TSUKUBA J. MATH.
Vol. 38 No. 1 (2014), 39–58

PARTIALLY ORDERED RINGS

By

Yoshimi Kitamura and Yoshio Tanaka

Abstract. We shall consider partially ordered rings as a generalization of ordered rings. We give properties or related matters of partially ordered rings, using convex ideals or non-negative semi-cones. Also, we consider order-preserving isomorphisms between residue class rings which are partially ordered rings.

1. Introduction

The symbol R means a non-zero commutative ring with the identity element denoted by 1, and I means an ideal of R with $I \neq R$ (similar, for R' and I'), unless otherwise stated.

The symbol $\mathbf{Z}$; $\mathbf{N}$ means the ring of integers; the set of natural numbers, respectively, and let $\mathbf{Z}^* = \mathbf{N} \cup \{0\}$. Also, $\mathbf{R}$; $\mathbf{Q}$ means the field of real numbers; the field of rational numbers, respectively.

As is well-known, the concept of "positive cone" of rings is useful in orderings on integral domains. This concept plays important roles in the study of ordered integral domains or ordered fields, as the positive parts which determine their orderings. The ring $\mathbf{Z}$, as well as the field $\mathbf{Q}$, has the unique positive cone. In [2], we induce the notion of "non-negative cone" of rings, and we study ordered rings, including integral domains.

In Section 2, we introduce the notion of "non-negative semi-cone" of rings. This notion will play important roles in the study of partially ordered rings as their non-negative parts. For each ring R, there exists a bijection between the class of non-negative semi-cones of R and the class of ordering relations on R each of which makes R a partially ordered ring. We give properties and related

2000 *Mathematics Subject Classification*: 06A06, 06F25, 46E25.
Key words and phrases: partially ordered ring, non-negative semi-cone, residue class ring, convex ideal, order-preserving homomorphism, continuous function.
Received May 13, 2013.

matters of partially ordered rings in terms of non-negative semi-cones. We consider conditions for a subset of a partially ordered ring to be a partial ordered subring (or totally ordered subring). We give a characterization of non-negative semi-cones of $\mathbf{Z}$. Also, we give a characterization of a partially ordered ring in which the ordered additive group (or ordered ring) $\mathbf{Z}$ is embeddable.

In Section 3, we deal with residue class rings of partially ordered rings. For a partially ordered ring R and an ideal I of R, the residue class ring R/I has the canonical ordering relation induced from the order of R. As is well-known, the ring R/I is a partially ordered ring by the ordering relation iff I is convex in R ([1]). We consider necessary and sufficient conditions for I to be convex, or R/I to be a partially ordered ring, using non-negative semi-cones. We give a characterization for an ideal of a partially ordered ring $\mathbf{Z}$ to be convex. Also, for a principal ideal I of the polynomial ring $R[x]$ over an ordered integral domain R, we give a characterization for I to be convex in $R[x]$ with respect to two typical orders on $R[x]$.

In Section 4, we consider order-preserving isomorphisms between residue class rings which are partially ordered rings. Let $(R, \leq)$ and $(R', \leq')$ be partially ordered rings, and $I; I'$ be an ideal of $R; R'$ respectively. For a homomorphism σ of $(R, \leq)$ to $(R', \leq')$, we naturally induce a homomorphism $\bar{\sigma}$ of R/I to R'/I' under $\sigma(I) \subset I'$. For R/I and R'/I' being partially ordered rings, we give a characterization of a homomorphism $\sigma : (R, \leq) \to (R', \leq)$ such that the map $\bar{\sigma}$ is an isomorphism with $\bar{\sigma}$ and $\bar{\sigma}^{-1}$ order-preserving, and we apply this to rings of continuous functions. Finally, related to certain properties on the map $\bar{\sigma}$, we give examples and matters on the map σ.

2. Partially Ordered Rings and Non-negative Semi-cones

For a ring R, we shall introduce a non-negative semi-cone of R. We give properties and related mattes of partially ordered rings in terms of non-negative semi-cones.

DEFINITION 2.1. Let A be a set, and $\leq$ be a binary relation on A. Then $\leq$ is a *partial order* (or *semi-order*) if it satisfies the following: (i) $x \leq x$ for all $x \in A$, (ii) $x \leq y$ and $y \leq x$ implies $x = y$, and (iii) $x \leq y$ and $y \leq z$ implies $x \leq z$.

For a partial order $\leq$ on R, $(R, \leq)$ is a *partially ordered ring* ([1]) if R satisfies the following conditions:

(a) $a \leq b$ implies $a + x \leq b + x$ for all x.

(b) $a \leq b$ and $0 \leq x$ implies $ax \leq bx$.

To define such a partial order on R, it is enough to specify the elements ≥ 0, subject to:

R has a binary relation $\leq$ satisfying the following conditions:

(a)′ $a \geq 0$ and $-a \geq 0$ iff $a = 0$.

(b)′ $a, b \geq 0$ implies $a + b \geq 0$.

(c)′ $a, b \geq 0$ implies $ab \geq 0$.

For a partially ordered ring $(R, \leq)$, if the order $\leq$ is a total order, then such an $(R, \leq)$ is called a *totally ordered ring* (abbreviated *ordered ring*), in particular, when R is a field (resp. integral domain), such an ordered ring $(R, \leq)$ is called an *ordered field* (resp. *ordered integral domain*).

Remark 2.2. (1) If $\leq$ is a partial order on a set A, then defining $x < y$ by $x \leq y$ and $x \neq y$, a binary relation $<$ on A satisfies the following: (i) for any $x, y \in A$, two of $x < y$, $y < x$, $x = y$ do not hold simultaneously, and (ii) $x < y$ and $y < z$ implies $x < z$. Conversely, for a binary relation $<$ on A satisfying (i) and (ii), define $x \leq y$ by $x < y$ or $x = y$, then $\leq$ is a partial order on A.

(2) For a binary relation (such as a partial order) $\leq$ on R, the following are equivalent.

(a) $a \leq b$ implies $a + x \leq b + x$ for all x (i.e., (a) in Definition 2.1).

(b) $a \leq b$ iff $0 \leq b - a$.

Let $A, B \subset R$. Define $-A = \{-x \mid x \in A\}$, $A + B = \{x + y \mid x \in A, y \in B\}$, and $A \cdot B = \{xy \mid x \in A, y \in B\}$; in particular, for $A = \{a\}$, let us denote $A \cdot B$ by aB. Also, define $A \backslash B = \{x \mid x \in A, x \notin B\}$.

Definition 2.3. For a subset S of a ring R, let us call S a *non-negative semi-cone* (resp. *non-negative cone* ([2])) of R if S satisfies (a), (b), and (c) (resp. (a), (b), (c), and (d)) below:

(a) $S \cap (-S) = \{0\}$.

(b) $S + S \subset S$.

(c) $S \cdot S \subset S$.

(d) $R = S \cup (-S)$.

For a non-negative semi-cone S of a ring R, we define $a \leq_S b$ by $b - a \in S$.

While, in a partially ordered ring $(R, \leq)$, the symbol S means its non-negative part $\{x \in R \mid 0 \leq x\}$, unless otherwise stated.

The part S in a partially ordered ring $(R, \leq)$ is a non-negative semi-cone of the ring $(R, \leq)$ with $\leq = \leq_S$. Conversely, for a non-negative semi-cone S of a ring R, $(R, \leq_S)$ is a partially ordered ring with $S = \{x \in R \mid 0 \leq_S x\}$. These are also valid for the relationship between "ordered rings" and "non-negative cones".

REMARK 2.4. (1) Every non-negative cone of a ring contains the identity element 1. But, this need not be valid for every non-negative semi-cone.

Indeed, for a non-negative semi-cone S of a ring R, S' defined by (a) or (b) below is a non-negative semi-cone of R satisfying $1 \notin S' \subset S$.

(a) (i) Define $S' = \{0\}$ trivially; or (ii) for an ideal I of R, put $S' = S \cap I$.

