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# GOLDIE EXTENDING MODULES AND GENERALIZATIONS OF QUASI-CONTINUOUS MODULES

By

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**Abstract.** A module $M$ is said to be *quasi-continuous* if it is extending with the condition $(C_3)$ (cf. [7], [10]). In this paper, by using the notion of a $\mathscr{G}$-extending module which is defined by E. Akalan, G. F. Birkenmeier and A. Tercan [1], we introduce a generalization of quasi-continuous "a GQC(generalized quasi-continuous)-module" and investigate some properties of GQC-modules. Initially we give some properties of a relative ejectivity which is useful in analyzing the structure of $\mathscr{G}$-extending modules and GQC-modules (cf. [1]). And we apply them to the study of direct sums of GQC-modules. We also prove that any direct summand of a GQC-module with the finite internal exchange property is GQC. Moreover, we show that a module $M$ is $\mathscr{G}$-extending modules with $(C_3)$ if and only if it is GQC-module with the finite internal exchange property.

## 1 Preliminaries

Throughout this paper $R$ is a ring with identity and all modules considered are unitary right $R$-modules. A submodule $X$ of a module $M$ is said to be *essential* in $M$ or an *essential submodule* of $M$, if $X \cap Y \neq 0$ for any non-zero submodule $Y$ of $M$ and we write $X \subseteq_e M$ in this case. $X <_{\oplus} M$ means that $X$ is a direct summand of $M$. Let $M = A \oplus B$ and let $\varphi : A \to B$ be a homomorphism. Then $\langle A \xrightarrow{\varphi} B \rangle = \{a - \varphi(a) \mid a \in A\}$ is a submodule of $M$. Note that $M = A \oplus B = \langle A \xrightarrow{\varphi} B \rangle \oplus B$.

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Let $\{M_i \mid i \in I\}$ be a family of modules. The direct sum decomposition $M = \bigoplus_{i \in I} M_i$ is said to be *exchangeable* if, for any direct summand $X$ of $M$, there exists $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\bigoplus_{i \in I} \overline{M_i})$. A module $M$ is said to have the *finite internal exchange property* (*FIEP*) if, any finite direct sum decomposition $M = M_1 \oplus \cdots \oplus M_n$ is exchangeable.

A module $M$ is said to be *extending* if, for any submodule $X$ of $M$, there exists a direct summand $A$ of $M$ such that $X$ is essential in $A$. A module $M$ is said to be *$\mathcal{G}$-extending* or *Goldie extending* if, for any submodule $X$ of $M$, there exist an essential submodule $X'$ of $X$ and a direct summand $A$ of $M$ such that $X'$ is essential in $A$. A module $M$ is said to be *$\mathcal{G}^+$-extending* if any direct summand of $M$ is $\mathcal{G}$-extending (cf. [1]). From [6], $\mathcal{G}$-extending modules with FIEP are $\mathcal{G}^+$-extending. Now we consider the following condition:

$(C_3)$ If $A$ and $B$ are direct summands of $M$ such that $A \cap B = 0$, then $A \oplus B$ is a direct summand of $M$.

A module $M$ is said to be *quasi-continuous* if $M$ is extending with $(C_3)$ (cf. [7], [10]). We obtain that $M$ is a quasi-continuous module if and only if, for every submodules $X_1$ and $X_2$ of $M$ with $X_1 \cap X_2 = 0$, there exists a decomposition $M = A_1 \oplus A_2$ such that $X_i \subseteq A_i$ $(i = 1, 2)$ (cf. [15, pp. 367–368]). Motivated by this result, we introduce a generalization of a quasi-continuous module as follows:

A module $M$ is said to be *GQC* (*generalized quasi-continuous*) if, for every submodules $X_1$ and $X_2$ with $X_1 \cap X_2 = 0$, there exist an essential submodule $X_i' \subseteq_e X_i$ and a decomposition $M = A_1 \oplus A_2$ such that $X_i'$ is a submodule of $A_i$ $(i = 1, 2)$. Note that any GQC-module is $\mathcal{G}$-extending (cf. Proposition 2.3).

Let $M_{\mathbf{Z}} = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/8\mathbf{Z}$. $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/8\mathbf{Z}$ are GQC-modules, but $M$ is not GQC (cf. Proposition 2.6). Hence a direct sum of GQC-modules need not be GQC. Moreover, it is unknown to the author whether or not the property GQC is inherited by direct summands.

In this paper, our main purpose is to show the following:

(I) Let $M_1$ and $M_2$ be GQC-modules with FIEP and put $M = M_1 \oplus M_2$. Then $M$ is a GQC-module with FIEP if and only if $M_i$ is $M_j$-ejective $(i \neq j)$ and the decomposition $M = M_1 \oplus M_2$ is exchangeable.

