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SOME TRANSFORMATIONS ON $(LCS)_n$-MANIFOLDS

By

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Abstract. The present paper deals with a study of certain transformations on an $(LCS)_n$-manifold. It is shown that an $(LCS)_n$-manifold remains invariant under a D-homothetic deformation. We also study an infinitesimal CL-transformation on an $(LCS)_n$-manifold and obtain a necessary and sufficient condition for such an infinitesimal transformation to be a Killing or a conformal Killing vector field. Finally, we study CL-transformation on an $(LCS)_n$-manifold and obtained a new tensor field, called *CL-curvature tensor field*, which is invariant under such a transformation.

1. Introduction

In 1968, Tanno [23] introduced and studied D-homothetic deformation on a contact metric manifold. By a D-homothetic deformation we mean a conformal change of structure on a contact metric manifold which is invariant under such change. Tanno [23] used D-homothetic deformation on Sasakian structure to get results on first Betti number, second Betti number and hormonic forms and hence D-homothetic deformation is an important transformation due to the invariance of a structure. Again, in [11] Olszak and in [18] Shaikh et al. are respectively studied the D-homothetic deformation on a quasi-Sasakian and a trans-Sasakian manifold, and both the structures remain invariant under such a deformation. In 1963, Tashiro and Tachibana [25] introduced a transformation, called CL-transformation, on a Sasakian manifold under which C-loxodrome remains invariant. We note that a C-loxodrome is a loxodrome cutting geodesic

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trajectories of the characterstic vector field ξ of the Sasakian manifold with constant angle. Again, Takamatsu and Mizusawa [24] studied an infinitesimal CL-transformation on a compact Sasakian manifold. In [6] Koto and Nagao obtained a tensor field on a Sasakian manifold which is invariant under a CL-transformation. Also, Matsumoto and Mihai [8] studied infinitesimal CL-transformation and CL-transformation on an LP-Sasakian manifold and obtained an invariant tensor field under a CL-transformation with many other interesting results.

On the other hand in 2003, the first author [14] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds), which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [7], Mihai and Rosca [9].

Motivating from the above studies, in the present paper, we study the D-homothetic deformation, infinitesimal CL-transformation and CL-transformation on an $(LCS)_n$-manifold. The paper is organized as follows. Section 2 provides the rudimentary facts of $(LCS)_n$-manifolds along with some curvature relations. Section 3 is devoted to the study of D-homothetic deformation on an $(LCS)_n$-manifold. It is proved that an $(LCS)_n$-manifold is invariant under a D-homothetic deformation (see, Theorem 3.1). However, under such a deformation an LP-Sasakian manifold is not invariant. We also prove that under a D-homothetic deformation an η-Einstein $(LCS)_n$-manifold is invariant and under such a deformation the ϕ-sectional curvature of an $(LCS)_n$-manifold is conformal (see, Theorem 3.3 and 3.4).

In 1966, Takamatsu and Mizusawa [24] studied an infinitesimal CL-transformation on a compact Sasakian manifold and proved that such a transformation is necessarily projective. Again in [8], Matsumoto and Mihai studied infinitesimal CL-transformation on an LP-Sasakian manifold. In Section 4 we study an infinitesimal CL-transformation on an $(LCS)_n$-manifold and obtain the expression of Lie derivative of the metric tensor with respect to such transformation (see, Theorem 4.1), which generalizes the corresponding result of LP-Sasakian manifold. We also obtain a necessary and sufficient condition for which an infinitesimal CL-transformation to be a Killing (resp. conformal Killing) vector field (see, Theorem 4.2 (resp. Theorem 4.3)).

In [25], Tashiro and Tachibana proved that if a Sasakian manifold is related to a locally Euclidean manifold by a CL-transformation, then it is a locally C-Fubinian manifold and vice-versa. In [6], Koto and Nagao obtained an invariant tensor field under a CL-transformation on a Sasakian manifold and in [8], Matsumoto and Mihai also obtained an invariant tensor field under a

CL-transformation on an LP-Sasakian manifold. Again in [2], Atceken proved that a conformally flat $(LCS)_n$-manifold is a manifold of quasi-constant curvature. In Section 5 we study CL-transformation on an $(LCS)_n$-manifold M and prove that if the Levi-Civita connection ∇ of M is transformed into a flat symmetric affine connection $\bar{\nabla}$ by a CL-transformation, then M is of quasi-constant curvature (see, Theorem 5.1). We also obtain a new tensor field A which is invariant under the CL-transformation and such an invariant tensor field on the manifold is said to be the *CL-curvature tensor field*. It is shown that the CL-curvature tensor field A is invariant under a D-homothetic deformation if and if the deformation is homothetic (see, Theorem 5.3).

If the CL-curvature tensor field A vanishes identically, then the $(LCS)_n$-manifold is said to be CL-flat [6]. Finally, in the last section we study CL-flat and CL-symmetric $(LCS)_n$-manifold. In [2] (Theorem 2 and Corollary 4), Atceken proved that a conformally flat as well as a quasi-conformally flat $(LCS)_n$-manifold M is an η-Einstein manifold. But in our paper it is proved that a CL-flat $(LCS)_n$-manifold is η-Einstein if $r \neq n(n-1)(\alpha^2 - \rho)$. However, if $r = n(n-1)(\alpha^2 - \rho)$, then the manifold is Einstein. It is shown that the scalar curvature of a CL-flat $(LCS)_n$-manifold is constant if and only if $2\alpha\rho - \beta = 0$. Again, in [2] (Theorem 3 and Theorem 6), Atceken proved that a quasi-conformally flat $(LCS)_n$-manifold is of constant curvature but a conformally flat $(LCS)_n$-manifold is of quasi-constant curvature. In our paper it is proved that a CL-flat $(LCS)_n$-manifold is of quasi-constant curvature if $r \neq n(n-1)(\alpha^2 - \rho)$. However, if $r = n(n-1)(\alpha^2 - \rho)$, then the manifold is of constant curvature. An $(LCS)_n$-manifold is said to be CL-symmetric if $\nabla A = 0$. It is proved that a CL-symmetric $(LCS)_n$-manifold is an η-Einstein manifold. Again, it is shown that in a CL-symmetric $(LCS)_n$-manifold, *grad* r is codirectional with ξ. It is also proved that a CL-symmetric $(LCS)_n$-manifold M is locally symmetric if and only if M is an Einstein manifold. We note that in Corollary 10 of [2], Atceken proved that a locally symmetric $(LCS)_n$-manifold is Einstein. But an Einstein manifold is not necessarily locally symmetric unless $n = 3$. However, our Theorem 6.5 ensures that if an Einstein $(LCS)_n$-manifold is CL-symmetric, then it is locally symmetric.

2. $(LCS)_n$-manifolds

An n-dimensional Lorentzian manifold M is a smooth connected para-compact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbf{R}$ is a non-degenerate inner product of signature

$(-+\cdots+)$, where T_pM denotes the tangent vector space of M at p and $\mathbf{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v,v) < 0$ (resp., ≤ 0, $= 0$, > 0) [12]. The category to which a given vector falls is called its causal character.