(b) If $S \ni 1$, then (i) for $m \in \mathbf{Z}^* \setminus \{1\}$, put $S' = \mathbf{Z}^* m \cdot 1$ $(= \{nm \cdot 1 \mid n \in \mathbf{Z}^*\})$; or (ii) for $a \in S \setminus \{0\}$, put $S' = \{x \in S \mid a + 1 \leq_S x\} \cup \{0\}$.

We note that every partially ordered ring $(R, \leq)$ satisfying $S \ni 1$ need not be an ordered ring (considering any field R of characteristic 0 having $\leq \, = \, \leq_{Z^* \cdot 1}$).

(2) For any non-negative semi-cone S of a ring R, $S \not\ni -1$ (indeed, suppose $S \ni -1$. Then $(-1)(-1) = 1 \in S$. Then $1 = 0$, a contradiction).

Let A be a subset of a partially ordered ring $(R, \leq)$. The symbol $\leq^*$ means the restriction order on A from $\leq$ (i.e., for $a, b \in A$, if $a \leq^* b$, then $a \leq b$). We shall say that the set A is a *partially ordered subring* of R if (i) A is a subring of R, and (ii) $(A, \leq^*)$ is a partially ordered ring. Here, we can omit (ii); see Proposition 2.5(1) below.

For a subset A of a partially ordered ring $(R, \leq)$ with $A \ni 0$, let us also use the same terminology "*non-negative semi-cone*" (or "*non-negative cone*") of A, as in Definition 2.3. Also, let us set $S_A = S \cap A$.

PROPOSITION 2.5. *Let A be a subset of a partially ordered ring $(R, \leq)$. Then the following hold.*

(1) *If A is a subring of R, then $(A, \leq^*)$ is a partially ordered subring of R with $\leq_{S_A} \, = \, \leq^*$ (hence, S_A is a non-negative semi-cone of A).*

(2) *$(A, \leq^*)$ is an ordered subring of R iff $A = S_A + (-S_A)$ such that S_A is a non-negative cone of A with $S_A \ni 1$.*

PROOF. For (1), since A is a subring of $(R, \leq)$, obviously $(A, \leq^*)$ is a partially ordered ring with $\leq_{S_A} \, = \, \leq^*$. Thus, S_A is a non-negative semi-cone of A. For the "if" part in (2), we show that $A = S_A + (-S_A)$ is a subring of R. Clearly $1 \in A$. Let $x = a - b$, $y = a' - b' \in A$ with $a, b, a', b' \in S_A$. Then $-x \in A$. Also, since S_A is a non-negative cone of A, $x + y \in A$ and xy $(= (aa' + bb') - (ab' + a'b)) \in A$. Thus A is a subring of R. Then $\leq_{S_A} \, = \, \leq^*$ on A by (1). Thus, $(A, \leq^*)$ is an ordered ring. For the "only if" part, since $\leq_{S_A} \, = \, \leq^*$ (by (1)), S_A is a non-negative cone of A with $S_A \ni 1$. Since $(A, \leq^*)$ is an ordered subring of R, obviously $A \subset S_A + (-S_A) \subset A$, hence $A = S_A + (-S_A)$. $\square$

Remark 2.6. In view of the proof of Proposition 2.5, we have the following: Let A be a subset of a partially ordered ring $(R, \leq)$ with $S_A \ni 1$. Assume (*) S_A is a non-negative semi-cone of A (in particular, A is a subring of R or A is a set containing S). Then $S_A + (-S_A)$ is a partially ordered subring by $\leq^*$. Concerning these, we have the following (a) and (b).

(a) $S_A \cup (-S_A)$ need not be a subring of R even if A is a subring of R containing S.

(b) Without the assumption (*), $S_A + (-S_A)$ need not be a subring of R, even if A is a group under addition (or $A \backslash \{0\}$ is a group under multiplication) such that $A = S_A \cup (-S_A) = S_A + (-S_A)$.

Indeed, for (a), let $R = \mathbf{R}$, $S = \{x \in \mathbf{Q} \mid x \geq 1\} \cup \{0\}$, and let $\leq = \leq_S$ in R. Let $A = \mathbf{Q}$. Then $(R, \leq)$ is a partially ordered ring with $S_A \ni 1$, and A is a subring of R containing S. But, $S_A \cup (-S_A)$ is not a subring of R. For (b), let $R = \mathbf{Q}$ be the usual ordered field. Let $A = \mathbf{Z} + \frac{1}{2}\mathbf{Z}$. Then A is a desired one, noting $\left(\frac{1}{2}\right)^2 = \frac{1}{4} \notin A$. For the parenthetic case, let $A = \{0, \pm 1\}$, then A is a desired one.

Corollary 2.7. *Let $(R, \leq)$ be a partially ordered ring with $S \ni 1$. Let $A = S + (-S)$, and $B = S \cup (-S)$. Then the following hold.*

(1) *$(A, \leq^*)$ is a partially ordered subring of R.*

(2) *$(B, \leq^*)$ is an ordered subring of R iff $B = A$.*

Proof. Since $S \subset A$, (1) holds by Remark 2.6. Since $S_B = S$, (2) holds by means of Proposition 2.5(2). $\square$

We assume that the ring $\mathbf{Z}$ has the usual order unless otherwise stated, but for non-negative semi-cones of $\mathbf{Z}$, consider them under $\mathbf{Z}$ being the ring. While, the set $\mathbf{Z}^*(= \mathbf{N} \cup \{0\})$ has the usual order when we consider its order.

Lemma 2.8. *For a non-negative semi-cone S of $\mathbf{Z}$, the following hold.*

(1) *For $a \in S$, $a\mathbf{Z}^* \subset S$, thus $S \cdot \mathbf{Z}^* = S$.*

(2) *$S \subset \mathbf{Z}^*$.*

Proof. (1) holds by (b) in Definition 2.3. For (2), let $x \in S \backslash \{0\}$. Suppose $x < 0$. Then $-x \in \mathbf{N}$. Thus, $(-x)x = -(xx) \in S \cap (-S) = \{0\}$, which yields $x = 0$, a contradiction. Thus, $S \subset \mathbf{Z}^*$. $\square$

PROPOSITION 2.9. *For a subset* S *of* $\mathbf{Z}$, S *is a non-negative semi-cone of* $\mathbf{Z}$ *iff* $S = a_1\mathbf{Z}^* + \cdots + a_n\mathbf{Z}^*$ *for some* $a_1, \ldots, a_n \in S$ $(\subset \mathbf{Z}^*)$.

PROOF. The "if" part is routinely shown. For the "only if" part, we consider S in $\mathbf{Z}^*$ by Lemma 2.8(2). Put $a_0 = 0$ and $S_0 = a_0\mathbf{Z}^*$. If $S \backslash S_0 = \emptyset$, then $S = S_0$. Otherwise, let $a_1 = \min(S \backslash S_0)$, and $S_1 = S_0 + a_1\mathbf{Z}^*$. Then $a_0 < a_1 \in S_1$. When this process proceeds to a_i, S_i $(i \geq 1)$, we consider the set $S \backslash S_i$. If $S \backslash S_i = \emptyset$, then $S = S_i$ (with Lemma 2.8(1)). Otherwise, let $a_{i+1} = \min S \backslash S_i$, and $S_{i+1} = S_i + a_{i+1}\mathbf{Z}^*$. Then $a_1 < \cdots < a_i < a_{i+1}$. Further, $a_k \not\equiv a_l \pmod{a_1}$ for any k, l $(1 \leq k < l \leq i+1)$. Indeed, if $a_k \equiv a_l \pmod{a_1}$ for $k < l$, then there exists $t \in \mathbf{N}$ with $a_l = a_1 t + a_k \in S_{l-1}$, a contradiction. We will show this process ends after a finite steps. Suppose that, for $m = a_1$, $S \backslash S_m \neq \emptyset$. Let $a_{m+1} = \min S \backslash S_m$, and $S_{m+1} = S_m + a_{m+1}\mathbf{Z}^*$. Then $a_1 < \cdots < a_m < a_{m+1}$, and $a_i \not\equiv a_j \pmod{m}$ for any i, j $(1 \leq i < j \leq m+1)$. But, for $m+1$ integers $a_1, \ldots, a_{m+1}$, $a_i \equiv a_j \pmod{m}$ for some i, j $(i < j)$, a contradiction. Therefore, there exists $n \in \mathbf{Z}^*$ with $S \backslash S_n = \emptyset$, thus $S = a_1\mathbf{Z}^* + \cdots + a_n\mathbf{Z}^*$. □