(II) If $M$ is a GQC-module with FIEP, then $A$ is GQC for any direct summand $A$ of $M$.

(III) A module $M$ is $\mathcal{G}$-extending with $(C_3)$ if and only if it is GQC with FIEP.

For undefined terminologies, the reader is referred to [2], [3], [7] and [15].

Many of the following lemmas can be found in the cited literature, but we list them here for easy reference.

Lemma 1.1. *Let $M$ be a module with a decomposition $M = A \oplus B$ and let $X$ be a submodule of $M$. If $A \cap X \subseteq_e A$, then $X \supseteq_e (A \cap X) \oplus (B \cap X)$.*

Proof. By [11, Lemma 2.2]. □

Lemma 1.2. *Let $M = A \oplus B$, $C \subseteq A$ and let $f : C \to B$ be a homomorphism. If $X \subseteq_e \langle C \xrightarrow{f} B\rangle$, then there exists $C' \subseteq_e C$ such that $X = \langle C' \xrightarrow{f|_{C'}} B\rangle$.*

Proof. Evident. □

Lemma 1.3. *If $M = A \oplus B = X \oplus Y \oplus B$, then there exists a homomorphism $\alpha : X \oplus Y \to B$ such that $A = \langle X \oplus Y \xrightarrow{\alpha} B\rangle = \langle X \xrightarrow{\alpha|_X} B\rangle \oplus \langle Y \xrightarrow{\alpha|_Y} B\rangle$.*

Proof. Let $p_1 : M = X \oplus Y \oplus B \to X \oplus Y$ and $p_2 : M = X \oplus Y \oplus B \to B$ be the projections. Define $\alpha : p_1(A) \to p_2(A)$ by $\alpha(p_1(a)) = p_2(a)$, where $a \in A$. Then $A = \langle X \oplus Y \xrightarrow{\alpha} B\rangle = \langle X \xrightarrow{\alpha|_X} B\rangle \oplus \langle Y \xrightarrow{\alpha|_Y} B\rangle$. □

Lemma 1.4 (cf. [4], [9, Proposition 2.5]). *Let $M = A \oplus B$. Then $M$ has FIEP if and only if $A$ and $B$ have FIEP and the decomposition $M = A \oplus B$ is exchangeable.*

Let $A$ and $B$ be modules. $A$ is said to be *essentially $B$-injective* if, for any submodule $X$ of $B$ and any homomorphism $f : X \to A$ with $\ker f \subseteq_e X$, there exists a homomorphism $g : B \to A$ such that $g|_X = f$.

Lemma 1.5. *Let $A$ and $B$ be modules. If $A$ is essentially $B$-injective, then $A'$ is essentially $B'$-injective for any $A' <_{\oplus} A$ and any $B' \subseteq B$.*

Proof. By [3, 2.15]. □

## 2 Ejective Modules and GQC-modules

Firstly, we recall a generalization of relative injectivity which is introduced by E. Akalan, G. F. Birkenmeier and A. Tercan [1].

DEFINITION. Let $A$ and $B$ be modules. $A$ is said to be *$B$-ejective* if, for any submodule $X$ of $B$ and any homomorphism $f: X \to A$, there exist an essnetial submodule $X'$ of $X$ and a homomorphism $g: B \to A$ such that $g|_{X'} = f|_{X'}$.

Now we consider some properties of relative ejectivities.

PROPOSITION 2.1. *Let $A$, $B$, $A_i$ and $B_i$ $(i = 1, 2)$ be modules.*

(1) *If $A$ is $B$-ejective, then $A'$ is $B'$-ejective for any $A' <_{\oplus} A$ and $B' \subseteq B$.*
(2) *If $A$ is $B_i$-ejective $(i = 1, 2)$, then $A$ is $B_1 \oplus B_2$-ejective.*
(3) *If $A_i$ is $B$-ejective $(i = 1, 2)$, then $A_1 \oplus A_2$ is $B$-ejective.*

PROOF. (1) is clear.

