.$$

We will prove that $A \in \mathscr{A}$. Fix any point $x \in X$.

(i) $A(x) \neq \emptyset$. Since each $A_n \in \mathscr{A}$, we can choose a point $y_n \in A_n(x) \neq \emptyset$. By the compactness of Y, we may assume that $\{y_n\}_{n\in\mathbf{N}}$ converges to some point $y \in Y$. Due to $(\star)$, $(x, y) \in A$. It follows that $A(x) \neq \emptyset$.

(ii) $[\mathbf{0}, y] \subset A(x)$ for every $y \in A(x)$. To show this, take any $z \in [\mathbf{0}, y]$. Then we can write $z = \gamma(\mathbf{0}, y, t)$ for some $t \in \mathbf{I}$, where $\gamma : Y^2 \times \mathbf{I} \to Y$ as in Lemma 2.1. Because $(x, y) \in A$, according to $(\star)$, there exists $(x_n, y_n) \in A_n$, $n \in \mathbf{N}$, such that $\lim_{n\to\infty}(x_n, y_n) = (x, y)$. Let $z_n = \gamma(\mathbf{0}, y_n, t)$ for each $n \in \mathbf{N}$. Since $A_n \in \mathscr{A}$ for every $n \in \mathbf{N}$, $z_n \in [\mathbf{0}, y_n] \subset A_n(x_n)$, and hence $(x_n, z_n) \in A_n$. It follows from Lemma 2.1 that $d_Y(z, z_n) \leq d_Y(y, y_n)$. Since $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$. Thus $\lim_{n\to\infty}(x_n, z_n) = (x, z)$. By $(\star)$, $(x, z) \in A$, namely $z \in A(x)$, which implies that $[\mathbf{0}, y] \subset A(x)$.

(iii) $A(x)$ is an arc or the singleton $\{\mathbf{0}\}$ if $x \in X_0$. Suppose the contrary, that is, x is an isolated point and $A(x)$ is neither arc nor the singleton $\{\mathbf{0}\}$. Then $A(x)$ contains a triod one of whose end points is $\mathbf{0}$. Let e_1, e_2 be the other end points and b be the branch point of the triod. Define $\delta_1 = \min\{d_Y(e_i, b), d_Y(e_2, b)\} > 0$. On the other hand, since $x \in X_0$, we can find $\delta_2 > 0$ such that $B_{d_X}(x, \delta_2) = \{x\}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Since $\lim_{n\to\infty} A_n = A$, there exists $n \in \mathbf{N}$ such that $\rho_H(A_n, A) < \delta$. Then we can choose points $y_1, y_2 \in Y$ so that $y_1, y_2 \in A_n(x)$ and $d_Y(y_i, e_i) < \delta$, $i = 1, 2$, respectively. Observe that $A_n(x)$ contains the triod whose end points are $\mathbf{0}$, y_1 and y_2, which contradicts to that $A_n \in \mathscr{A}$. Therefore $A(x)$ is an arc or the singleton $\{\mathbf{0}\}$.

By (i), (ii) and (iii), $A \in \mathscr{A}$. Consequently, $\mathscr{A}$ is closed in $\mathrm{Cld}(X \times Y)$.

Next, we will prove that $\downarrow\mathrm{C}(X, Y)$ is dense in $\mathscr{A}$. Take any $A \in \mathscr{A}$ and $\varepsilon > 0$. We need only to construct a map $f : X \to Y$ such that $\rho_H(\downarrow f, A) < \varepsilon$. Since A is compact, it has a finite subset $A' = \{(x_i, y_i) \in A \mid i = 1, \dots, n\}$ such that $A \subset \bigcup_{i=1}^{n} B_\rho((x_i, y_i), \varepsilon/4)$. Recall that if $x_i \in X_0$ for some $i = 1, \dots, n$, then $A(x_i)$ is an arc or the singleton $\{\mathbf{0}\}$. In the case that there are $1 \le i < j \le n$ such that $x_i = x_j$ and $x_i \notin X_0$, replace x_j with a point $x_j' \in X$ such that $x_j' \neq x_i$ and $d_X(x_i, x_j') < \varepsilon/4$. Moreover, if there are $1 \le i < j \le n$ such that $x_i = x_j$ and $x_i \in X_0$, we may assume that $y_j \in [\mathbf{0}, y_i]$. Then remove (x_j, y_j) from A'. Repeating these operations, we can obtain $\{(x_i, y_i) \in X \times Y \mid i = 1, \dots, m\}$ for some $m \le n$ such that $x_i \neq x_j$ if $i \neq j$, and letting