REMARK 2.10. (1) Let S be a non-negative semi-cone of $\mathbf{Z}$ with $S \neq \{0\}$. Let $m = \min(S \backslash \{0\})$. Then there exist $a_1, \ldots, a_n$ $(n \leq m)$ in S such that (a) $S = a_1\mathbf{Z}^* + \cdots + a_n\mathbf{Z}^*$, (b) $0 < a_1 < a_2 < \cdots < a_n$, and (c) $a_i \in S \backslash (a_0\mathbf{Z}^* + \cdots + a_{i-1}\mathbf{Z}^*)$ $(a_0 = 0; 1 \leq i \leq n)$. Moreover, such integers n, and $a_1, \ldots, a_n$ are determined uniquely under (a), (b), and (c).

Indeed, there exist such $a_1, \ldots, a_n$ in S by the proof of Proposition 2.9. By (a) and (b), we can replace (c) by (c$'$) $a_i = \min S \backslash (a_0\mathbf{Z}^* + \cdots + a_{i-1}\mathbf{Z}^*)$ $(a_0 = 0;$ $1 \leq i \leq n)$. Thus, for $0 < b_1 < \cdots < b_k$ in S, if $S = b_1\mathbf{Z}^* + \cdots + b_k\mathbf{Z}^*$, and $b_i \in S \backslash (b_0\mathbf{Z}^* + b_1\mathbf{Z}^* + \cdots + b_{i-1}\mathbf{Z}^*)$ $(b_0 = 0)$, then $a_i = b_i$ $(1 \leq i \leq k = n)$.

(2) Let $S = c_1\mathbf{Z}^* + \cdots + c_n\mathbf{Z}^*$ $(0 < c_1 < \cdots < c_n; n \geq 2)$ be a non-negative semi-cone of $\mathbf{Z}$. Then $S = c\mathbf{Z}^*$ for some $c \in \mathbf{N}$ iff $c_1 | c_i$ (i.e., c_1 is a divisor of c_i) holds for all $i \geq 2$ (indeed, for the "if" part, put $c = c_1$. For the "only if" part, $\min(S \backslash \{0\}) = c_1 = c$, and $c | c_i$ holds for all $i \geq 1$).

COROLLARY 2.11. *Let* T *be a non-negative semi-cone of* $\mathbf{Z}$, *and let* $n \in \mathbf{N}$. *Then* $T \subset n\mathbf{Z}$ *iff* $T = n(a_1\mathbf{Z}^* + \cdots + a_m\mathbf{Z}^*)$ *for some* $a_1, \ldots, a_m \in \mathbf{Z}^*$.

PROOF. The "if" part is clear. For the "only if" part, $T = (k_1\mathbf{Z}^* + \cdots + k_m\mathbf{Z}^*)$ for some $k_1, \ldots, k_m \in \mathbf{Z}^*$ by Proposition 2.9. Since $T \subset n\mathbf{Z}$, each $k_i\mathbf{Z}^* \subset n\mathbf{Z}$, so $n | k_i$. Put $k_i = na_i$, then $T = n(a_1\mathbf{Z}^* + \cdots + a_m\mathbf{Z}^*)$ $(a_i \in \mathbf{Z}^*)$. □

Definition 2.12. (1) We recall that a map $f : R \to R'$ is a *ring homomorphism* (abbreviated *homomorphism*) if f satisfies the following conditions: (i) $f(x+y) = f(x) + f(y)$; (ii) $f(xy) = f(x)f(y)$, and (iii) $f(1) = 1'$. Also, a homomorphism is a *monomorphism*; *epimorphism*; *isomorphism* if it is injective; surjective; bijective, respectively. We shall say that a map $f : R \to R'$ is an *additive homomorphism* if f satisfies the above condition (i) (hence, $f(0) = 0$, but $f(1) = 1'$ need not hold). For an additive homomorphism, *additive monomorphism*, etc., are similarly defined.

(2) For partially ordered rings $(R, \leq)$ and $(R', \leq')$, a map $f : (R, \leq) \to (R', \leq')$ is *order-preserving* if f satisfies: if $x \leq y$, then $f(x) \leq' f(y)$. For an additive homomorphism f, f is order-preserving iff $f(S) \subset S'$.

Remark 2.13. (1) Let R be a ring. As is well-known, there exists uniquely a homomorphism $h : \mathbf{Z} \to R$ (actually, given by $h(n) = n \cdot 1$). The homomorphism h is a monomorphism if there exists an additive monomorphism $f : \mathbf{Z} \to R$ (indeed, for $n \in \mathbf{Z} \backslash \{0\}$, $f(n) = nf(1) = h(n)f(1) \neq 0$, so $h(n) \neq 0$).

(2) If there exists an order-preserving monomorphism $f : \mathbf{Z} \to (R, \leq)$, then the homomorphism $h : \mathbf{Z} \to (R, \leq)$ is an order-preserving monomorphism since $f(n) = h(n)$ for $n \in \mathbf{Z}$. However, the map h need not be order-preserving even if there exists an order-preserving additive monomorphism $f : \mathbf{Z} \to (R, \leq)$. Indeed, for a partially ordered ring $(\mathbf{Z}, \leq_S)$, $S = 2\mathbf{Z}^*$, define $f : \mathbf{Z} \to (\mathbf{Z}, \leq_S)$ by $f(n) = 2n$. Then f is an order-preserving additive monomorphism, but the map h is never order-preserving since $0 \leq_S 1$ doesn't hold.

We shall say that the ring (resp. the ordered ring) $\mathbf{Z}$ *is embeddable in* R (resp. $(R, \leq)$) if the homomorphism $h : \mathbf{Z} \to R$ is a monomorphism (resp. order-preserving monomorphism), and that the ordered additive group $\mathbf{Z}$ *is embeddable in* $(R, \leq)$ if there exists an order-preserving additive monomorphism of $\mathbf{Z}$ to R.

We shall say that a partially ordered ring $(R, \leq)$ is *trivial* if $S = \{0\}$. For $S \neq \{0\}$, obviously S is infinite. Every ordered ring is a non-trivial partially ordered ring, and every non-trivial partially ordered ring is infinite.

Proposition 2.14. *For a partially ordered ring* $(R, \leq)$, *the following hold.*
(1) $(R, \leq)$ *is trivial iff any distinct points in* R *are incomparable.*
(2) $(R, \leq)$ *is non-trivial iff the ordered additive group* $\mathbf{Z}$ *is embeddable in* $(R, \leq)$.

Proof. (1) is obvious. For (2), assume $(R, \leq)$ is non-trivial, and take $a \in S$ with $a \neq 0$. Then a map $f : \mathbf{Z} \to R$ defined by $f(m) = ma$ is an order-preserving additive monomorphism. Conversely, for an order-preserving additive monomorphism $g : \mathbf{Z} \to R$, $0 = g(0) < g(1)$, which implies $S \neq \{0\}$. □

The following holds in view of the proof of Proposition 2.14(2).

COROLLARY 2.15. *For a partially ordered ring $(R, \leq)$, $S \ni 1$ iff the ordered ring $\mathbf{Z}$ is embeddable in $(R, \leq)$. In particular, the ordered ring $\mathbf{Z}$ is embeddable in any ordered ring.*

REMARK 2.16. The ordered ring $\mathbf{Z}$ need not be embeddable in every non-trivial partially ordered ring by Corollary 2.15 with Remark 2.13(2). While, the ring $\mathbf{Z}$ need not be embeddable in every infinite field K (indeed, let $R = \mathbf{Z}/m\mathbf{Z}$ for a prime number $m > 1$, then R is a field. Let $K = R(x)$ be the infinite field of all rational functions over R in one variable x. Since $m1 = 0$ in K, the ring $\mathbf{Z}$ is not embeddable in the field K).