(2) Put $B = B_1 \oplus B_2$, let $X$ be a submodule of $B$ and let $f: X \to A$ be a homomorphism. Let $Y$ be a complement of $X$ in $B$. Define $f^*: X \oplus Y \to A$ by $f^*(x + y) = f(x)$, where $x \in X$ and $y \in Y$. By $X \oplus Y \subseteq_e B$, $(X \oplus Y) \cap B_i \subseteq_e B_i$ $(i = 1, 2)$. Since $A$ is $B_i$-ejective $(i = 1, 2)$, there exist an essential submodule $B_i'$ of $(X \oplus Y) \cap B_i$ and a homomorphism $g_i: B_i \to A$ such that $g_i|_{B_i'} = f^*|_{B_i'}$ $(i = 1, 2)$. By $B_1' \oplus B_2' \subseteq_e X \oplus Y$, we see $(B_1' \oplus B_2') \cap X \subseteq_e X$. Put $g = g_1 + g_2 : B \to A$. Let $x = b_1' + b_2' \in (B_1' \oplus B_2') \cap X$, where $b_i' \in B_i'$ $(i = 1, 2)$. Then

$$f(x) = f^*(x) = f^*(b_1') + f^*(b_2') = g_1(b_1') + g_2(b_2') = g(b_1' + b_2') = g(x).$$

Thus $A$ is $B$-ejective.

(3) Put $A = A_1 \oplus A_2$, let $X$ be a submodule of $B$ and let $f: X \to A$ be a homomorphism. Let $p_i: A \to A_i$ be the projection $(i = 1, 2)$. Since $A_i$ is $B$-ejective, for $p_i f: X \to A_i$, there exist an essential submodule $X_i$ of $X$ and a homomorphism $g_i: B \to A_i$ such that $g_i|_{X_i} = p_i f|_{X_i}$ $(i = 1, 2)$. By $X_i \subseteq_e X$ $(i = 1, 2)$, $X_1 \cap X_2 \subseteq_e X$. Put $g = g_1 + g_2 : B \to A$. Then, for any $x \in X_1 \cap X_2$,

$$f(x) = p_1 f(x) + p_2 f(x) = g_1(x) + g_2(x) = g(x).$$

Thus $A$ is $B$-ejective. $\square$

Let $A$ and $B$ be modules. $A$ is said to be *mono-$B$-injective* if, for any submodule $X$ of $B$ and any monomorphism $f: X \to A$, there exists a homomorphim $g: B \to A$ such that $g|_X = f$ (cf. [5]). The following is a connection between relative mono-injectivities and relative ejectivities.

PROPOSITION 2.2. *Let $A$ be a module and let $B$ be a $\mathcal{G}$-extending module. If $A$ is mono-$B$-injective, then $A$ is $B$-ejective.*

Proof. Let $X$ be a submodule of $B$ and let $f : X \to A$ be a homomorphism. As $B$ is $\mathscr{G}$-extending, there exist an essential submodule $K$ of $\ker f$ and a decomposition $B = B_1 \oplus B_2$ such that $K \subseteq_e B_1$. By Lemma 1.1, $X \supseteq_e K \oplus (B_2 \cap X)$. Then $f|_{B_2 \cap X}$ is a monomorphism. Since $A$ is mono-$B_2$-injective, there exists a homomorphism $g : B_2 \to A$ with $g|_{B_2 \cap X} = f|_{B_2 \cap X}$. Define $h : B \to A$ by $h(b_1 + b_2) = g(b_2)$, where $b_i \in B_i$ $(i = 1, 2)$. Let $k + b_2 \in K \oplus (B_2 \cap X)$, where $k \in K$ and $b_2 \in B_2 \cap X$. Then $h(k + b_2) = g(b_2) = f(b_2) = f(k + b_2)$. Hence $A$ is $B$-ejective. □

Now we show that any GQC-module is $\mathscr{G}$-extending.

Proposition 2.3. *If $M$ is a GQC-module, then it is $\mathscr{G}$-extending.*

Proof. Let $M$ be a GQC-module and let $X$ be a submodule of $M$. Let $Y$ be a complement of $X$ in $M$. By $X \cap Y = 0$, there exists essential submodules $X'$ of $X$ and $Y'$ of $Y$ and a decomposition $M = A \oplus B$ such that $X' \subseteq A$ and $Y' \subseteq B$. Then $X' = (X' \oplus Y') \cap A \subseteq_e M \cap A = A$. Thus $M$ is $\mathscr{G}$-extending. □

Proposition 2.4. *If $M$ is a $\mathscr{G}$-extending module with $(C_3)$, then it is GQC.*

Proof. Obvious. □

By [1, Example 3.20], $M_{\mathbf{Z}} = \mathbf{Q} \oplus \mathbf{Z}/p\mathbf{Z}$ is a $\mathscr{G}$-extending module with $(C_3)$ but not quasi-continuous, where $p$ is a prime number.

Now we give a characterlization for any direct summand of a GQC-module to be GQC.