$$A_0 = X \times \{\mathbf{0}\} \cup \bigcup_{i=1}^{m} \{x_i\} \times [\mathbf{0}, y_i],$$

we get $\rho_H(A_0, A) < \varepsilon/2$. Let $\lambda = \min\{\varepsilon, d_X(x_i, x_j) \mid 1 \le i < j \le m\}/3 > 0$. Using the map $\gamma : Y^2 \times \mathbf{I} \to Y$ as in Lemma 2.1, we can define a map $f : X \to Y$ as follows:

$$f(x) = \begin{cases} \gamma(\mathbf{0}, y_i, (\lambda - d_X(x, x_i))/\lambda) & \text{if } x \in B_{d_X}(x_i, \lambda),\ i = 1, \dots, m, \\ \mathbf{0} & \text{if } x \in X \setminus \bigcup_{i=1}^{m} B_{d_X}(x_i, \lambda). \end{cases}$$

Then $\rho_H(\downarrow f, A_0) \le \lambda \le \varepsilon/3$. It follows that

$$\rho_H(\downarrow f, A) \le \rho_H(\downarrow f, A_0) + \rho_H(A_0, A) \le \varepsilon/3 + \varepsilon/2 < \varepsilon,$$

which means that $\downarrow\mathrm{C}(X \times Y)$ is dense in $\mathscr{A}$. The proof is complete. □

We show the implication (3) ⇒ (1) in the main theorem.

Proposition 3.2. *Suppose that X_0 is dense in X. Then $\downarrow\mathrm{C}(X, Y)$ is a Baire space.*

Proof. Let $\mathscr{F}$ be the collection of finite subsets of X_0. For each $F \in \mathscr{F}$ and $n \in \mathbf{N}$, we define

$$\mathscr{U}_{F,n} = \{A \in \overline{\downarrow\mathrm{C}(X, Y)} \mid A(x) \subset B_{d_Y}(\mathbf{0}, 1/n) \text{ for all } x \in X \setminus F\}.$$

Observe that $\mathscr{U}_{F,n}$ is open in $\overline{\downarrow\mathrm{C}(X, Y)}$ because $F \subset X_0$. Let $\mathscr{U}_n = \bigcup_{F \in \mathscr{F}} \mathscr{U}_{F,n}$. First, we shall prove that each $\mathscr{U}_n$ is dense in $\overline{\downarrow\mathrm{C}(X, Y)}$. For each $\downarrow f \in \downarrow\mathrm{C}(X, Y)$

and $\varepsilon > 0$, we can obtain $F \in \mathscr{F}$ so that $\rho_H(\downarrow f|_F, \downarrow f) < \varepsilon$ because $\downarrow f$ is compact and X_0 is dense in X. Define a map $g : X \to Y$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in F, \\ \mathbf{0} & \text{if } x \in X \backslash F. \end{cases}$$

Then $\downarrow g \in \mathscr{U}_{F,n} \subset \mathscr{U}_n$ and $\rho_H(\downarrow g, \downarrow f) \leq \rho_H(\downarrow f|_F, \downarrow f) < \varepsilon$. Hence $\mathscr{U}_n$ is dense in $\overline{\downarrow \mathrm{C}(X, Y)}$.