REMARK 2.17. We can make any ring R to be a trivial partially ordered ring (putting $S = \{0\}$). Also, for a ring R, we can make R to be a non-trivial partially ordered ring $(R, \leq)$ iff the ring $\mathbf{Z}$ is embeddable in R. Indeed, for the "if" part, let $S' = \{n1 \mid n \in Z^*\}$, and $\leq \, = \, \leq_{S'}$. Then $(R, \leq)$ is a non-trivial partially ordered ring. For the "only if" part, take $a \in S \backslash \{0\}$. Then for $n \in \mathbf{N}$, $na \neq 0$, so $n1 \neq 0$. Thus, the ring $\mathbf{Z}$ is embeddable in R (putting $a = 1$ in the proof of Proposition 2.14(2)).

3. Residue Class Rings and Convex Ideals

For a partially ordered ring $(R, \leq)$, let R/I be the residue class ring having a canonical ordering relation induced by $\leq$. As is well-known, the convexity of I in $(R, \leq)$ gives a characterization for R/I to be a partially ordered ring. For an ideal I in $(R, \leq)$, we will give characterizations for I to be convex, or R/I to be a partially ordered ring.

DEFINITION 3.1 ([1]). Let $(R, \leq)$ be a partially ordered ring. For an ideal I of $(R, \leq)$, let R/I be the residue class ring.

(1) I is *convex* in $(R, \leq)$ if whenever $0 \leq x \leq y$ and $y \in I$, then $x \in I$.

(2) We induce a canonical ordering relation in R/I as follows: For $a \in R$, define $[a](= a + I) \geq 0$ in R/I if there exists $x \geq 0$ in R with $[a] = [x]$ (we use the same symbol $\leq$ in R/I without confusion).

$(R/I, \leq)$ need not be a partially ordered ring; see Example 3.10(2) later. We recall that $(R/I, \leq)$ is a partially ordered ring iff I is convex ([1], etc.).

(3) Let $\varphi : (R, \leq) \to (R/I, \leq)$ be the natural homomorphism defined by $\varphi(a) = [a]$. Then $\varphi(S) = \{[a] \in R/I \mid [a] \geq 0\}$. Also, φ is order-preserving (that is, if $a \leq b$, then $[a] \leq [b]$, here we define $[a] \leq [b]$ by $[b] - [a](= [b - a]) \geq 0$).

In what follows, we assume that any residue class ring R/I of a partially ordered ring R has the ordering relation in Definition 3.1(2), unless otherwise stated. Also, the map φ means the natural homomorphism of R to R/I in Definition 3.1(3).

Remark 3.2. For a partially ordered ring $(R, \leq)$, let R/I be a partially ordered ring. Then the following hold.

(1) There exists no $x \in I$ with $x \geq 1$ by Remark 2.4(2).

(2) If $S \backslash I \neq \emptyset$, then R/I is infinite by Proposition 2.14(2).

In (1) or (2), the assumption that R/I is a partially ordered ring or $S \backslash I \neq \emptyset$ is essential; see Example 3.10 later.

Remark 3.3. For a partially ordered ring $(R, \leq)$, the following hold.

(1) $S \subset I$ iff R/I is a trivial partially ordered ring (equivalently, for $a > 0$, $[a] = 0$). In particular, put $S' = S \cap I$, then for a partially ordered ring $(R, \leq_{S'})$, R/I is a trivial partially ordered ring (by $\leq_{S'}$).

(2) $S \cap I = \{0\}$ iff R/I is a partially ordered ring such that for $a > 0$, $[a] > 0$ (equivalently, for $a < b$, $\varphi(a) < \varphi(b)$).

(3) $I \subset S \cup (-S)$ iff I has a total order $\leq^*$ (equivalently, for $[a] = 0$ in R/I, $a \geq 0$ or $a \leq 0$). In particular, $I = S \cup (-S)$ iff R/I is a partially ordered ring such that $[a] = 0$ iff $a \geq 0$ or $a \leq 0$.

Indeed, (1) is routinely shown, and (3) are obvious. For (2), the "if" part is clear, so we see the "only if" part. Obviously, for $a > 0$, $[a] > 0$. To see R/I is a partially ordered ring, it suffices to show that $[a] \geq 0$ and $[a] \leq 0$ implies $[a] = 0$. There exist $x \in S$ with $[a] = [x]$. Suppose $x \neq 0$. Then $x > 0$, so $[a] = [x] > 0$, a contradiction. Then $x = 0$, hence $[a] = 0$.

Proposition 3.4. *Let S be a non-negative semi-cone of $\mathbf{Z}$. Let I be a non-zero ideal of $\mathbf{Z}$, so we can put $I = n\mathbf{Z}$ $(n > 1)$. Then the following are equivalent.*

(a) *I is convex in $(\mathbf{Z}, \leq_S)$.*

(b) *$S \subset I$.*

(c) *$n | x$ holds for all $x \in S$.*

PROOF. (b) $\Rightarrow$ (a) is clear. For (a) $\Rightarrow$ (b), $x \in S$. Then $0 \leq_S x \leq_S nx$ and $nx \in I$, thus $x \in I$. Hence, $S \subset I$. (b) $\Leftrightarrow$ (c) is routinely shown. $\square$

EXAMPLE 3.5. For $m \in \mathbf{Z}^*$, $m\mathbf{Z}^*$ and $S(m) = \{n \in \mathbf{Z} \mid n \geq m\} \cup \{0\}$ are non-negative semi-cones of $\mathbf{Z}$. Let $I = n\mathbf{Z}$ $(n > 1)$ be an ideal of $\mathbf{Z}$. Then Proposition 3.4 implies that I is convex in $(\mathbf{Z}, \leq_{m\mathbf{Z}^*})$ iff $n|m$ holds, but I is not convex in $(\mathbf{Z}, \leq_{S(m)})$. In particular, any non-zero ideal of $\mathbf{Z}$ is not convex in $\mathbf{Z}$.

For an ordered integral domain R, let $R[x]$ be the polynomial ring over R in one variable x. For $f(x) \in R[x]$, let $I = (f(x))$ be the ideal of $R[x]$ generated by $f(x)$. We recall two orders on $R[x]$. For a non-zero polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ in $R[x]$, define $0 <_1 f$ and $0 <_2 f$ as follows.

$0 <_1 f(x)$ if the leading coefficient a_n is positive in R.

$0 <_2 f(x)$ if the first nonzero coefficient a_k is positive in R.

Then the rings $(R[x], \leq_1)$ and $(R[x], \leq_2)$ are ordered integral domains (as is well-know). But, for R being an ordered ring, this need not valid by the following Example 3.6.

EXAMPLE 3.6. An ordered ring R such that neither $(R[x], \leq_1)$ nor $(R[x], \leq_2)$ is a partially ordered ring.

Indeed, let $(R, \leq)$ be an ordered ring having elements a, b with $ab = 0$ $(a \neq 0, b \neq 0)$ (see [2, Example 1], for example). Since $ab = (-a)b = a(-b) = (-a)(-b) = 0$, we can assume $a > 0$ and $b > 0$. Put $u = ax - 1$, $v = b$, $w = -x + a \in R[x]$. Then $0 <_1 u$, $0 <_1 v$, but $uv <_1 0$. Also, $0 <_2 v$, $0 <_2 w$, but $vw <_2 0$. Hence, neither $(R[x], \leq_1)$ nor $(R[x], \leq_2)$ is a partially ordered ring.