Proposition 2.5. *If $M$ is a GQC-module with FIEP, then $A$ is GQC for any direct summand $A$ of $M$.*

Proof. Let $A$ be a direct summand of $M$ and let $X$ and $Y$ be submodules of $A$ with $X \cap Y = 0$. As $M$ is GQC, there exist essential submodules $X'$ of $X$ and $Y'$ of $Y$ and a decomposition $M = M_1 \oplus M_2$ such that $X' \subseteq M_1$ and $Y' \subseteq M_2$. Since $M$ satisfies FIEP, there exists a direct summand $M_i'$ of $M_i$ $(i = 1, 2)$ such that $M = A \oplus M_1' \oplus M_2'$. Put $M_i = M_i' \oplus M_i''$ $(i = 1, 2)$. By Lemma 1.3, there exists a homomorphism $\alpha : M_1'' \oplus M_2'' \to M_1' \oplus M_2'$ such that $A = \langle M_1'' \xrightarrow{\alpha|_{M_1''}} M_1' \oplus M_2' \rangle \oplus \langle M_2'' \xrightarrow{\alpha|_{M_2''}} M_1' \oplus M_2' \rangle$. By $X' \subseteq A \cap M_1$, $X' \subseteq \langle M_1'' \xrightarrow{\alpha|_{M_1''}} M_1' \oplus M_2' \rangle$. Similarly, we obtain $Y' \subseteq \langle M_2'' \xrightarrow{\alpha|_{M_2''}} M_1' \oplus M_2' \rangle$. Thus $A$ is GQC. □

Let $M$ be a finitely generated torsion-free abelian group with rank $\geq 2$. Then $M$ is ($\mathscr{G}$-)*extending* but not satisfy FIEP (cf. [3, p. 56]). For $\mathscr{G}$-extending modules with FIEP, we can give a characterization of GQC-module in a term of a relative ejectivity as follows:

Proposition 2.6. *Let $M$ be a $\mathscr{G}$-extending module with FIEP. Then $M$ is GQC if and only if $A$ is $B$-ejective for any decomposition $M = A \oplus B$.*

Proof. ($\Rightarrow$) Let $M = A \oplus B$, let $X$ be a submodule of $B$ and let $f : X \to A$ be a homomorphism. As $B$ is $\mathscr{G}$-extending, there exist an essential submodule $X'$ of $X$ and a decomposition $B = B_1 \oplus B_2$ such that $X' \subseteq_e B_1$. By Proposition 2.5, $A \oplus B_1$ is GQC and so we may assume that $X \subseteq_e B$.

By $\langle X \xrightarrow{f} A \rangle \cap A = 0$, there exist essential submodules $T$ of $\langle X \xrightarrow{f} A \rangle$ and $L$ of $A$ and a decomposition $M = M_1 \oplus M_2$ such that $T \subseteq M_1$ and $L \subseteq M_2$. By $\langle X \xrightarrow{f} A \rangle \oplus A = X \oplus A \subseteq_e M$, we see $T \subseteq_e M_1$ and $L \subseteq_e M_2$. Since $M$ satisfies FIEP, there exists $M_i' \subseteq M_i$ $(i = 1, 2)$ such that $M = A \oplus M_1' \oplus M_2'$. By $L \subseteq_e M_2 \cap A$, we get $M_2' = 0$ and so $M = A \oplus M_1'$. By $A \oplus M_1 \subseteq_e M$, we obtain $M = A \oplus M_1$. By Lemma 1.3, there exists a homomorphism $g : B \to A$ such that $M_1 = \langle B \xrightarrow{g} A \rangle$.

As $T \subseteq_e \langle X \xrightarrow{f} A \rangle$, by Lemma 1.2, there exists an essential submodule $X'$ of $X$ such that $T = \langle X' \xrightarrow{f|_{X'}} A \rangle$. Thus $\langle X' \xrightarrow{f|_{X'}} A \rangle = T \subseteq_e M_1 = \langle B \xrightarrow{g} A \rangle$. Then, for any $x' \in X'$, there exists $b \in B$ such that $x' - f(x') = b - g(b)$. By $x' = b$ and $f(x') = g(b)$, we obtain $g(x') = f(x')$. Therefore $A$ is $B$-ejective.