In a semi-Riemannian manifold M a vector field P defined by $g(X,P) = A(X)$ for any X on M, is said to to be a concircular vector field [28] if

$$(\nabla_X A)(Y) = \alpha\{g(X,Y) + \omega(X)A(Y)\}, \tag{2.1}$$

where α is a non-zero scalar and ω is a closed 1-form. Let M be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the structure vector field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{2.2}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X,\xi) = \eta(X), \tag{2.3}$$

the following equation

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \tag{2.4}$$

holds for all vector fields X, Y on M and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.6}$$

then from (2.4) and (2.6) we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

from which it follows that

$$\phi^2 X = X + \eta(X)\xi, \tag{2.8}$$

that is, ϕ is a symmetric (1,1) tensor field, called the structure tensor of the manifold. The n-dimensional Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η, and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) [14]. Especially, if $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [7]. The $(LCS)_n$-manifold have also been studied in ([1], [3], [13], [15], [16], [17], [19], [20], [21], [22]).

In an $(LCS)_n$-manifold, the following relations hold (see [14], [15]):

$$\text{(a)}\ \eta(\xi) = -1, \quad \text{(b)}\ \phi \circ \xi = 0, \quad \text{(c)}\ \eta \circ \phi = 0, \tag{2.9}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.10}$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \tag{2.11}$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)\{g(Y, Z)\xi - \eta(Z)Y\}, \tag{2.12}$$

$$S(X, \xi) = (\alpha^2 - \rho)(n-1)\eta(X) \tag{2.13}$$

for any vector fields X, Y, Z on M and $\alpha^2 - \rho \neq 0$, where R and S denotes respectively the curvature tensor and the Ricci tensor of the manifold.

In an $(LCS)_n$-manifold, we also have the following relations:

$$(\nabla_X \eta)(Y) = (\nabla_Y \eta)(X), \tag{2.14}$$

$$d\eta(X, Y) = 0. \tag{2.15}$$

We also mention that, in an $(LCS)_n$-manifold the symmetric (1,1) tensor field ϕ is idempotent and hence the eigenvalue of ϕ is either 1 or 0.

3. *D*-homothetic Deformation on an $(LCS)_n$-manifold

An odd dimensional smooth manifold M is said to be an almost contact metric manifold [30] if there exist an (1,1) tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g on M such that $\eta(\xi) = 1$, $g(X, \xi) = \eta(X)$, $\phi^2 X = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X, Y on M.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). A transformation on M is said to be a D-homothetic deformation [11] if the almost contact metric structure (ϕ, ξ, η, g) is transformed into (ϕ', ξ', η', g') such that

$$\phi' = \phi, \quad \xi' = \frac{1}{a}\xi, \quad \eta' = a\eta, \quad g' = bg + (a^2 - b)\eta \otimes \eta,$$

where a and b are constants such that $a \neq 0$ and $b > 0$. If $a^2 = b$, then the transformation is called a homothetic deformation. It can be easily seen that (ϕ', ξ', η', g') is also an almost contact metric structure on M.

Now we make a little change in the definition of a D-homothetic deformation for Lorentzian metric.

DEFINITION 3.1. *Let M be an $(LCS)_n$-manifold with structure (ϕ, ξ, η, g). If the Lorentzian concircular structure (ϕ, ξ, η, g) on M is transformed into (ϕ', ξ', η', g') such that*

$$\phi' = \phi, \quad \xi' = \frac{1}{a}\xi, \quad \eta' = a\eta, \quad g' = bg - (a^2 - b)\eta \otimes \eta \tag{3.1}$$

for certain constants a and b such that $a \neq 0$ and $b > 0$, then the transformation is called a D-homothetic deformation on M.

PROPOSITION 3.1. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on an $(LCS)_n$-manifold M is transformed into (ϕ', ξ', η', g') under a D-homothetic deformation, then*

$$\text{(a)}\ \eta'(\xi') = -1, \quad \text{(b)}\ g'(X, \xi') = \eta'(X). \tag{3.2}$$

PROOF. (3.2) follows from (3.1), (2.3) and (2.9).

LEMMA 3.1. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on an $(LCS)_n$-manifold M is transformed into (ϕ', ξ', η', g') under a D-homothetic deformation, then*

$$\nabla'_X Y = \nabla_X Y - \frac{(a^2 - b)\alpha}{a^2}\{g(X, Y) + \eta(X)\eta(Y)\}\xi, \tag{3.3}$$

where ∇ and ∇' are the Levi-Civita connections of g and g' respectively.

PROOF. Using Koszul formula for ∇' we get

$$\begin{aligned} 2g'(\nabla'_X Y, Z) = {} & Xg'(Y, Z) + Yg'(Z, X) - Zg'(X, Y) + g'([X, Y], Z) \\ & - g'([Y, Z], X) + g'([Z, X], Y) \end{aligned} \tag{3.4}$$

for all X, Y and Z on M.

In view of (3.1), (2.14) and (2.15), (3.4) yields

$$\begin{aligned} & bg(\nabla'_X Y, Z) - (a^2 - b)\eta(\nabla'_X Y)\eta(Z) \\ & \quad = bg(\nabla_X Y, Z) - (a^2 - b)\eta(Z)\{\eta(\nabla_X Y) + (\nabla_X \eta)(Y)\}. \end{aligned} \tag{3.5}$$

Setting $Z = \xi$ in (3.5), we get

$$\eta(\nabla'_X Y) = \eta(\nabla_X Y) + \frac{a^2 - b}{a^2}(\nabla_X \eta)(Y). \tag{3.6}$$

Using (3.6) and (2.4) in (3.5), we obtain (3.3).

Lemma 3.2. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on an $(LCS)_n$-manifold M is transformed into (ϕ', ξ', η', g') under a D-homothetic deformation, then*

$$\nabla'_X \xi' = \frac{1}{a}\nabla_X \xi, \tag{3.7}$$

$$(\nabla'_X \eta')(Y) = \frac{b}{a}(\nabla_X \eta)(Y). \tag{3.8}$$

Proof. From (3.3) we have

$$\nabla'_X \xi = \nabla_X \xi. \tag{3.9}$$

Then in view of (3.1) and (3.9), we have (3.7). Now using (3.7) we can easily prove (3.8).

Proposition 3.2. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on an $(LCS)_n$-manifold M is transformed into (ϕ', ξ', η', g') under a D-homothetic deformation, then*

$$(\nabla'_X \eta')(Y) = \alpha'\{g'(X, Y) + \eta'(X)\eta'(Y)\}, \tag{3.10}$$

where $\alpha' = \frac{\alpha}{a}$ is a non-zero scalar function such that

$$\nabla'_X \alpha' = (X\alpha') = d\alpha'(X) = \rho'\eta'(X), \tag{3.11}$$

ρ' being a certain scalar function given by $\rho' = -(\xi'\alpha')$.