Next, we will show that $\mathscr{G} = \bigcap_{n \in \mathbf{N}} \mathscr{U}_n \subset \downarrow \mathrm{C}(X, Y)$. Let $A \in \mathscr{G}$. Observe that for each $x \in X \backslash X_0$, $A(x) = \{\mathbf{0}\}$. According to Lemma 3.1, for any $x \in X_0$, $A(x)$ is an arc or the singleton $\{\mathbf{0}\}$. Therefore A is a hypo-graph of some function $f : X \to Y$. It remains to show that f is continuous at each $x \in X \backslash X_0$. For each $n \in \mathbf{N}$, we can find $F \in \mathscr{F}$ such that $A \in \mathscr{U}_{F,n}$. Then $X \backslash F$ is a neighborhood of x in X and $A(y) \subset B_{d_Y}(\mathbf{0}, 1/n)$ for all $y \in X \backslash F$, which means that f is continuous at x. Therefore $A = \downarrow f \in \downarrow \mathrm{C}(X, Y)$, so $\mathscr{G} \subset \downarrow \mathrm{C}(X, Y)$. Since $\overline{\downarrow \mathrm{C}(X, Y)}$ is compact, the G_δ-set $\mathscr{G} = \bigcap_{n \in \mathbf{N}} \mathscr{U}_n$ is a Baire space and dense in $\overline{\downarrow \mathrm{C}(X, Y)}$, so it is also dense in $\downarrow \mathrm{C}(X, Y)$. Consequently, $\downarrow \mathrm{C}(X, Y)$ is a Baire space. □

The following lemma is a counterpart to Lemma 5 of [7], but we can not prove it by the same argument. The reason is because for hypo-graphs $\downarrow f, \downarrow g \in \downarrow \mathrm{C}(X, Y)$ and a point $x \in X$, the union $\downarrow f(x) \cup \downarrow g(x)$ of values of x is not necessarily an arc or the singleton $\{\mathbf{0}\}$ in Y, so $\downarrow f \cup \downarrow g \notin \downarrow \mathrm{C}(X, Y)$ in general. Using the Digging Lemma 2.2, we can prove the following:

Lemma 3.3. *Suppose that $\mathscr{A} = \mathscr{B} \cup \mathscr{Z} \subset \downarrow \mathrm{C}(X, Y)$ is a closed set such that $\mathscr{Z}$ is a Z-set in $\downarrow \mathrm{C}(X, Y)$, and there exists a point $x \in X$ such that for every $\downarrow f \in \mathscr{B}$, $\downarrow f(x) = \{\mathbf{0}\}$. Then $\mathscr{A}$ is a Z-set in $\downarrow \mathrm{C}(X, Y)$.*

Proof. Let $\varepsilon : \downarrow \mathrm{C}(X, Y) \to (0, 1)$. It is sufficient to construct a map $\phi : \downarrow \mathrm{C}(X, Y) \to \downarrow \mathrm{C}(X, Y)$ such that $\phi(\downarrow \mathrm{C}(X, Y)) \cap \mathscr{A} = \emptyset$ and $\rho_H(\phi(\downarrow f), \downarrow f) < \varepsilon(\downarrow f)$ for each $\downarrow f \in \downarrow \mathrm{C}(X, Y)$. Since $\mathscr{Z}$ is a Z-set, there is a map $\psi : \downarrow \mathrm{C}(X, Y) \to \downarrow \mathrm{C}(X, Y) \backslash \mathscr{Z}$ such that $\rho_H(\psi(\downarrow f), \downarrow f) < \varepsilon(\downarrow f)/2$ for every $\downarrow f \in \downarrow \mathrm{C}(X, Y)$. Fix a point $y_0 \in Y \backslash \{\mathbf{0}\}$ with $d_Y(\mathbf{0}, y_0) \leq 1$ and let

$$t(\downarrow f) = \min\{\varepsilon(\downarrow f), \rho_H(\psi(\downarrow f), \mathscr{Z}), \operatorname{diam} Y\}/2 > 0$$

for each $\downarrow f \in \downarrow \mathrm{C}(X, Y)$, where $\rho_H(\psi(\downarrow f), \mathscr{Z})$ means the usual distance between the point $\psi(\downarrow f)$ and the set $\mathscr{Z}$ in $\downarrow \mathrm{C}(X, Y)$ and diam Y means the diameter of Y.