($\Leftarrow$) Let $X$ and $Y$ be submodules of $M$ with $X \cap Y = 0$. As $M$ is $\mathscr{G}$-extending, there exist an essential submodule $X'$ of $X$ and a decomposition $M = A \oplus B$ such that $X' \subseteq_e A$. Let $p_A : M = A \oplus B \to A$ and $p_B : M = A \oplus B \to B$ be the projections. By $Y \cap A = 0$, we can define a homomorphism $f : p_B(Y) \to p_A(Y)$ by $f(p_B(y)) = p_A(y)$, where $y \in Y$. Since $A$ is $B$-ejective, there exist an essential submodule $B'$ of $p_B(Y)$ and a homomorphism $g : B \to A$ such that $g|_{B'} = f|_{B'}$. Then we see

$$\langle B' \xrightarrow{f|_{B'}} A \rangle \subseteq_e \langle p_B(Y) \xrightarrow{f} p_A(Y) \rangle = Y \quad \text{and} \quad \langle B' \xrightarrow{f|_{B'}} A \rangle \subseteq \langle B \xrightarrow{g} A \rangle.$$

Thus $M$ is GQC. □

Let $M_{\mathbf{Z}} = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/8\mathbf{Z}$. From [1, Example 3.4] or [6], we see that $M$ is $\mathcal{G}$-extending with FIEP. However, by Proposition 2.6, $M$ is not GQC since $\mathbf{Z}/2\mathbf{Z}$ is not $\mathbf{Z}/8\mathbf{Z}$-ejective. Next we show a characterization for a GQC-module to be quasi-continuous.

Proposition 2.7. *Let $M$ be a $\mathcal{G}$-extending module. Assume that $A$ is essentially $B$-injective for any decomposition $M = A \oplus B$. Then*

(1) *$M$ is extending.*
(2) *$M$ satisfies $(C_3)$ if and only if it is GQC.*

Proof. (1) By [6, Proposition 2.1].

(2) $(\Rightarrow)$ By Proposition 2.4.

$(\Leftarrow)$ Let $X$ and $Y$ be direct summands of $M$ with $X \cap Y = 0$. Since $M$ is GQC, there exist essential submodules $X' \subseteq_e X$ and $Y' \subseteq_e Y$ and a decomposition $M = A \oplus B$ such that $X' \subseteq A$ and $Y' \subseteq_e B$. Let $p_A : M = A \oplus B \to A$ and $p_B : M = A \oplus B \to B$ be the projections. By $Y \cap A = 0$ and $Y' \subseteq_e Y$, the canonical map $f : p_B(Y) \to p_A(Y)$ is a homomorphism with $\ker f \subseteq_e p_B(Y)$. Since $A$ is essentially $B$-injective, there exists a homomorphism $f^* : B \to A$ such that $f^*|_{p_B(Y)} = f$ and then

$$M = \langle B \xrightarrow{f^*} A \rangle \oplus A \quad \text{and} \quad Y = \langle p_B(Y) \xrightarrow{f} p_A(Y) \rangle \subseteq_e \langle B \xrightarrow{f^*} A \rangle.$$

Hence $M = Y \oplus A$. By (1), there exists a decomposition $A = A' \oplus A''$ such that $X' \subseteq_e A'$. By the same argument above, there exists a homomorphism $g^* : A' \to Y \oplus A''$ such that $M = \langle A' \xrightarrow{g^*} Y \oplus A'' \rangle \oplus Y \oplus A''$ and $X \subseteq_e \langle A' \xrightarrow{g^*} Y \oplus A'' \rangle$. Thus $M = X \oplus Y \oplus A''$. □

Corollary 2.8. *Let $M$ be a module. Then $M$ is quasi-continuous if and only if $M$ is GQC and $A$ is essentially $B$-injective for any decomposition $M = A \oplus B$.*

Proof. By Proposition 2.7 and [7, Proposition 2.10]. □

Now we give a necessary and sufficient condition for a direct sum of GQC-modules with FIEP to be GQC with FIEP. First, we show the following lemma which is due to E. Akalan, G. F. Birkenmeier and A. Tercan [1, Theorem 3.1].

LEMMA 2.9 ([1, Theorem 3.1]). *Let $A$ and $B$ be $\mathcal{G}$-extending modules and put $M = A \oplus B$. If $A$ is $B$-ejective, then $M$ is $\mathcal{G}$-extending. In general, the converse is not true.*

The following is an immediate consequence of Proposition 2.2 and Lemma 2.9.

COROLLARY 2.10. *Let $A$ and $B$ be $\mathcal{G}$-extending modules and put $M = A \oplus B$. If $A$ is mono-$B$-injective, then $M$ is $\mathcal{G}$-extending.*

By Proposition 2.6 and Lemma 2.9, we obtain the following result.