Proof. By virtue of (2.4), (3.8) yields

$$(\nabla'_X \eta')(Y) = \frac{b\alpha}{a}\{g(X, Y) + \eta(X)\eta(Y)\}. \tag{3.12}$$

On the other hand, from (3.1) we have

$$g'(X, Y) + \eta'(X)\eta'(Y) = b\{g(X, Y) + \eta(X)\eta(Y)\}. \tag{3.13}$$

Hence in view of (3.13) and (3.12), we obtain (3.10). Again, since $\alpha' = \frac{\alpha}{a}$ and a is a constant, in view of (2.5), (3.11) holds where $\rho' = \frac{\rho}{a^2} = -(\xi'\alpha')$.

PROPOSITION 3.3. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on an $(LCS)_n$-manifold M is transformed into (ϕ', ξ', η', g') under a D-homothetic deformation, then*

$$\nabla'_X \xi' = \alpha' \phi' X, \quad \alpha' = \frac{\alpha}{a}, \tag{3.14}$$

$$\phi' X = X + \eta'(X)\xi'. \tag{3.15}$$

PROOF. In view of (2.6) and (3.1), (3.7) gives us (3.14). Also from (3.1) and (2.7), we obtain (3.15).

THEOREM 3.1. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on an $(LCS)_n$-manifold M is transformed into (ϕ', ξ', η', g') under a D-homothetic deformation, then (ϕ', ξ', η', g') is also a Lorentzian concircular structure on M.*

PROOF. By the above propositions and lemmas it follows that (ϕ', ξ', η', g') is a Lorentzian concircular structure on M.

COROLLARY 3.1. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on an LP-Sasakian manifold M is transformed into (ϕ', ξ', η', g') under a D-homothetic deformation, then (ϕ', ξ', η', g') is not an LP-Sasakian structure on M.*

THEOREM 3.2. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on an $(LCS)_n$-manifold M is transformed into (ϕ', ξ', η', g') under a D-homothetic deformation, then the curvature tensors R and R' with respect to the metric g and g' are related by*

$$\begin{aligned} R'(X, Y)Z = R(X, Y)Z - \frac{(a^2 - b)\alpha^2}{a^2}[&\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \{\eta(Y)X - \eta(X)Y\}\eta(Z)]. \end{aligned} \tag{3.16}$$

PROOF. For the curvature tensor R and R' we have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \tag{3.17}$$

$$R'(X, Y)Z = \nabla'_X \nabla'_Y Z - \nabla'_Y \nabla'_X Z - \nabla'_{[X, Y]} Z, \tag{3.18}$$

where ∇ and ∇' are Levi-Civita connection for g and g' respectively.

Using (2.4), (3.3), (3.9) and (3.17) in (3.18), we have

$$R'(X,Y)Z = R(X,Y)Z - \frac{(a^2-b)\alpha}{a^2}[\alpha\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi + \{g(Y,Z)+\eta(Y)\eta(Z)\}\nabla_X\xi - \{g(X,Z)+\eta(X)\eta(Z)\}\nabla_Y\xi]. \tag{3.19}$$

Using (2.6) and (2.7) in (3.19), we obtain (3.16).

THEOREM 3.3. *Under a D-homothetic deformation an η-Einstein $(LCS)_n$-manifold is invariant.*

PROOF. If an $(LCS)_n$-manifold M with the structure (ϕ,ξ,η,g) is η-Einstein, then the Ricci tensor S satisfies the relation

$$S(Y,Z) = lg(Y,Z) + m\eta(Y)\eta(Z) \tag{3.20}$$

where l and m are smooth functions given by ([14]) $l = \frac{r}{n-1} - (\alpha^2-\rho)$ and $m = \frac{r}{n-1} - n(\alpha^2-\rho)$. Now from (3.16) we have

$$S'(Y,Z) = S(Y,Z) - \frac{(a^2-b)\{(n-3)\alpha^2 b + 2a^2\rho\}}{a^2 b} \times \{g(Y,Z)+\eta(Y)\eta(Z)\}. \tag{3.21}$$

In view of (3.1) and (3.20), (3.21) yields

$$S'(Y,Z) = l'g'(Y,Z) + m'\eta'(Y)\eta'(Z)$$

where

$$l' = \frac{l}{b} - \frac{(a^2-b)\{(n-3)\alpha^2 b + 2a^2\rho\}}{a^2 b^2}$$

and

$$m' = \frac{1}{a^2}\left[m + \frac{l(a^2-b)}{b} - \frac{(a^2-b)\{(n-3)\alpha^2 b + 2a^2\rho\}}{b^2}\right].$$

This completes the proof.

DEFINITION 3.2. *A plane section of the tangent space $T_x(M)$ is called a ϕ-section if there exists a unit vector X in $T_x(M)$ orthogonal to ξ such that*

$\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature $K(X, \phi X) = g(R(X, \phi X)\phi X, X)$ is called a ϕ-sectional curvature.

THEOREM 3.4. *Under a D-homothetic deformation the ϕ-sectional curvature of an $(LCS)_n$-manifold M is conformal.*

PROOF. In view of (3.1) and (2.11), (3.16) yields

$$
\begin{aligned}
(3.22)\qquad & R'(X, Y, Z, W) \\
&= bR(X, Y, Z, W) - \frac{b(a^2-b)\alpha^2}{a^2}[\{g(Y, Z)g(X, W) \\
&\quad - g(X, Z)g(Y, W)\} + \{g(X, W)\eta(Y) - g(Y, W)\eta(X)\}\eta(Z)] \\
&\quad + \frac{(a^2-b)(a^2\rho - b\alpha^2)}{a^2}\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\eta(W).
\end{aligned}
$$

Now if X is a non-zero unit vector tangent to $M(\phi', \xi', \eta', g')$ and orthogonal to ξ', then (3.22) entails

$$
K'(X, \phi' X) = \frac{1}{b}K(X, \phi X). \tag{3.23}
$$

This completes the proof.

4. Infinitesimal CL-transformation in an $(LCS)_n$-manifold

DEFINITION 4.1. *A vector field V in an $(LCS)_n$-manifold M is said to be an infinitesimal CL-transformation* [8] *if it satisfies*

$$
\pounds_V\left\{{h \atop ji}\right\} = \mu_j\delta_i^h + \mu_i\delta_j^h + a(\eta_j\phi_i^h + \eta_i\phi_j^h) + b\phi_{ji}\xi^h, \quad \phi_{ji} = \phi_j^l g_{li} \tag{4.1}
$$

for certain constants a and b, where μ_i are components of the 1-form μ, $\pounds_V$ denotes the Lie derivative with respect to V and $\left\{{h \atop ji}\right\}$ is the Christoffel symbol of the Lorentzian metric g.