PROPOSITION 3.7. *Let R be an ordered integral domain. For a non-zero ideal $I = (h(x))$ of $R[x]$, the following hold.*

(1) *I is not convex in $(R[x], \leq_1)$.*

(2) *I is convex in $(R[x], \leq_2)$ iff $h(x)$ is a monomial with* $\deg h(x) > 0$, *and its coefficient is invertible in R.*

PROOF. Since $I = (h(x)) = (-h(x))$, we can assume that the leading coefficient, say a, of $h(x)$ is positive in R.

For (1), $0 <_1 1 <_1 xh(x) \in I$ (possibly, $h(x) = a \in R$), but $1 \notin I$. Hence, I is not convex in $(R[x], \leq_1)$.

For (2), assume that I is convex in $(R[x], \leq_2)$. Let $n = \deg h(x)$. To see $n > 0$, suppose $n = 0$. Then $h(x) = a \in R$, and $0 <_2 x <_2 a \in I$, but $x \notin I$ (indeed, suppose $x \in I$, then $ag(x) = x$ for some $g(x) \in R[x]$, so $ab = 1$ for some $b \in R$. Hence $1 \in I$, a contradiction). Thus, I is not convex in $(R[x], \leq_2)$, a contradiction. Hence, $n > 0$. To see $h(x)$ is a monomial, suppose not. Put $h(x) = ax^n + bx^{n-1} + \cdots + cx^k$ $(a, b, \dots, c \in R; c \neq 0; n > k)$. If $0 < c$, then $0 <_2 ax^n <_2 h(x) \in I$. Since I is convex in $(R[x], \leq_2)$, $ax^n \in I$, and so $ax^n = h(x)h_1(x)$ for some $h_1(x) \in R[x]$. Then $\deg h_1(x) = 0$. Thus $h_1(x) = 1$, so $h(x) = ax^n$, a contradiction. If $c < 0$, then we have also a contradiction, replacing "$h(x)$" by "$-h(x)$". Hence, $h(x)$ is a monomial; that is, $h(x) = ax^n$ with $a > 0$. We show the coefficient a is invertible. If $a = 1$, then this is obvious. If $1 < a$, then $0 <_2 (a-1)x^{n+1} <_2 ax^n \in I$, and so $(a-1)x^{n+1} \in I$ by the convexity of I. Hence, there exists $g(x) \in R[x]$ with $(a-1)x^{n+1} = ax^n g(x)$. Thus $a - 1 = ad$ for some $d \in R$, and so $a(1-d) = 1$. Hence a is invertible in R. If $0 < a < 1$, then $0 <_2 (1-a)x^{n+1} <_2 ax^n \in I$, and so $(1-a)x^{n+1} \in I$. Thus, similarly a is invertible in R. Hence, the coefficient a is invertible in R. Conversely, assume that $h(x)$ is a monomial with $\deg h(x) > 0$, and its coefficient a is invertible. Since $I = (a^{-1}h(x))$ with $\deg h(x) > 0$, we can assume that $a = 1$. Put $h(x) = x^n$ $(n > 0)$. Let $0 <_2 f(x) <_2 g(x)$ and $g(x) \in I$. Since $g(x) \in I$, $g(x) = h(x)g_1(x)$ for some $g_1(x) \in R[x]$. Let $f(x) = x^n f_1(x) + r(x)$ $(f_1(x), r(x) \in R[x])$ with $\deg r(x) < n$. Suppose $r(x) \neq 0$. Since $0 <_2 f(x)$, $0 <_2 r(x)$. But $0 <_2 g(x) - f(x) = x^n(g_1(x) - f_1(x)) - r(x) <_2 0$, a contradiction. Hence $r(x) = 0$, and so $f(x) = x^n f_1(x) \in I$. Thus I is convex in $(R[x], \leq_2)$. $\square$

Remark 3.8. (1) In Proposition 3.7, put $R = \mathbf{Z}$. Then the non-zero ideal I $(= (h(x)))$ is not convex in $(\mathbf{Z}[x], \leq_1)$. While, I is convex in $(\mathbf{Z}[x], \leq_2)$ iff $h(x) = \pm x^n$ $(n > 0)$. Thus, for any integers $m > 1$ and $n \geq 0$, $I' = (mx^n)$ is not convex in $(\mathbf{Z}[x], \leq_2)$.

(2) Any ideal I of $\mathbf{Z}[x]$ with $I \cap \mathbf{Z} \neq (0)$ is not convex in $(\mathbf{Z}[x], \leq)$, where $\leq = \leq_1$ or $\leq_2$ (indeed, $I \cap \mathbf{Z} = n\mathbf{Z}$ for some integer $n > 1$. Thus $0 < 1 < n$ and $n \in I$, but $1 \notin I$. Then I is not convex in $(\mathbf{Z}[x], \leq)$). In particular, for a prime number p, let $I = (p, x)$ be an ideal of $\mathbf{Z}[x]$ generated by p, x. Then I is a maximal ideal of $\mathbf{Z}[x]$ (note that $I' \cap \mathbf{Z} = p\mathbf{Z}$ for any ideal $I' \supset I$ in $\mathbf{Z}[x]$), but I is not convex in $(\mathbf{Z}[x], \leq)$.

The symbol K means an ordered field. As is well-known, the ring $K[x]$ is a principal ideal domain. Thus Proposition 3.7 implies the following ([2]).

COROLLARY 3.9. *Let I be a non-zero ideal of $K[x]$ (with $I \neq K[x]$). Then the following hold.*

(1) *I is not convex in $(K[x], \leq_1)$.*

(2) *I is convex in $(K[x], \leq_2)$ iff I is generated by a monomial.*

EXAMPLE 3.10. (1) A partially ordered ring $(R, \leq)$ and an ideal $I \supset S$ such that R/I is a finite partially ordered ring which doesn't satisfy $[0] < [1]$.

(2) A partially ordered ring $(R, \leq)$ and an ideal I containing $x > 1$ such that R/I is a finite field which is not a partially ordered ring.

(3) In (1) and (2), we can take R/I to be infinite.

Indeed, for (1), let $R = \mathbf{Z}$, $I = 2\mathbf{Z}$, and $S = \mathbf{Z}^* \cap I$. Then $(R, \leq_S)$ and I satisfy conditions in (1) by Remark 3.3(1). For (2), let $R = \mathbf{Z}$, $I = 3\mathbf{Z}$ and $S = 2\mathbf{Z}^*$. Then $(R, \leq_S)$ and I satisfy conditions in (2) by Remark 3.2(1). For (3), in (1), let $R = K[x]$ and $I = (x)$. Then R/I is isomorphic to K, so it is an infinite field. Let S be the positive part of $(R, \leq_1)$ or $(R, \leq_2)$, and $S' = S \cap I$. Then $(R, \leq_{S'})$ and I are desired ones by Remark 3.3(1). In (2), let $R = K[x]$, $I = (x)$. Then $(R, \leq_1)$ and I are desired ones by Corollary 3.9(1) ([2, Example 4(1)]).

The following basic result is routinely shown, referring to the proof of [1, Theorem 5.2].

PROPOSITION 3.11. *Let $(R, \leq)$ be a partially ordered ring, and I be an ideal of R. Then the following are equivalent.*

(a) *I is convex.*

(b) *$(S \backslash I) + (S \backslash I) \subset (S \backslash I)$.*

(c) *If $[a] \geq 0$ and $-[a] \geq 0$, then $[a] = 0$*

(d) *$\varphi(S) \cap \varphi(-S) = \{0\}$.*

(e) *R/I is a partially ordered ring.*

In view of Proposition 3.11, let us give the following review of [2, Lemma 2 and Corollary 1], in terms of the sets S and I in R.

PROPOSITION 3.12. *For a partially ordered ring $(R, \leq)$, the following hold.*

(1) *R/I is a partially ordered ring iff $(S \backslash I) + (S \backslash I) \subset (S \backslash I)$.*

(2) *R/I is an ordered ring iff $R = (S \cup -S) + I$, and $(S \backslash I) + (S \backslash I) \subset (S \backslash I)$.*

(3) *R/I is an ordered integral domain iff $R = (S \cup -S) + I$, $(S \backslash I) + (S \backslash I) \subset (S \backslash I)$, and $(S \backslash I) \cdot (S \backslash I) \subset (S \backslash I)$.*

The following holds by Propositions 2.14 and 3.11 (with Remark 3.3(1)).