THEOREM 2.11. *Let $M_1$ and $M_2$ be GQC-modules with FIEP and put $M = M_1 \oplus M_2$. Then $M$ is a GQC-module with FIEP if and only if $M_i$ is $M_j$-ejective $(i \neq j)$ and the decomposition $M = M_1 \oplus M_2$ is exchangeable.*

PROOF. $(\Rightarrow)$ By Proposition 2.6.

$(\Leftarrow)$ By Lemmas 1.4 and 2.9, $M$ is $\mathcal{G}$-extending with FIEP. By Proposition 2.6, we may prove that $A$ is $B$-ejective for any decomposition $M = A \oplus B$. Let $M = A \oplus B$. Since the decomposition $M = M_1 \oplus M_2$ is exchangeable, there exists $M_i' \subseteq M_i$ $(i = 1, 2)$ such that $M = A \oplus M_1' \oplus M_2'$. Put $M_i = M_i' \oplus M_i''$ $(i = 1, 2)$. Then

$$A \simeq M_1'' \oplus M_2'' \quad \text{and} \quad B \simeq M_1' \oplus M_2'.$$

As $M_i$ is GQC-module with FIEP, by Proposition 2.6, $M_i''$ is $M_i'$-ejective $(i = 1, 2)$. By Proposition 2.1, we see that $M_1'' \oplus M_2''$ is $M_1' \oplus M_2'$-ejective. Hence $A$ is $B$-ejective. Thus $M$ is a GQC-module with FIEP. □

By results above, we can easily prove the following result which is well known:

COROLLARY 2.12 ([7, Theorem 2.13]). Let $M_1$ and $M_2$ be quasi-continuous modules and put $M = M_1 \oplus M_2$. Then $M$ is quasi-continuous if and only if $M_i$ is $M_j$-injective $(i \neq j)$.

Proof. ($\Rightarrow$) By Theorem 2.11 and [4, Proposition 1.4 and Theorem 2.1], $M_i$ is $M_j$-ejective and essentially $M_j$-injective $(i \neq j)$. Thus $M_i$ is $M_j$-injective $(i \neq j)$ by [6, Proposition 2.2].

($\Leftarrow$) By [4, Proposition 1.4, Theorems 2.1 and 2.15], we see that $M$ is extending with FIEP and $A$ is essentially $B$-injective for any decomposition $M = A \oplus B$. Thus, by Proposition 2.7 and Theorem 2.11, $M$ is quasi-continuous. □

## 3 $\mathscr{G}$-extending Modules with $(C_3)$

Firstly we recall the condition $(C_{11})$ from [12], which can be considered as a generalization of $\mathscr{G}$-extending.

Definition (cf. [12]). Let $M$ be a module. $M$ is said to be a $(C_{11})$-*module* if any submodule $X$ of $M$ has a complement which is a direct summand of $M$.

From [13, Example 4], there exists a $(C_{11})$-module which has a direct summand that does not satisfy $(C_{11})$. However, any direct summand of $(C_{11})$-modules with $(C_3)$ satisfies $(C_{11})$.

Proposition 3.1. *Let $M$ be a $(C_{11})$-module with $(C_3)$ and let $A$ be a direct summand of $M$. Then $A$ is a $(C_{11})$-module with $(C_3)$.*

Proof. Let $M$ be a $(C_{11})$-module with $(C_3)$ and let $M = A \oplus B$. From [7, Proposition 2.7], we may show that $A$ satisfies $(C_{11})$. Let $X$ be a submodule of $A$. Since $M$ satisfies $(C_{11})$, there exists a direct summand $N$ of $M$ such that $(X \oplus B) \oplus N \subseteq_e M$. By $(C_3)$, $B \oplus N$ is a direct summand of $M$. Put $M = T \oplus N \oplus B$. By Lemma 1.3, there exists a homomorphism $\alpha : T \oplus N \to B$ such that $A = \langle T \overset{\alpha|_T}{\to} B \rangle \oplus \langle N \overset{\alpha|_N}{\to} B \rangle$. Put $A_1 = \langle T \overset{\alpha|_T}{\to} B \rangle$ and $A_2 = (N \oplus B) \cap A$. Then we see $M = A_1 \oplus N \oplus B$ and $A = A_1 \oplus A_2$.

Now we prove that $X \oplus A_2 \subseteq_e A$. Given $0 \neq a \in A$ and express $a$ in $A = A_1 \oplus A_2$ as $a = a_1 + a_2$ $(a_i \in A_i)$. If $a_1 = 0$, then $0 \neq a = a_2 \in A_2$. Let $a_1 \neq 0$. By $X \oplus B \oplus N \subseteq_e M$, there exists $r \in R$ such that $0 \neq a_1 r = x + b + n$ for some $x \in X$, $b \in B$ and $n \in N$. So $n + b = a_1 r - x \in (N \oplus B) \cap A = A_2$. Thus $0 \neq ar = a_1 r + a_2 r = x + (b + n + a_2 r) \in X \oplus A_2$. Hence $X \oplus A_2 \subseteq_e A$.