PROPOSITION 4.1. *If V is an infinitesimal CL-transformation on an $(LCS)_n$-manifold, then the 1-form μ is closed.*

PROOF. From (4.1) and (2.7) we have

$$
\nabla_j\nabla_i V^h + R^h_{kji}V^k = (\mu_j + a\eta_j)\delta_i^h + (\mu_i + a\eta_i)\delta_j^h + (2a+b)\eta_j\eta_i\xi^h + bg_{ji}\xi^h. \tag{4.2}
$$

Contracting h and i in (4.2) we get

$$\nabla_i \nabla_l V^l = (n+1)\mu_i + a(n-1)\eta_i,$$

which yields by virtue of (2.4)

$$\nabla_j \nabla_i \nabla_l V^l = (n+1)\nabla_j \mu_i + a(n-1)\alpha(g_{ji} + \eta_j \eta_i). \tag{4.3}$$

Taking skew-symmetric part of (4.3) we get the result.

Theorem 4.1. *If V is an infinitesimal CL-transformation on an $(LCS)_n$-manifold M, then the relation*

$$\begin{aligned}(\alpha^2 - \rho)(\pounds_V g)(Y, Z) = {} & -(\nabla_Y \mu)(Z) + \{\alpha(a+b) - (2\alpha\rho - \beta)\eta(V)\}g(Y, Z) \\ & + \alpha(3a+b)\eta(Y)\eta(Z)\end{aligned} \tag{4.4}$$

holds for any vector fields Y and Z on M.

Proof. We know from [29] that

$$\pounds_V R^h_{kji} = \nabla_k \pounds_V \{^{\,h}_{ji}\} - \nabla_j \pounds_V \{^{\,h}_{ki}\}. \tag{4.5}$$

Substituting (4.1) in (4.5) and then using (2.4), (2.6) and (2.7), we obtain

$$\begin{aligned}(\pounds_V R)(X, Y)Z = {} & (\nabla_X \mu)(Z)Y - (\nabla_Y \mu)(Z)X \\ & + \alpha(a-b)\{g(X, Z)Y - g(Y, Z)X\} \\ & + \alpha(a+b)\{\eta(Y)X - \eta(X)Y\}\eta(Z) \\ & + 2a\alpha\{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\}\xi.\end{aligned} \tag{4.6}$$

Applying η on (4.6) we get

$$\begin{aligned}\eta((\pounds_V R)(X, Y)Z) = {} & (\nabla_X \mu)(Z)\eta(Y) - (\nabla_Y \mu)(Z)\eta(X) \\ & + \alpha(a+b)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}.\end{aligned} \tag{4.7}$$

Taking Lie derivative of (2.11) with respect to V and using (4.7) and then setting $Y = \xi$, we get

$$\begin{aligned}&(\alpha^2 - \rho)(\pounds_V g)(Y, Z) \\ &= -(\nabla_Y \mu)(Z) - \{(\nabla_\xi \mu)(Z) + (\alpha^2 - \rho)(\pounds_V g)(\xi, Z)\}\eta(Y) \\ &\quad + \{\alpha(a+b) - (2\alpha\rho - \beta)\eta(V)\}[g(Y, Z) + \eta(Y)\eta(Z)].\end{aligned} \tag{4.8}$$

Interchanging Y and Z in (4.8) and then subtracting from (4.8), we get

$$\{(\nabla_\xi \mu)(Z) + (\alpha^2 - \rho)(\pounds_V g)(\xi, Z)\}\eta(Y) = \{(\nabla_\xi \mu)(Y) + (\alpha^2 - \rho)(\pounds_V g)(\xi, Y)\}\eta(Z). \tag{4.9}$$

Putting $Y = \xi$ in (4.9) we obtain by virtue of (4.8)

$$\begin{aligned}(\alpha^2 - \rho)(\pounds_V g)(Y, Z) &= -(\nabla_Y \mu)(Z) + \{(\nabla_\xi \mu)(\xi) - (\alpha^2 - \rho)2\eta(\pounds_V \xi)\}\eta(Y)\eta(Z) \\ &\quad + \{\alpha(a + b) - (2\alpha\rho - \beta)\eta(V)\}[g(Y, Z) + \eta(Y)\eta(Z)].\end{aligned} \tag{4.10}$$

Now taking inner product of (4.6) with W and then contracting X and W, we get

$$\begin{aligned}(\pounds_V S)(Y, Z) &= -(n - 1)(\nabla_Y \mu)(Z) + \alpha\{(n + 1)a + (n - 1)b\}\eta(Y)\eta(Z) \\ &\quad - \alpha\{(n - 3)a - (n - 1)b\}g(Y, Z).\end{aligned} \tag{4.11}$$

Setting $Y = \xi$ in (4.11), we have

$$(\pounds_V S)(\xi, Z) = -(n - 1)\{(\nabla_\xi \mu)(Z) + 2a\alpha\eta(Z)\}. \tag{4.12}$$

Taking Lie derivative of (2.12) with respect to V and using (4.12) and then setting $Z = \xi$, we obtain

$$(\nabla_\xi \mu)(\xi) - (\alpha^2 - \rho)2\eta(\pounds_V \xi) = 2a\alpha + (2\alpha\rho - \beta)\eta(V). \tag{4.13}$$

Using (4.13) in (4.10), we obtain (4.4). This completes the proof.

In an $(LCS)_n$-manifold if we take $\alpha = 1$, then $\rho = 0$ and hence the manifold is LP-Sasakian. Thus we have

COROLLARY 4.1 [8]. *If V is an infinitesimal CL-transformation on an LP-Sasakian manifold, then the relation*

$$(\pounds_V g)(Y, Z) = -(\nabla_Y \mu)(Z) + (a + b)g(Y, Z) + (3a + b)\eta(Y)\eta(Z) \tag{4.14}$$

holds.

From (4.4) we can state the following:

THEOREM 4.2. *An infinitesimal CL-transformation V on an $(LCS)_n$-manifold is a Killing vector field if and only if*

$$(\nabla_Y \mu)(Z) = \{\alpha(a + b) - (2\alpha\rho - \beta)\eta(V)\}g(Y, Z) + \alpha(3a + b)\eta(Y)\eta(Z). \tag{4.15}$$

Corollary 4.2. *If an infinitesimal CL-transformation V on an $(LCS)_n$-manifold is a Killing vector field such that μ is codirectional with η, then μ is concircular.*

A vector field Z on M is said to be conformal Killing [30] if $(\pounds_Z g)(X, Y) = \sigma g(X, Y)$, where σ is a scalar. By virtue of (4.4), this leads to the following:

Theorem 4.3. *An infinitesimal CL-transformation V on an $(LCS)_n$-manifold is a conformal Killing vector field if and only if*

$$(\nabla_Y \mu)(Z) = \{\alpha(a+b) - (\alpha^2 - \rho)\sigma - (2\alpha\rho - \beta)\eta(V)\}g(Y,Z) + \alpha(3a+b)\eta(Y)\eta(Z). \tag{4.16}$$

5. CL-transformation on an $(LCS)_n$-manifold

Definition 5.1. *A transformation on an $(LCS)_n$-manifold M, $n > 3$, with structure (ϕ, ξ, η, g) is said to be a CL-transformation* [8] if the Levi-Civita connection ∇ is transformed into a symmetric affine connection $\bar{\nabla}$ such that

$$\bar{\nabla}_X Y = \nabla_X Y + \mu(X)Y + \mu(Y)X + c\{\eta(X)\phi Y + \eta(Y)\phi X\} + 2g(\phi X, Y)\xi, \tag{5.1}$$

where μ is the associated 1-form and c is a constant.