Theorem 3.13. *Let* $(R, \leq)$ *be a partially ordered ring. For an ideal* I *of* R, R/I *is a partially ordered ring iff the following case* (a) *or* (b) *holds.*

(a) I *is convex, and the ordered additive subgroup* $\mathbf{Z}$ *is embeddable in* R/I.

(b) *Any distinct points in* R/I *are incomparable* (*equivalently*, $S \subset I$).

In the following corollary, (1) holds by Theorem 3.13, and (2) is directly shown by (1), using Proposition 3.11.

Corollary 3.14. *Let* $(R, \leq)$ *be a partially ordered ring.*

(1) *The following are equivalent.*

(a) R/I *is a partially ordered ring in which the ordered additive group* $\mathbf{Z}$ *can't be embeddable.*

(b) *Any distinct points in* R/I *are incomparable* (*equivalently*, $S \subset I$).

(2) *The following are equivalent.*

(a) R/I *is a partially ordered ring in which the ordered additive group* $\mathbf{Z}$ *is embeddable.*

(b) $(S \backslash I) + (S \backslash I) \subset (S \backslash I)$, *and* $S \backslash I \neq \emptyset$.

The following corollary holds by Corollaries 2.11 and 3.14(1).

Corollary 3.15. *Let* $(\mathbf{Z}, \leq)$ *be a partially ordered ring, and* $I = n\mathbf{Z}$ $(1 < n \in \mathbf{N})$. *Then the following are equivalent.*

(a) $\mathbf{Z}/I$ *is a partially ordered ring.*

(b) *Any distinct points in* $\mathbf{Z}/I$ *are incomparable.*

(c) $S = n(a_1\mathbf{Z}^* + \cdots + a_m\mathbf{Z}^*)$ *for some* $a_1, \ldots, a_m \in \mathbf{Z}^*$.

4. Order-preserving Isomorphisms

We consider order-preserving isomorphisms between residue class rings which are partially ordered rings. In this section, the symbols $(R, \leq)$ and $(R', \leq')$ mean partially ordered rings, unless otherwise stated.

Definition 4.1. For an isomorphism $\sigma : (R, \leq) \to (R', \leq')$, let us say that σ is *isomorphic as partially ordered rings* if σ and σ^{-1} are order-preserving (equivalently, $\sigma(S) = S'$). If there exists such an isomorphism σ, we shall say that $(R, \leq)$ is *isomorphic to* $(R', \leq')$ *as partially ordered rings*. When $(R, \leq)$ and

$(R', \leq')$ are ordered rings (or ordered fields, etc.), we say that $(R, \leq)$ is *isomorphic to* $(R', \leq')$ *as ordered rings* (or *ordered fields*, etc.).

REMARK 4.2. For an isomorphism $\sigma : (R, \leq) \to (R', \leq')$, let us consider (i) σ is order-preserving; (ii) σ^{-1} is order-preserving. Then (i) need not imply (ii), and vice versa. Indeed, for the first, consider the identity map $\sigma : (\mathbf{Z}, \leq_S) \to \mathbf{Z}$, $S = 2\mathbf{Z}^*$, and for the latter, consider the identity map $\sigma' = \sigma^{-1}$. If $(R, \leq)$ is an ordered ring, (i) holds iff (ii) holds and $(R', \leq')$ is an ordered ring (but (ii) need not imply (i)).

DEFINITION 4.3. For a homomorphism $\sigma : (R, \leq) \to (R', \leq')$, we induce a homomorphism $\bar{\sigma}$; $R/I \to R'/I'$ by $\bar{\sigma}([a]) = [\sigma(a)]$ under $\sigma(I) \subset I'$.

We note that $\bar{\sigma}$ is well-defined iff $\sigma(I) \subset I'$. Thus, we assume $\sigma(I) \subset I'$ for the induced homomorphism $\bar{\sigma}$.

The following diagram is commutative (i.e., $\bar{\sigma} \circ \varphi = \varphi' \circ \sigma$) for the induced homomorphism $\bar{\sigma}$ and the natural homomorphism φ, etc.

$$(*) \qquad \begin{array}{ccc} R & \xrightarrow{\sigma} & R' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ R/I & \xrightarrow{\bar{\sigma}} & R'/I' \end{array}$$

Let us observe the map $\bar{\sigma}$ induced by σ, and give a characterization for the map $\bar{\sigma}$ to be an isomorphism as partially ordered rings.

The following lemma is routinely shown.

LEMMA 4.4. *Let $\sigma : R \to R'$ be a homomorphism, and $I; I'$ be an ideal of $R; R'$ respectively. Then the following hold.*

(1) *$\bar{\sigma}$ is an epimorphism iff $\sigma(R) + I' = R'$. In particular, if σ is an epimorphism, then so is $\bar{\sigma}$.*

(2) *$\bar{\sigma}$ is a monomorphism iff $\sigma^{-1}(I') \subset I$ (equivalently, $\sigma^{-1}(I') = I$).*

(3) *$\bar{\sigma}$ is an isomorphism iff $\sigma(R) + I' = R'$ and $\sigma^{-1}(I') = I$.*

THEOREM 4.5. *Let $\sigma : (R, \leq) \to (R', \leq')$ be a homomorphism, and $I; I'$ be convex in $R; R'$ respectively. Then the following hold.*

(1) *$\bar{\sigma}$ is order-preserving iff $\sigma(S) \subset S' + I'$. In particular, if σ is order-preserving, then so is $\bar{\sigma}$.*

(2) $\bar{\sigma}$ *is an order-preserving isomorphism iff* $\sigma(R) + I' = R'$, $\sigma^{-1}(I') = I$ *and* $\sigma(S) \subset S' + I'$.

(3) $\bar{\sigma}$ *is an isomorphism as partially ordered rings iff* $\sigma(R) + I' = R'$, $\sigma^{-1}(I') = I$ *and* $\sigma(S) + I' = S' + I'$.

Proof. For (1), noting the commutative diagram $(*)$,

$$\begin{aligned}\bar{\sigma} \text{ is order-preserving} &\Leftrightarrow \bar{\sigma}(\varphi(S)) \subset \varphi'(S') \\ &\Leftrightarrow \varphi'(\sigma(S)) \subset \varphi'(S') \Leftrightarrow \sigma(S) \subset S' + I'.\end{aligned}$$

Hence (1) holds.

For (2), this is a consequence of (1) and Lemma 4.4(3).

For (3), noting also the commutative diagram $(*)$, for $\bar{\sigma}$ being an isomorphism, the inverse map $\bar{\sigma}^{-1} : (R'/I', \leq') \to (R/I, \leq)$ is order-preserving $\Leftrightarrow \bar{\sigma}^{-1}(\varphi'(S')) \subset \varphi(S) \Leftrightarrow \bar{\sigma}(\bar{\sigma}^{-1}(\varphi'(S'))) \subset \bar{\sigma}(\varphi(S)) \Leftrightarrow \varphi'(S') \subset \varphi'(\sigma(S)) \Leftrightarrow S' + I' \subset \sigma(S) + I'$. Hence (3) holds by means of (2). □

Remark 4.6. (1) The conditions (i) $\sigma(R) + I' = R'$, (ii) $\sigma^{-1}(I') = I$ and (iii) $\sigma(S) + I' = S' + I'$ in Theorem 4.5(3) are independent. Indeed, let $\sigma : R \to R'$ be a monomorphism, but not an epimorphism, between partially ordered rings which are trivial and let I, I' zero ideals. Then (ii) and (iii) hold, but (i) doesn't hold. Further, (i) and (ii) (resp. (i) and (iii)) need not imply (iii) (resp. (ii)) by (1) (resp. (4)) of Example 4.13 later.