First, we consider the case that $x \in X_0$. For each $\downarrow f \in \downarrow \mathrm{C}(X, Y)$, we have a map $g(\downarrow f) : X \to Y$ such that $\downarrow g(\downarrow f) = \psi(\downarrow f) \in \downarrow \mathrm{C}(X, Y)$. Define a map $\phi : \downarrow \mathrm{C}(X, Y) \to \downarrow \mathrm{C}(X, Y)$ by

$$\phi(\downarrow f) = \psi(\downarrow f)|_{X \setminus \{x\}} \cup \{x\} \times [\mathbf{0}, \gamma(g(\downarrow f)(x), y_0, t(\downarrow f)/\mathrm{diam}\ Y)],$$

where $\gamma : Y^2 \times \mathbf{I} \to Y$ is as in Lemma 2.1. Obviously, $\phi(\downarrow f)(x) \neq \{\mathbf{0}\}$, that is, $\phi(\downarrow f) \notin \mathscr{B}$. Since d_Y is convex, we have

$$\begin{aligned}\rho_H(\phi(\downarrow f), \psi(\downarrow f)) &\le d_Y(\gamma(g(\downarrow f)(x), y_0, t(\downarrow f)/\mathrm{diam}\ Y), g(\downarrow f)(x))\\ &= d_Y(g(\downarrow f)(x), y_0) \times t(\downarrow f)/\mathrm{diam}\ Y \le t(\downarrow f)\\ &\le \rho_H(\psi(\downarrow f), \mathscr{Z})/2,\end{aligned}$$

which implies that $\phi(\downarrow f) \notin \mathscr{Z}$. Moreover,

$$\begin{aligned}\rho_H(\phi(\downarrow f), \downarrow f) &\le \rho_H(\phi(\downarrow f), \psi(\downarrow f)) + \rho_H(\psi(\downarrow f), \downarrow f) < t(\downarrow f) + \varepsilon(\downarrow f)/2\\ &\le \varepsilon(\downarrow f)/2 + \varepsilon(\downarrow f)/2 = \varepsilon(\downarrow f).\end{aligned}$$

Next, we consider the case that $x \notin X_0$. Using the Digging Lemma 2.2, we can obtain maps $\xi : \downarrow \mathrm{C}(X, Y) \to \downarrow \mathrm{C}(X, Y)$ and $\delta : \downarrow \mathrm{C}(X, Y) \to (0, 1)$ such that for each $\downarrow f \in \downarrow \mathrm{C}(X, Y)$,

(a) $\rho_H(\xi(\downarrow f), \psi(\downarrow f)) < t(\downarrow f)/2$,
(b) $\xi(\downarrow f)(B_{d_X}(x, \delta(\downarrow f))) = \{\{\mathbf{0}\}\}$.

For each $\downarrow f \in \downarrow \mathrm{C}(X, Y)$, let

$$\eta(\downarrow f) = \bigcup_{x' \in B_{d_X}(x, \delta(\downarrow f))} \{x'\} \times [\mathbf{0}, \gamma(\mathbf{0}, y_0, t(\downarrow f)(\delta(\downarrow f) - d_X(x, x'))/(2\delta(\downarrow f)))].$$

We define a map $\phi : \downarrow \mathrm{C}(X, Y) \to \downarrow \mathrm{C}(X, Y)$ by $\phi(\downarrow f) = \xi(\downarrow f) \cup \eta(\downarrow f)$. Note that $\phi(\downarrow f)(x) \neq \{\mathbf{0}\}$, and hence $\phi(\downarrow \mathrm{C}(X, Y)) \cap \mathscr{B} = \varnothing$. For every $\downarrow f \in \downarrow \mathrm{C}(X, Y)$, we have

$$\begin{aligned}&\rho_H(\phi(\downarrow f), \psi(\downarrow f))\\ &\quad\le \rho_H(\phi(\downarrow f), \xi(\downarrow f)) + \rho_H(\xi(\downarrow f), \psi(\downarrow f))\\ &\quad< \max\{d_Y(\mathbf{0}, \gamma(\mathbf{0}, y_0, t(\downarrow f)(\delta(\downarrow f) - d_X(x, x'))/(2\delta(\downarrow f)))) \mid d_X(x, x') < \delta(\downarrow f)\}\\ &\qquad + t(\downarrow f)/2\\ &\quad= d_Y(\mathbf{0}, y_0) \times t(\downarrow f)/2 + t(\downarrow f)/2 \le t(\downarrow f)/2 + t(\downarrow f)/2 = t(\downarrow f)\\ &\quad\le \rho_H(\psi(\downarrow f), \mathscr{Z})/2.\end{aligned}$$