Throughout the section '–' represents the geometric objects with respect to the symmetric affine connection $\bar{\nabla}$ and other notations have their usual meaning. Also throughout the section 5 and 6, we will assume an $(LCS)_n$-manifold M with $n > 3$.

In view of (2.7), (5.1) yields

$$\bar{\nabla}_X Y = \nabla_X Y + \{\mu(X) + c\eta(X)\}Y + \{\mu(Y) + c\eta(Y)\}X + 2(c+1)\eta(X)\eta(Y)\xi + 2g(X, Y)\xi. \tag{5.2}$$

If a symmetric affine connection $\bar{\nabla}$ is related with the Levi-Civita connection ∇ on an $(LCS)_n$-manifold M by a CL-transformation, then by virtue of (5.2), (2.4), (2.6) and (2.7), the curvature tensor $\bar{R}(X, Y)Z$ of the connection $\bar{\nabla}$ is given by

$$\bar{R}(X, Y)Z = R(X, Y)Z + \{P(X, Y) - P(Y, X)\}Z + P(X,Z)Y - P(Y,Z)X - 2c(\alpha+2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \tag{5.3}$$

for any vector fields X, Y, Z on M, where the tensor field $P(X, Y)$ is defined by

$$P(X, Y) = (\nabla_X \mu)(Y) + [c(\alpha + 2) - 2(\alpha + \mu(\xi))]g(X, Y) + [(c+2)(c-\alpha) - 2\mu(\xi)(c+1)]\eta(X)\eta(Y) - \mu(X)\mu(Y) - c\{\mu(X)\eta(Y) + \eta(X)\mu(Y)\}. \tag{5.4}$$

Proposition 5.1. *In an $(LCS)_n$-manifold M, the tensor field $P(X, Y)$ is symmetric if and only if the 1-form μ is closed.*

Proof. Interchanging X and Y in (5.4) and then subtracting from (5.4), we get the result.

A symmetric affine connection $\bar{\nabla}$ on M is said to be flat if the corresponding curvature tensor $\bar{R}$ vanishes identically on M.

Proposition 5.2. *In an $(LCS)_n$-manifold M, if the symmetric affine connection $\bar{\nabla}$ is flat, then the tensor field $P(X, Y)$ is symmetric.*

Proof. If the symmetric affine connection $\bar{\nabla}$ is flat, then from (5.3) we have

$$R(X, Y)Z = \{P(Y, X) - P(X, Y)\}Z - P(X, Z)Y + P(Y, Z)X + 2c(\alpha + 2)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi. \tag{5.5}$$

From (5.5), it follows that

$$S(Y, Z) = nP(Y, Z) - P(Z, Y) - 2c(\alpha + 2)\{g(Y, Z) + \eta(Y)\eta(Z)\}. \tag{5.6}$$

Interchanging Y and Z in (5.6) and then subtracting from (5.6), we get $P(Y, Z) = P(Z, Y)$. This completes the proof.

The Weyl conformal curvature tensor C of type (1,3) of an n-dimensional Riemannian manifold M, $n > 3$, is given by [27]

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y], \tag{5.7}$$

where R, S, Q, r denote respectively the curvature tensor, the Ricci tensor of type (0,2), the Ricci operator and the scalar curvature of the manifold. The manifold

M is said to be conformally flat if the conformal curvature tensor C vanishes identically on M.

Definition 5.2. *A semi-Riemannian manifold is said to be a manifold of quasi-constant curvature* [5] *if it is conformally flat and its curvature tensor R of type* (0,4) *is of the form*

$$R(X,Y,Z,U) = p\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} + q\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) + g(X,U)A(Y)A(Z) - g(Y,U)A(X)A(Z)\} \tag{5.8}$$

for any vector fields X, Y, Z and U on M, where p and q are scalars such that $q \neq 0$ and A is a non-zero 1*-form*. If $q = 0$, then the manifold reduces to a manifold of constant curvature.

It is easy to check that if the curvature tensor R is of the form (5.8), then the manifold is conformally flat. Hence a semi-Riemannian manifold is a manifold of quasi-constant curvature only if its curvature tensor is of the form (5.8). Thus a manifold of quasi-constant curvature is conformally flat, but the converse is not true, in general. However, the converse is true if the manifold is quasi-Einstein. We also note that, in [26], Vranceanu defined the notion of almost constant curvature by the same expression of (5.8). However, Mocanu [10] showed that both the notions of almost constant curvature by Vranceanu [26] and quasi-constant curvature by Chen and Yano [5] are the same.

Theorem 5.1. *If the Levi-Civita connection ∇ on an $(LCS)_n$-manifold M is transformed into a symmetric affine connection $\bar{\nabla}$ by a CL-transformation such that $\bar{\nabla}$ is flat, then M is of quasi-constant curvature.*

Proof. Since the connection $\bar{\nabla}$ on M is flat, on account of Proposition 5.2, (5.5) and (5.6) turns into

$$R(X,Y)Z = P(Y,Z)X - P(X,Z)Y + 2c(\alpha+2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi, \tag{5.9}$$

$$S(Y,Z) = (n-1)P(Y,Z) - 2c(\alpha+2)\{g(Y,Z) + \eta(Y)\eta(Z)\}. \tag{5.10}$$

Taking inner product of (5.9) with U and then using (5.10), (5.9) yields

$$(5.11)\qquad R(X,Y,Z,U)=\frac{1}{n-1}\{S(Y,Z)g(X,U)-S(X,Z)g(Y,U)\}$$
$$+\frac{2c(\alpha+2)}{n-1}[\{g(Y,Z)+\eta(Y)\eta(Z)\}g(X,U)$$
$$-\{g(X,Z)+\eta(X)\eta(Z)\}g(Y,U)]$$
$$+2c(\alpha+2)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\eta(U).$$

Using (5.11) in the relation $R(X,Y,Z,U)+R(X,Y,U,Z)=0$ and then setting $X=U=\xi$ and using (2.13), we obtain

$$(5.12)\qquad \frac{1}{n-1}[S(Y,Z)-2c(\alpha+2)(n-2)\{g(Y,Z)+\eta(Y)\eta(Z)\}]$$
$$=(\alpha^2-\rho)g(Y,Z).$$

In view of (5.12), (5.11) yields

$$(5.13)\qquad R(X,Y,Z,U)$$
$$=\{(\alpha^2-\rho)+2c(\alpha+2)\}\{g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\}$$
$$+2c(\alpha+2)\{g(Y,Z)\eta(X)\eta(U)-g(X,Z)\eta(Y)\eta(U)$$
$$+g(X,U)\eta(Y)\eta(Z)-g(Y,U)\eta(X)\eta(Z)\}.$$

Moreover, from (5.7), (5.12) and (5.13), it can be easily seen that the manifold M is conformally flat and hence the manifold M is of quasi-constant curvature.