Therefore $A_2$ is a complement of $X$ in $A$. □

Now we consider the following conditions for a module $M$ (cf. [1], [14]):

($*$) For any decomposition $M = A \oplus B$, $A$ is $B$-ejective.
($SIP$) For any direct summands $A$ and $B$ of $M$, $A \cap B$ is a direct summand.

PROPOSITION 3.2. *If $M$ is module with the conditions* ($*$) *and* ($SIP$), *then $M$ satisfies* ($C_3$).

PROOF. Let $A$ and $B$ be direct summands of $M$ with $A \cap B = 0$. Put $M = A \oplus C$. Let $p_A : M \to A$ and $p_C : M \to C$ be the projections. Define $f : p_C(B) \to p_A(B)$ by $f(p_C(b)) = p_A(b)$, where $b \in B$. Since $A$ is $C$-ejective, there exist an essntial submodule $C'$ of $p_C(B)$ and a homomorphism $g : C \to A$ such that $g|_{C'} = f|_{C'}$. Then

$$\langle C' \xrightarrow{f|_{C'}} A \rangle \subseteq \langle C \xrightarrow{g} A \rangle \quad \text{and} \quad \langle C' \xrightarrow{f|_{C'}} A \rangle \subseteq_e \langle p_C(B) \xrightarrow{f} p_A(B) \rangle = B.$$

So we see $\langle C \xrightarrow{g} A \rangle \cap B \subseteq_e B$. By ($SIP$), $\langle C \xrightarrow{g} A \rangle \cap B$ is a direct summand of $M$ and hence $\langle C \xrightarrow{g} A \rangle \cap B = B$. As $B \subseteq \langle C \xrightarrow{g} A \rangle$, there exists a direct summand $T$ of $M$ such that $\langle C \xrightarrow{g} A \rangle = B \oplus T$. Thus $M = A \oplus B \oplus T$. □

Next we show that any ($C_{11}$)-module with ($*$) is $\mathcal{G}$-extending.

PROPOSITION 3.3. *Let $M$ be a module with* ($*$). *Then $M$ is $\mathcal{G}$-extending if and only if $M$ satisfies* ($C_{11}$).

PROOF. ($\Rightarrow$) is clear.
($\Leftarrow$) Let $X$ be a submodule of $M$. Then there exists a direct summand $A$ of $M$ such that $A$ is a complement of $X$ in $M$. Put $M = A \oplus B$. By the similar proof of Proposition 3.2, there exist an essential submodule $B'$ of $B$ and homomorphisms $f : B' \to A$, $g : B \to A$ such that

$$\langle B' \xrightarrow{f} A \rangle \subseteq_e \langle B \xrightarrow{g} A \rangle <_{\oplus} M \quad \text{and} \quad \langle B' \xrightarrow{f} A \rangle \subseteq_e X.$$

Thus $M$ is $\mathcal{G}$-extending. □

Now we show that a $\mathcal{G}$-extending module with ($C_3$) is just GQC with FIEP.

Theorem 3.4. *Let $M$ be a module. Then*

(1) *If $M$ is $\mathcal{G}$-extending with $(C_3)$, then any direct summand of $M$ is $\mathcal{G}$-extending.*
(2) *$M$ is $\mathcal{G}$-extending with $(C_3)$ if and only if it is GQC with FIEP.*

Proof. (1) Let $M = A \oplus B$ and let $X$ be a submodule of $A$. Since $M$ is $\mathcal{G}$-extending, there exist an essential submodule $X'$ of $X$ and a direct summand $X^*$ of $M$ such that $X' \subseteq_e X^*$. As $X^* \cap B = 0$, $X^* \oplus B <_{\oplus} M$. So there exists a direct summand $K$ of $M$ such that $M = X^* \oplus K \oplus B$. By Lemma 1.3, there exists a homomorphism $\alpha : X^* \oplus K \to B$ such that $A = \langle X^* \xrightarrow{\alpha|_{X^*}} B\rangle \oplus \langle K \xrightarrow{\alpha|_K} B\rangle$. By $X' \subseteq A \cap X^* \subseteq \ker \alpha$ and $X' \subseteq_e X^*$, we see

$$X' \subseteq_e \langle X^* \xrightarrow{\alpha|_{X^*}} B\rangle.$$

Thus $A$ is $\mathcal{G}$-extending.