(2) The "if" part of Theorem 4.5(3) holds if the ideal I or I' is convex (indeed, the proof there shows that the conditions (i), (ii) and (iii) in (1) imply that for the isomorphism $\bar{\sigma} : R/I \to R'/I'$, $\bar{\sigma}(\varphi(S)) = \varphi'(S')$. Thus, I is convex $\Leftrightarrow R/I$ is a partially ordered ring $\Leftrightarrow R'/I'$ is a partially ordered ring $\Leftrightarrow I'$ is convex. Therefore, I and I' are convex iff so is either I or I').

Remark 4.7. Every monomorphism $\sigma : (R, \leq) \to (R', \leq')$ need not be an epimorphism even if $\bar{\sigma} : R/I \to R'/I'$ is an isomorphism as partially ordered rings. Indeed, let $R = \mathbf{Z}$, $I = (0)$; $R' = \mathbf{Z}[x]$, $I' = (x)$. Consider the usual order $\leq$ on R, and the order $\leq' = \leq_2$ on R' given in Proposition 3.7(2). Then $I; I'$ is convex in $R; R'$, respectively. Let $\sigma : \mathbf{Z} \to \mathbf{Z}[x]$ be the injection (defined by $\sigma(a) = a$). Then $\sigma(\mathbf{Z}) + I' = \mathbf{Z}[x]$, $\sigma^{-1}(I') = I$, and $\sigma(S) + I' = S' + I'$. Thus, $\bar{\sigma} : R/I \to R'/I'$ is an isomorphism as partially ordered rings by Theorem 4.5(3), but σ is not an epimorphism.

The following corollary holds by Theorem 4.5.

Corollary 4.8. *Let $\sigma : (R, \leq) \to (R', \leq')$ be an epimorphism, and $I; I'$ be convex in $R; R'$ respectively. Then $\bar{\sigma} : R/I \to R'/I'$ is an isomorphism as partially ordered rings iff $I = \sigma^{-1}(I')$ and $\sigma(S) + I' = S' + I'$.*

Corollary 4.9. (1) *Let $\sigma : (R, \leq) \to (R', \leq')$ be an epimorphism with $\sigma(S) = S'$. Let $I = \sigma^{-1}(I')$, and assume that I is convex in R, or so is I' in R'. Then R/I is isomorphic to R'/I' as partially ordered rings.*

(2) *Let $\sigma : (R, \leq) \to (R', \leq')$ be a monomorphism with $\sigma(S) = S' \cap \sigma(R)$. Let $I' = \sigma(I)$, and assume that I is convex in R, or so is I' in $\sigma(R)$. Then R/I is isomorphic to $\sigma(R)/I'$ as partially ordered rings.*

Proof. For (1), $I = \sigma^{-1}(I')$. While, $\sigma(S) + I' = S' + I'$ by $\sigma(S) = S'$. Thus, (1) holds by Theorem 4.5(3) with Remark 4.6(2). For (2), $\sigma(R)$ is a partially ordered ring by Proposition 2.5(1). Thus (2) holds by replacing R' with $\sigma(R)$ in (1). □

Corollary 4.10. (1) *Let $\sigma : (R, \leq) \to (R', \leq')$ be an order-preserving epimorphism. Let $I = \sigma^{-1}(I')$, and assume that I is convex in R, or so is I' in R'. If $(R, \leq)$ is an ordered ring, then R/I is isomorphic to R'/I' as ordered rings.*

(2) *Let $\sigma : (R, \leq) \to (R', \leq')$ be an order-preserving monomorphism. Let $I' = \sigma(I)$, and assume that I is convex in R, or so is I' in $\sigma(R)$. If $(R, \leq)$ is an ordered ring, then R/I is isomorphic to $\sigma(R)/I'$ as ordered rings.*

Proof. (1) is similarly shown as in (2), so we will show (2) holds. Since $(R, \leq)$ is an ordered ring and σ is order-preserving, then $\sigma(R)$ is an ordered ring, and $\sigma(S) = S' \cap \sigma(R)$ holds. Then, by Corollary 4.9(2), R/I is isomorphic to $\sigma(R)/I'$ as partially ordered rings. But, $(R, \leq)$ and $(\sigma(R), \leq')$ are ordered rings such that I is convex in R and so is I' in $\sigma(R)$ (in view of Remark 4.6(2)). Then, R/I and $\sigma(R)/I'$ are ordered rings by [2, Theorem 1] (cf. Proposition 3.12(2)). Hence the corollary holds. □

For a (completely regular) space X, let $C(X)$ be the set of all continuous maps from X to the usual space $\mathbf{R}$ of real numbers. Then $C(X)$ is a partially ordered ring (indeed, for $f, g \in C(X)$, define $(f + g)(x) = f(x) + g(x)$, $fg(x) = f(x)g(x)$; and for $r \in \mathbf{R}$, $\mathbf{r} \in C(X)$ is the constant map $\mathbf{r}(x) \equiv r$. Define a partial order $\leq$ on $C(X)$ by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$).

Lemma 4.11. (1) *For a prime ideal I of $C(X)$, $C(X)/I$ is an ordered integral domain with I convex in $C(X)$.*

(2) *Every homomorphism $H : C(X) \to C(Y)$ is order-preserving.*

Proof. (1) holds in view of [1, Theorem 5.5], here $C(X)/I$ is an integral domain iff I is a prime ideal of $C(X)$. For (2), let $f \geq \mathbf{0}$, and take $g \in C(X)$ with $f = g^2$. Thus, $H(f) = H(g^2) = (H(g))^2 \geq \mathbf{0}$. Hence H is order-preserving. $\square$

Corollary 4.12. *For a continuous surjection $t : X \to Y$, define a homomorphism $H : C(Y) \to C(X)$ by $H(g) = g \circ t$. For a maximal* (*resp. prime*) *ideal L of $C(Y)$, let $M = H(L)$. Then $C(Y)/L$ is isomorphic to $H(C(Y))/M$ as ordered fields* (*resp. ordered integral domains*).

Proof. The map $H : C(Y) \to C(X)$ is a monomorphism. Let $S = \{g \in C(Y) \mid \mathbf{0} \leq g\}$, and $S' = \{f \in C(X) \mid \mathbf{0} \leq f\}$. Let us show that $H(S) = S' \cap H(C(Y))$. The homomorphism H is order-preserving by Lemma 4.11(2). Hence $H(S) \subset S' \cap H(C(Y))$. Let $f \in S' \cap H(C(Y))$, and $f = H(g)$ $(g \in C(Y))$. Since $\mathbf{0} \leq f$, for all $x \in X$, $0 \leq f(x)$, thus $0 \leq g(t(x))$. But the map t is surjective, then $0 \leq g(y)$ for all y in Y, so $\mathbf{0} \leq g$. Hence $H(g) \in H(S)$. Thus, $H(S) = S' \cap H(C(Y))$. While, the ideal L is convex in $C(Y)$ by Lemma 4.11(1). Thus, $C(Y)/L$ is isomorphic to $H(C(Y))/M$ as partially ordered rings by Corollary 4.9(2). But, for L being maximal (resp. prime) in $C(Y)$, $C(Y)/L$ and $H(C(Y))/M$ are ordered fields (resp. ordered integral domains), using Lemma 4.11(1). Hence, Corollary 4.12 holds. $\square$

Example 4.13. Let $\sigma : (R, \leq) \to (R', \leq')$ be an epimorphism, $I; I'$ be convex in $R; R'$ respectively, and $\sigma(I) \subset I'$. Let us consider the following conditions on σ related to Theorem 4.5 (note that the induced homomorphism $\bar{\sigma}$ is an isomorphism as partially ordered rings iff (a) and (b) hold (Corollary 4.8)).

(a) $\sigma^{-1}(I') = I$.

(a*) $\sigma(I) = I'$.

(b) $\sigma(S) + I' = S' + I'$.

(b*) $\sigma(S + I) = S' + I'$.

(c) $\sigma(S) = S'$.

(c*) $\sigma(S) \subset S'$.

(c**) $\sigma^{-1}(S') \subset S$.