THEOREM 5.2. *If the Levi-Civita connection ∇ on an $(LCS)_n$-manifold M is transformed into a symmetric affine connection $\bar{\nabla}$ by a CL-transformation such that its associated 1-form μ is closed, then a tensor field A of type (1,3) is invariant under the CL-transformation, where A is given by*

$$(5.14)\qquad A(X,Y)Z=R(X,Y)Z-\frac{1}{n-2}[\{S(Y,Z)X-S(X,Z)Y\}$$
$$+\{S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\}\xi]$$
$$+\frac{(\alpha^2-\rho)}{n-2}[\{g(Y,Z)X-g(X,Z)Y\}$$
$$+(n-1)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi].$$

Proof. Since the associated 1-form μ of the transformation is closed, taking account of Proposition 5.1, (5.3) can be written as

$$\bar{R}(X,Y)Z = R(X,Y)Z + P(X,Z)Y - P(Y,Z)X - 2c(\alpha+2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi, \tag{5.15}$$

which yields

$$(n-1)P(Y,Z) = S(Y,Z) - \bar{S}(Y,Z) + 2c(\alpha+2)\{g(Y,Z)) + \eta(Y)\eta(Z)\}. \tag{5.16}$$

Inserting (5.16) in (5.15), we obtain

$$\begin{aligned}\bar{H}(X,Y)Z = H(X,Y)Z &- 2c(\alpha+2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \\ &- \frac{2c(\alpha+2)}{n-1}[\{g(Y,Z)) + \eta(Y)\eta(Z)\}X \\ &- \{g(X,Z)) + \eta(X)\eta(Z)\}Y],\end{aligned} \tag{5.17}$$

where we put

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}. \tag{5.18}$$

Setting $X = \xi$ in (5.17) and then applying η, we obtain

$$\eta(\bar{H}(\xi,Y)Z) - \eta(H(\xi,Y)Z) = -\frac{2c(\alpha+2)(n-2)}{n-1}\{g(Y,Z) + \eta(Y)\eta(Z)\}. \tag{5.19}$$

Using (5.19) in (5.17), we have

$$\bar{T}(X,Y)Z = T(X,Y)Z - 2c(\alpha+2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi, \tag{5.20}$$

where we put

$$T(X,Y)Z = H(X,Y)Z - \frac{1}{n-2}\{\eta(H(\xi,Y)Z)X - \eta(H(\xi,X)Z)Y\}. \tag{5.21}$$

From (5.20), it follows that

$$g(Y,Z) + \eta(Y)\eta(Z) = \frac{1}{2c(\alpha+2)}\{\bar{T}(Y,Z) - T(Y,Z)\}, \tag{5.22}$$

where

$$T(Y,Z) = \sum_{i=1}^{n} \varepsilon_i g(T(e_i,Y)Z, e_i), \quad \varepsilon_i = g(e_i,e_i),$$

$\{e_i : i = 1, 2, \ldots, n\}$ being an orthonormal frame of the tangent space at any point of the manifold.

Substituting (5.22) in (5.20), we obtain

$$\bar{A}(X, Y)Z = A(X, Y)Z, \tag{5.23}$$

where the tensor field A is defined by

$$A(X, Y)Z = T(X, Y)Z + \{T(Y, Z)\eta(X) - T(X, Z)\eta(Y)\}\xi. \tag{5.24}$$

Hence the tensor field A is invariant. Using (5.18), (5.21), (2.12) and (2.13) in (5.24) we get (5.14). This completes the proof.

The invariant tensor field A on an $(LCS)_n$-manifold M obtained under a CL-transformation is said to be the CL-curvature tensor field on M.

Theorem 5.3. *In an $(LCS)_n$-manifold M, the CL-curvature tensor field remains invariant under a D-homothetic deformation if and only if the deformation is homothetic.*

Proof. From Theorem 3.1, it follows that under a D-homothetic deformation defined by (3.1) an $(LCS)_n$-manifold $M_\alpha(\phi, \xi, \eta, g)$ is again an $(LCS)_n$-manifold $M_{\alpha'}(\phi', \xi', \eta', g')$, where $\alpha' = \frac{\alpha}{a}$. Hence from (5.14), the CL-curvature tensor field on $M_{\alpha'}(\phi', \xi', \eta', g')$ can be written as

$$\begin{aligned} A'(X, Y)Z = R'(X, Y)Z - \frac{1}{n-2}[&\{S'(Y, Z)X - S'(X, Z)Y\} \\ &+ \{S'(Y, Z)\eta'(X) - S'(X, Z)\eta'(Y)\}\xi'] \\ &+ \frac{(\alpha'^2 - \rho')}{n-2}[\{g'(Y, Z)X - g'(X, Z)Y\} \\ &+ (n-1)\{g'(Y, Z)\eta'(X) - g'(X, Z)\eta'(Y)\}\xi'], \end{aligned} \tag{5.25}$$

where ρ' is a scalar such that $\rho' = -(\xi'\alpha')$.

Using (3.1), (3.16), (3.20) and (5.14) in (5.25), we obtain

$$\begin{aligned} A'(X, Y)Z - A(X, Y)Z = \frac{(a^2 - b)\rho}{a^2(n-2)}[&\{g(Y, Z)X - g(X, Z)Y\} \\ &+ (n-1)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \\ &+ \{\eta(Y)X - \eta(X)Y\}\eta(Z)]. \end{aligned} \tag{5.26}$$

Now we assume that the CL-curvature tensor field A remains invariant under a D-homothetic deformation. Then right hand side of (5.26) is equal to zero, which yields, for $X = \xi$,

$$(a^2 - b)\rho = 0, \tag{5.27}$$

which implies that either $(a^2 - b) = 0$ or $\rho = 0$. If $\rho = 0$, then $(X\alpha) = 0$ and hence α is constant, which is inadmissible. Thus we must have $(a^2 - b) = 0$ and hence the deformation is homothetic.

Next we suppose that the deformation is homothetic, that is, $a^2 = b$. Hence the right hand side of (5.26) is equal to zero. Therefore $A' = A$.

6. CL-flat and CL-symmetric $(LCS)_n$-manifold

Definition 6.1. *An $(LCS)_n$-manifold M is said to be CL-flat if the CL-curvature tensor field A of type* (1,3) *vanishes identically on M.*

We mention that CL-flat manifold was introduced by Koto and Nagao in [6] for a Sasakian manifold.