Therefore $\phi(\downarrow f) \notin \mathscr{Z}$. It follows that

$$\begin{aligned}\rho_H(\phi(\downarrow f), \downarrow f) &\le \rho_H(\phi(\downarrow f), \psi(\downarrow f)) + \rho_H(\psi(\downarrow f), \downarrow f) < t(\downarrow f) + \varepsilon(\downarrow f)/2 \\ &\le \varepsilon(\downarrow f)/2 + \varepsilon(\downarrow f)/2 = \varepsilon(\downarrow f).\end{aligned}$$

This completes the proof. $\square$

PROPOSITION 3.4. *If X_0 is not dense in X, then $\downarrow\mathrm{C}(X, Y)$ is a Z_σ-set in itself.*

PROOF. Take a countable dense set $D = \{x_n \mid n \in \mathbf{N}\}$ in $X \setminus X_0$. For each $n, m \in \mathbf{N}$, let

$$\mathscr{F}_{n,m} = \{\downarrow f \in \downarrow\mathrm{C}(X, Y) \mid d_Y(f(x_n), \mathbf{0}) \ge 1/m\}.$$

We will show that each $\mathscr{F}_{n,m}$ is a Z-set in $\downarrow\mathrm{C}(X, Y)$. Observe that $\mathscr{F}_{n,m}$ is closed in $\downarrow\mathrm{C}(X, Y)$. Applying the Digging Lemma 2.2, for each map $\varepsilon : \downarrow\mathrm{C}(X, Y) \to (0, 1)$, we can find a map $\phi : \downarrow\mathrm{C}(X, Y) \to \downarrow\mathrm{C}(X, Y)$ such that $\rho_H(\phi(\downarrow f), \downarrow f) < \varepsilon(\downarrow f)$ and $\phi(\downarrow f)(x_n) = \{\mathbf{0}\}$ for every $\downarrow f \in \downarrow\mathrm{C}(X, Y)$. Then $\phi(\downarrow\mathrm{C}(X, Y))$ misses $\mathscr{F}_{n,m}$. It follows that $\mathscr{F}_{n,m}$ is a Z-set in $\downarrow\mathrm{C}(X, Y)$.

Let $\mathscr{F} = \bigcap_{n \in \mathbf{N}} \bigcap_{m \in \mathbf{N}} (\downarrow\mathrm{C}(X, Y) \setminus \mathscr{F}_{n,m})$. We need only to prove that the closure $\overline{\mathscr{F}}$ of $\mathscr{F}$ in $\downarrow\mathrm{C}(X, Y)$ is a Z-set. As is easily observed,

$$\mathscr{F} = \{\downarrow f \in \downarrow\mathrm{C}(X, Y) \mid f(x_n) = \mathbf{0} \text{ for each } n \in \mathbf{N}\},$$

which implies that $f(x) = \mathbf{0}$ for all $\downarrow f \in \mathscr{F}$ and all $x \in X \setminus \overline{X_0}$, where $\overline{X_0}$ is the closure of X_0. Since X_0 is not dense in X, we can choose a point $x \in X \setminus \overline{X_0}$. Fix $\delta > 0$ such that $B_{d_X}(x, \delta) \subset X \setminus \overline{X_0}$. For every $\downarrow f \in \overline{\mathscr{F}}$, we have $\downarrow f(x) = \{\mathbf{0}\}$. Indeed, for each $\varepsilon \in (0, \delta)$, there exists $\downarrow g \in \mathscr{F}$ such that $\rho_H(\downarrow f, \downarrow g) < \varepsilon$. Then we can find $(a, b) \in \downarrow g$ such that $\rho((x, f(x)), (a, b)) < \varepsilon$. Since $d_X(x, a) < \varepsilon < \delta$, we get $g(a) = \mathbf{0}$. Hence $d_Y(f(x), \mathbf{0}) = d_Y(f(x), b) < \varepsilon$, which implies that $\downarrow f(x) = \{\mathbf{0}\}$. According to Lemma 3.3, $\overline{\mathscr{F}}$ is a Z-set in $\downarrow\mathrm{C}(X, Y)$. Consequently, $\downarrow\mathrm{C}(X, Y) = \overline{\mathscr{F}} \cup \bigcup_{m, n \in \mathbf{N}} \mathscr{F}_{n,m}$ is a Z_σ-set in itself. $\square$