Clearly, (a) $\Rightarrow$ (a*); (c) $\Rightarrow$ (b); (a*) & (c) $\Rightarrow$ (b*); and (c*) & (c**) $\Rightarrow$ (c) hold.

Obviously, (c) $\Rightarrow$ (c*) holds, and the reverse holds if $(R, \leq)$ is an ordered ring.

Also, (b*) $\Rightarrow$ (b) holds, and the reverse holds if (a*) holds (indeed, $\sigma(S) \subset \sigma(S+I) = S' + I'$, so $\sigma(S) + I' \subset \sigma(S+I) = S' + I'$. But, $\sigma(S+I) = \sigma(S) + \sigma(I) \subset \sigma(S) + I'$. Thus, we have $\sigma(S) + I' = \sigma(S+I)$ (which also holds under (a*)).

However, we have the following examples related to the above.

(1) (a) need not imply (b), (c*) or (c**).

(2) (a), (b*), and (c*) need not imply (c) or (c**).

(3) (a), (b*), and (c**) need not imply (c*).

(4) (a*), (b*), and (c) need not imply (a) or (c**).

(5) (a) and (c*) need not imply (b) or (c**).

(6) (c) (hence (b)) need not imply (a*) or (b*).

Indeed, for (1), let $K = \mathbf{Q}(\pi)$, and let $(K, \leq), (K, \preceq)$ be ordered fields in [3, Example 3.2], and $I = I' = (0)$. Consider the identity map σ of $(K, \leq)$ to $(K, \preceq)$. Then $\sigma(I) = I'$, but neither σ nor σ^{-1} is ordered-preserving.

For (2), let $R = \mathbf{Z}$. Let $S = 4\mathbf{Z}^*$, and $S' = 2\mathbf{Z}^*$. Then $(\mathbf{Z}, \leq_S)$ and $(\mathbf{Z}, \leq_{S'})$ are partially ordered rings. Let $I = I' = 2\mathbf{Z}$. Then I (resp. I') is convex in $(\mathbf{Z}, \leq_S)$ (resp. $(\mathbf{Z}, \leq_{S'})$) by Proposition 3.4. Let $\sigma : (\mathbf{Z}, \leq_S) \to (\mathbf{Z}, \leq_{S'})$ be the identity map. Then $\sigma^{-1}(I') = I$, $\sigma(S+I) = S' + I'$, and $\sigma(S) \subset S'$, but $\sigma(S) \neq S'$ (hence, $\sigma^{-1}(S') \not\subset S$).

For (3), consider (2), putting $I = 2\mathbf{Z}$, $S = 2\mathbf{Z}^*$; and $I' = 2\mathbf{Z}$, $S' = 4\mathbf{Z}^*$.

For (4), let $(R, \leq) = (K[x], \leq_2)$. Let $I = (x^2)$. Then I is convex in $(R, \leq)$ by Corollary 3.9. Let $R' = K$, and σ be the map of R to R' defined by $\sigma(f(x)) = f(0)$. Let $I' = (0)$. Clearly, I' is convex in R', and σ is an epimorphism. Obviously, $\sigma(I) = I'$, $\sigma(S) = S'$ (i.e., S' is the non-negative part in K), and thus $\sigma(S+I) = S' + I'$. But, $\sigma^{-1}(I') = (x) \neq I$. Also, $\sigma^{-1}(S') \not\subset S$, for $x^2 - x \in \sigma^{-1}(S') \backslash S$.

For (5), let $\sigma : K[x] \to K[x]$ be the identity map. Let $S = \{0\}$, $S' = \{f \in K[x] \mid 0 \leq_2 f\}$, and $I = I' = (x)$. Then $(K[x], \leq_S)$ is a partially ordered ring with I convex, and $(K[x], \leq_{S'})$ is an ordered integral domain with I' convex. Clearly, $\sigma^{-1}(I') = I$ and $\sigma(S) \subset S'$. Obviously, $\sigma(S) + I' \neq S' + I'$, and $\sigma^{-1}(S') \not\subset S$.

For (6), let $R = R' = (K[x], \leq_2)$. Let $S = S'$, and $I = (x^2)$, and $I' = (x)$. Then I and I' are convex in $R = R'$ by Corollary 3.9. Let σ be the identity map of R. Clearly, $\sigma(S) = S'$, but $\sigma(I) \neq I'$. We will show that $\sigma(S+I) \neq S' + I'$. Since $I \subset I'$, $S + I = \sigma(S+I) \subset S' + I'$. Since $-x \in I'$ and $0 \in S'$, $-x \in S' + I'$. But, $-x$ never belongs to $S + I$. Because, if $-x \in S + I$, then $-x = f + g$ for some $f \in S$ and some $g \in I$. Let $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Since $-x = f + g$, then $a_0 = 0$, $a_1 = -1$. Thus $f \notin S$, a contradiction. Hence, $-x \notin S + I$, thus $\sigma(S + I) \neq S' + I'$.

Related to Example 4.13(6), let us consider a question whether (b*) and (c) imply (a*). We will show that this question is positive under $\sigma(I)$ being convex in R', or R being an ordered ring.

Lemma 4.14. *Let $(R, \leq)$ be a partially ordered ring, and I, I' be convex in R. If $S + I = S + I'$, then $I = I'$.*

Proof. To see $I \subset I'$, let $x \in I$. Since $I \subset S + I'$, there exist $s \in S$ and $a \in I'$ with $x = s + a$. Since $-x \in I$, there exist $t \in S$ and $b \in I'$ with $-x = t + b$. Then $s + t = -(a + b) \in I'$. Since $0 \leq s \leq s + t$, $s \in I'$ by the convexity of I'. Thus, $x = s + a \in I'$. This shows $I \subset I'$ holds. Similarly, $I \supset I'$ holds. Hence, $I = I'$ holds. □

Proposition 4.15. *Let $\sigma : (R, \leq) \to (R', \leq')$ be an epimorphism, and $I; I'$ be a convex ideal in $R; R'$ respectively. If $\sigma(S + I) = S' + I'$ and $\sigma(S) = S'$, then $\sigma(I) = I'$ holds when* (i) *$\sigma(I)$ is convex in $(R', \leq')$, or* (ii) *$(R, \leq)$ is an ordered ring.*

Proof. For case (i), since $\sigma(S + I) = \sigma(S) + \sigma(I)$, $S' + \sigma(I) = S' + I'$. Since $\sigma(I)$ is convex in R', $\sigma(I) = I'$ by Lemma 4.14. For case (ii), R is an ordered ring, and σ is order-preserving by $\sigma(S) = S'$. Then, since I is convex in R, so is $\sigma(I)$ in R'. Thus, $\sigma(I) = I'$ by case (i). □

Remark 4.16. In Proposition 4.15 (or Lemma 4.14), if the convexity of I' is omitted, then the result need not hold. Indeed, let K be an ordered field, and $R = R' = (K[x], \leq_2)$ be the ordered integral domain. Let $I = (x)$ and $I' = (x^2 + x)$ be ideals in R. Then I is convex, but I' is not convex by Corollary 3.9(2). It is easy to see that $S + I = S + I'$, but $I \neq I'$. Then the identity map σ of R to R is a desired one.

References

[1] Gillman, L. and Jerison, M., Rings of continuous functions, Van Nostrand Reinhold company, 1960.

[2] Kitamura, Y. and Tanaka, Y., Ordered rings and order-preservation, Bull. Tokyo Gakugei Univ., Nat. Sci., **64** (2012), 5–13.

[3] Tanaka, Y., Topology on ordered fields, Comment Math. Univ. Carolin., **53** (2012), 139–147.

Yoshimi Kitamura
Department of Mathematics
Tokyo Gakugei University
Tokyo 184-8501, Japan
(Professor Emeritus)
E-mail: kitamura@u-gakugei.ac.jp

Yoshio Tanaka
Department of Mathematics
Tokyo Gakugei University
Tokyo 184-8501, Japan
(Professor Emeritus)
E-mail: ytanaka@u-gakugei.ac.jp