Theorem 6.1. *A CL-flat $(LCS)_n$-manifold M is an η-Einstein manifold if $r \neq n(n-1)(\alpha^2 - \rho)$.*

Proof. Let M be a CL-flat $(LCS)_n$-manifold. Then from (5.14) we have

$$\begin{aligned} R(X,Y)Z = {} & \frac{1}{n-2}[\{S(Y,Z)X - S(X,Z)Y\} \\ & + \{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}\xi] \\ & - \frac{(\alpha^2 - \rho)}{n-2}[\{g(Y,Z)X - g(X,Z)Y\} \\ & + (n-1)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi]. \end{aligned} \tag{6.1}$$

Taking inner product of (6.1) with U and then contracting over Y and Z, we get

$$S(X,U) = \left\{\frac{r}{n-1} - (\alpha^2 - \rho)\right\}g(X,U) + \left\{\frac{r}{n-1} - n(\alpha^2 - \rho)\right\}\eta(X)\eta(U), \tag{6.2}$$

where r is the scalar curvature of the manifold. Thus the manifold is η-Einstein. This completes the proof.

COROLLARY 6.1. *If* $r = n(n-1)(\alpha^2 - \rho)$ *then a CL-flat* $(LCS)_n$*-manifold* M *is an Einstein manifold.*

COROLLARY 6.2. *A CL-flat LP-Sasakian manifold is an* η*-Einstein.*

COROLLARY 6.3. *In a CL-flat* $(LCS)_n$*-manifold* M*, the scalar curvature of the manifold is constant if and only if* $2\alpha\rho - \beta = 0$.

PROOF. The result follows from Theorem 3.2 of [14] and Theorem 6.1.

THEOREM 6.2. *A CL-flat* $(LCS)_n$*-manifold* M *is a manifold of quasi-constant curvature if* $r \neq n(n-1)(\alpha^2 - \rho)$.

PROOF. Let M be a CL-flat $(LCS)_n$-manifold. Then (6.1) and (6.2) holds on M. Inserting (6.2) in (6.1) and then taking inner product with U, we obtain

$$
\begin{aligned}
R(X,Y,Z,U) &= p\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \\
&\quad + q\{g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U) \\
&\quad + g(X,U)\eta(Y)\eta(Z) - g(Y,U)\eta(X)\eta(Z)\},
\end{aligned}
\tag{6.3}
$$

where $p = \frac{1}{n-2}\left\{\frac{r}{n-1} - 2(\alpha^2 - \rho)\right\}$ and $q = \frac{1}{n-2}\left\{\frac{r}{n-1} - n(\alpha^2 - \rho)\right\}$. Also in view of (6.1) and (6.2), it is clear from (5.7) that the manifold is conformally flat. Hence the manifold is of quasi-constant curvature. This completes the proof.

COROLLARY 6.4. *If* $r = n(n-1)(\alpha^2 - \rho)$ *then a CL-flat* $(LCS)_n$*-manifold* M *is of constant curvature.*

DEFINITION 6.2. *An* $(LCS)_n$*-manifold* M *is said to be a CL-symmetric if* $(\nabla_U A)(X,Y)Z = 0$ *for all* X, Y, Z *and* U *on* M.

Differentiating (5.14) covariantly with respect to U, we obtain

$$
\begin{aligned}
(\nabla_U A)(X,Y)Z &= (\nabla_U R)(X,Y)Z - \frac{1}{n-2}[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y \\
&\quad + \{(\nabla_U S)(Y,Z)\eta(X) - (\nabla_U S)(X,Z)\eta(Y)\}\xi] \\
&\quad - \frac{\alpha}{n-2}[(S(Y,Z)\{g(X,U) + 2\eta(X)\eta(U)\} \\
&\quad - S(X,Z)\{g(Y,U) + 2\eta(Y)\eta(U)\})\xi \\
&\quad + \{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}U]
\end{aligned}
\tag{6.4}
$$

$$+\frac{2\alpha\rho-\beta}{n-2}[\{g(Y,Z)X-g(X,Z)Y\}$$
$$+(n-1)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi]\eta(U)$$
$$+\frac{(n-1)(\alpha^2-\rho)\alpha}{n-2}[(g(Y,Z)\{g(X,U)+2\eta(X)\eta(U)\}$$
$$-g(X,Z)\{g(Y,U)+2\eta(Y)\eta(U)\})\xi$$
$$+\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}U].$$

Theorem 6.3. *A CL-symmetric $(LCS)_n$-manifold M is an η-Einstein manifold.*

Proof. Let M be a CL-symmetric $(LCS)_n$-manifold. Then $(\nabla_U A)(X,Y)Z = 0$ for all X, Y, Z and U on M and hence (6.4) yields

$$(6.5)\qquad (\nabla_U S)(X,W)=\frac{dr(U)}{n-1}\{g(X,W)+\eta(X)\eta(W)\}$$
$$+\alpha\left\{\frac{r}{n-1}-n(\alpha^2-\rho)\right\}$$
$$\times\{g(X,U)\eta(W)+g(U,W)\eta(X)+2\eta(X)\eta(U)\eta(W)\}$$
$$-(2\alpha\rho-\beta)\{g(X,W)+n\eta(X)\eta(W)\}\eta(U).$$

Putting $W=\xi$ in (6.5), we obtain

$$(6.6)\qquad S(X,U)=\left\{\frac{r}{n-1}-(\alpha^2-\rho)\right\}g(X,U)+\left\{\frac{r}{n-1}-n(\alpha^2-\rho)\right\}\eta(X)\eta(U),$$

that is, the manifold is η-Einstein.

Theorem 6.4. *In a CL-symmetric $(LCS)_n$-manifold M, grad r is codirectional with the structure vector field ξ, r being the scalar curvature of the manifold.*

Proof. Contracting X and U in (6.5), we get

$$(6.7)\qquad (n-3)\,dr(X)=2\{dr(\xi)+(n-1)^2(2\alpha\rho-\beta)$$
$$+(n-1)\alpha r-n(n-1)^2(\alpha^2-\rho)\alpha\}\eta(X).$$

Setting $X = \xi$ in (6.7), we get

$$dr(\xi) = -2\{\alpha r + (n-1)(2\alpha\rho - \beta) - n(n-1)(\alpha^2 - \rho)\alpha\}. \tag{6.8}$$

In view of (6.8), (6.7) yields

$$dr(X) = -dr(\xi)\eta(X). \tag{6.9}$$

Thus the result follows from (6.9).

A semi-Riemannian manifold M is said to be locally symmetric due to Cartan [4] if it satisfies $\nabla R = 0$.