Combining Propositions 3.2 and 3.4, we can prove the main theorem.

4. Topological Type of $\downarrow\mathrm{C}(X, Y)$

The theory of infinite-dimensional topology has made meaningful contributions to the study on function spaces because they are frequently infinite-dimensional. Indeed, several function spaces have been shown to be homeomorphic to typical infinite-dimensional spaces. From the end of 1980s to the beginning of 1990s, many researchers investigated topological types of function spaces of real-valued continuous functions on countable spaces endowed with the pointwise convergence topology, see [4].

We can consider that hypo-graph spaces give certain geometric aspect to function spaces with the pointwise convergence topology. Let $\mathbf{Q} = \mathbf{I}^{\mathbf{N}}$ be the Hilbert cube, where $\mathbf{N} = \{1, 2, \dots\}$ is the natural numbers, and $\mathbf{c}_0 = \{(x_i)_{i \in \mathbf{N}} \in \mathbf{Q} \mid \lim_{i \to \infty} x_i = 0\}$. In the case that $Y = \mathbf{I}$ and $\mathbf{0} = 0$, we can regard

$$\downarrow\mathrm{USC}(X, \mathbf{I}) = \{\downarrow f \mid f : X \to \mathbf{I} \text{ is upper semi-continuous}\}$$

as a subspace in $\mathrm{Cld}(X \times \mathbf{I})$. Z. Yang [7] showed the following theorem:

Theorem 4.1. *Suppose that X is infinite and locally connected. Then $\downarrow\mathrm{USC}(X, \mathbf{I}) = \overline{\downarrow\mathrm{C}(X, \mathbf{I})}$ and the pair $(\downarrow\mathrm{USC}(X, \mathbf{I}), \downarrow\mathrm{C}(X, \mathbf{I}))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$.*

For spaces W_1 and W_2, the symbol (W_1, W_2) means that $W_2 \subset W_1$. A pair (W_1, W_2) of spaces is homeomorphic to (Z_1, Z_2) if there exists a homeomorphism $f : W_1 \to Z_1$ such that $f(W_2) = Z_2$. In the paper [3], his result is generalized as follows:

Theorem 4.2. *If X is infinite and has only a finite number of isolated points, then the pair $(\overline{\downarrow\mathrm{C}(X, Y)}, \downarrow\mathrm{C}(X, Y))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$.*

The space $\mathbf{c}_0$ is not a Baire space. In fact, it is a Z_σ-set in itself. According to the main theorem, we can establish the following immediately.

Corollary 4.3. *If $\downarrow\mathrm{C}(X, Y)$ is homeomorphic to $\mathbf{c}_0$, then the set of isolated points is not dense in X.*

Z. Yang and X. Zhou [8] strengthened Theorem 4.1 as follows:

Theorem 4.4. *The pair $(\downarrow\mathrm{USC}(X, \mathbf{I}), \downarrow\mathrm{C}(X, \mathbf{I}))$ is homeomorphic to $(\mathbf{Q}, \mathbf{c}_0)$ if and only if the set of isolated points of X is not dense.*

It is still unknown whether the same result holds or not in our setting, that is, the case that Y is a non-degenerate dendrite.

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Division of Mathematics
Pure and Applied Sciences
University of Tsukuba
Tsukuba, 305-8571, Japan
E-mail: kakoshino@math.tsukuba.ac.jp

Present Address:
Faculty of Engineering
Kanagawa University
Yokohama, 221-8686, Japan
E-mail: pt120401we@kanagawa-u.ac.jp