(2) $(\Rightarrow)$ From [8, Proposition 16] (cf. [4, Theorem 2.15]), we may show that any decomposition $M = M_1 \oplus M_2$ is exchangeable. Let $M = M_1 \oplus M_2$ and let $X$ be a direct summand of $M$. By (1), $M_i$ is $\mathcal{G}$-extending and hence there exist an essential submodule $X_i'$ of $M_i \cap X$ and a decomposition $M_i = A_i \oplus B_i$ such that $X_i' \subseteq_e A_i$ $(i = 1, 2)$. By Lemma 1.1, $X \supseteq_e X_1' \oplus X_2' \oplus (B_1 \oplus B_2) \cap X$. As $B_1 \oplus B_2$ is $\mathcal{G}$-extending, there exist an essential submodule $Y$ of $(B_1 \oplus B_2) \cap X$ and a direct summand $T$ of $B_1 \oplus B_2$ with $Y \subseteq_e T$. By $B_1 \cap X = 0$, we see $B_1 \cap T = 0$. Thus $B_1 \oplus T$ is a direct summand of $B_1 \oplus B_2$. Put $B_1 \oplus B_2 = L \oplus T \oplus B_1$. By Lemma 1.3, there exists a homomorphism $\alpha : L \oplus T \to B_1$ such that $B_2 = \langle L \xrightarrow{\alpha|_L} B_1\rangle \oplus \langle T \xrightarrow{\alpha|_T} B_1\rangle$. Put $B_2' = \langle L \xrightarrow{\alpha|_L} B_1\rangle$. Then $B_1 \oplus B_2 = T \oplus B_1 \oplus B_2'$. Thus we see

$$X \supseteq_e X_1' \oplus X_2' \oplus Y \quad \text{and} \quad A_1 \oplus A_2 \oplus T \supseteq_e X_1' \oplus X_2' \oplus Y.$$

So we see $(B_1 \oplus B_2') \cap X = 0$. By $(C_3)$, $X \oplus B_1 \oplus B_2'$ is a direct summand of $M$. As $X \oplus B_1 \oplus B_2' \subseteq_e M$, we obtain $M = X \oplus B_1 \oplus B_2'$.

Therefore $M$ satisfies FIEP.

$(\Leftarrow)$ Let $A$ and $B$ be direct summands of $M$ with $A \cap B = 0$. As $M$ is GQC, there exist essential submodules $A' \subseteq_e A$ and $B' \subseteq_e B$ and a decomposition $M = M_1 \oplus M_2$ such that $A' \subseteq_e M_1$ and $B' \subseteq M_2$. As $M_2$ is $\mathcal{G}$-extending, we may assume that $M = M_1 \oplus M_2 \oplus M_3$ with $A' \subseteq_e M_1$ and $B' \subseteq_e M_2$. As $M$ satisfies FIEP, there exists $M_i' \subseteq M_i$ $(i = 1, 2, 3)$ with $M = A \oplus M_1' \oplus M_2' \oplus M_3'$. By $M_1 \supseteq_e A' \subseteq_e A$, $M = A \oplus M_2' \oplus M_3'$. By $M_3 \cap (A \oplus M_2') = 0$, $M = A \oplus M_2' \oplus M_3$. As $M$ satisfies FIEP, there exist $\bar{A} \subseteq A$, $\overline{M_2'} \subseteq M_2'$ and $\overline{M_3} \subseteq M_3$ with $M =$

$B \oplus \bar{A} \oplus \overline{M_2'} \oplus \overline{M_3}$. By $M_2 \supseteq_e B' \subseteq_e B$, $M = B \oplus \bar{A} \oplus \overline{M_3}$. As $A \cap (B \oplus \overline{M_3}) = 0$, we see $\bar{A} = A$. Thus $A \oplus B$ is a direct summand of $M$. $\square$

Finally, we touch on the relations of modules which are generalizations of quasi-continuous modules.

$$\begin{tikzcd}
\text{quasi-continuous} \arrow[r] \arrow[d] & \text{extending with FIEP} \arrow[rr] \arrow[dr] & & \text{extending} \arrow[rr] \arrow[dr] & & (C_{11})\text{-module} \\
\mathscr{G}\text{-extending with } (C_3) \arrow[rr] \arrow[d, leftrightarrow] & & \mathscr{G}\text{-extending with FIEP} \arrow[rr] & & \mathscr{G}\text{-extending} \arrow[ur] & \\
\text{GQC-module with FIEP} \arrow[rrr] & & & \text{GQC-module} \arrow[ur] & &
\end{tikzcd}$$

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