THEOREM 6.5. *A CL-symmetric $(LCS)_n$-manifold M is locally symmetric if and only if M is an Einstein manifold such that*

$$S(X, Y) = (n-1)(\alpha^2 - \rho)g(X, Y). \tag{6.10}$$

PROOF. First we suppose that a CL-symmetric $(LCS)_n$-manifold M is locally symmetric. Then from (6.4) we have

$$\begin{aligned} &(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y \\ &\quad = -\{(\nabla_U S)(Y,Z)\eta(X) - (\nabla_U S)(X,Z)\eta(Y)\}\xi \\ &\qquad - \alpha[\{S(Y,Z)\{g(X,U) + 2\eta(X)\eta(U)\} \\ &\qquad - S(X,Z)\{g(Y,U) + 2\eta(Y)\eta(U)\}\}\xi \\ &\qquad + \{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}U] \\ &\qquad + (2\alpha\rho - \beta)[\{g(Y,Z)X - g(X,Z)Y\} \\ &\qquad + (n-1)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi]\eta(U) \\ &\qquad + (n-1)(\alpha^2 - \rho)\alpha[\{g(Y,Z)\{g(X,U) + 2\eta(X)\eta(U)\} \\ &\qquad - g(X,Z)\{g(Y,U) + 2\eta(Y)\eta(U)\}\}\xi \\ &\qquad + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}U]. \end{aligned} \tag{6.11}$$

Taking inner product of (6.11) with W and then contracting X and W and using (2.13), we get

$$(\nabla_U S)(Y,Z) = 0. \tag{6.12}$$

In view of (6.12), (6.11) yields

$$
\begin{aligned}
(6.13)\qquad & \alpha S(Y,Z)\{g(X,U)+2\eta(X)\eta(U)\}\xi \\
&= \alpha[S(X,Z)\{g(Y,U)+2\eta(Y)\eta(U)\}\xi \\
&\quad - \{S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\}U] \\
&\quad + (2\alpha\rho-\beta)[\{g(Y,Z)X-g(X,Z)Y\} \\
&\quad + (n-1)\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi]\eta(U) \\
&\quad + (n-1)(\alpha^2-\rho)\alpha[\{g(Y,Z)\{g(X,U)+2\eta(X)\eta(U)\} \\
&\quad - g(X,Z)\{g(Y,U)+2\eta(Y)\eta(U)\}\}\xi \\
&\quad + \{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}U].
\end{aligned}
$$

Again, taking inner product of (6.13) with W and contracting X and U and then setting $Y=\xi$ and using (2.13), we obtain (6.10) and hence the manifold is Einstein.

Conversely, if a CL-symmetric $(LCS)_n$-manifold M is an Einstein manifold with the Ricci tensor given as (6.10), then (6.4) entails that M is locally symmetric.

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References

[1] Atceken, M., On geometry of submanifolds of $(LCS)_n$-manifolds, Int. J. Math. and Math. Sci., **2012**, 1–11, Hindawi Publishing Corporation, doi: 10.1155/2012/304647.
[2] Atceken, M., Some curvature properties of $(LCS)_n$-manifolds, Abstract and Applied Analysis, Hindawi Publishing Corporation, Vol. 2013, Article ID 380657.
[3] Atceken, M. and Hui, S. K., Contact warped product semi-slant submanifolds of $(LCS)_n$-manifolds, Acta Univ. Sapientiae Mathematica, **3**, 2 (2011), 212–224.
[4] Cartan, E., Sur une classe remarquable d'espaces de Riemannian, Bull. Soc. Math., France, **54** (1926), 214–264.
[5] Chen, B. Y. and Yano, K., Hypersurfaces of a conformally flat space, Tensor N. S., **26** (1972), 315–321.
[6] Koto, S. and Nagao, M., On an invariant tensor under a CL-transformation, Kōdai Math. Sem. Rep., **18** (1966), 87–95.
[7] Matsumoto, K., On Lorentzian almost paracontact manifolds, Bull. of Yamagata Univ., Nat. Sci., **12** (1989), 151–156.
[8] Matsumoto, K. and Mihai, I., On a certain transformation in a Lorentzian para-Sasakain manifold, Tensor, N. S., **47** (1988).
[9] Mihai, I. and Rosca, R., On Lorentzian P-Sasakain manifolds, Clasical Analysis, World Sci. Publi., Singapore, **1992**, 155–169.

[10] Mocanu, A. L., Les varietes a courbure quasi-constante de type Vrănceanu, Lucr. Conf. Nat. de Geom. Si Top. Trigoviste, 1987.
[11] Olszak, Z., Curvature properties of quasi-Sasakian manifolds, Tensor, N. S., **38** (1982), 19–28.
[12] O'Neill, B., Semi-Riemannian geometry with application to the relativity, Academic Press, New York, **1983**.
[13] Prakasha, D. G., On Ricci η-recurrent $(LCS)_n$-manifolds, Acta Universitatis Apulensis, **24** (2010), 109–118.
[14] Shaikh, A. A., On Lorentzian almost paracontact manifold with a structure of the concircular type, Kyungpook Math. J., **43** (2003), 305–314.
[15] Shaikh, A. A., Some results on $(LCS)_n$-manifolds, J. Korean Math. Soc., **46** (2009), No. 3, 449–461.
[16] Shaikh, A. A. and Baishya, K. K., On concircular structure spacetimes, J. of Math. and Stat., **1** (2005), 129–132.
[17] Shaikh, A. A. and Baishya, K. K., On concircular structure spacetimes II, American J. of Appl. Sci., **3** no. **4** (2006), 1790–1794.
[18] Shaikh, A. A., Baishya, K. K. and Eyasmin, S., On D-homothetic deformation of trans-Sasakian structure, Demonstratio Mathematica, **41**, No. 1, (2008), 171–188.
[19] Shaikh, A. A., Basu, T. and Eyasmin, S., On the existence of ϕ-recurrent $(LCS)_n$-manifolds, Extracta Mathematicae, **23(1)** (2008), 71–83.
[20] Shaikh, A. A., Basu, T. and Eyasmin, S., On locally ϕ-symmetric $(LCS)_n$-manifolds, Int. J. Pure Appl. Math., **48(8)** (2007), 1161–1170.
[21] Shaikh, A. A. and Binh, T. Q., On weakly symmetric $(LCS)_n$-manifolds, J. Adv. Math. Studies, **2** (2009), 103–118.
[22] Sreenivasa, G. T., Venkatesha and Bagewadi, C. S., Some results on $(LCS)_{2n+1}$-manifolds, Bull. of Math. Analysis. and Appl., **1**, Issue **3** (2009), 64–70.
[23] Tanno, S., The topology of contact Riemannian manifolds, Illinois J. Math., **12** (1968), 700–717.
[24] Takamatsu, K. and Mizusawa, H., On infinitesimal CL-transformations of normal contact metric spaces, Sci. Rep. Niigata Univ., Ser. A, **3** (1966), 31–40.
[25] Tashiro, Y. and Tachibana, S., On Fubinian and C-Fubinian manifold, Kōdai Math. Sem. Rep., **15** (1963), 176–183.
[26] Vrănceanu, G., Lecons des Geometrie Differential, **4**, Ed. de l'Academie, Bucharest, **1968**.
[27] Weyl, H., Reine infinitesimal geometrie, Math. Z., **2** (1918), 384–411.
[28] Yano, K., Concircular geometry, I–IV, Proc. Imp. Acad. Tokyo, **16** (1940), 195–200, 354–360, 442–448, 505–511.
[29] Yano, K., Differential geometry on complex and almost complex spaces, Pergamon Press, **1965**.
[30] Yano, K. and Kon, M., Structures on manifolds, World Sci. Publi., Singapore, **1984**.

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