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THE BOX TOPOLOGY OF INFINITE SIMPLICIAL COMPLEXES

By

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Abstract. In this paper, realizing an infinite simplicial complex K in the linear space $\mathbf{R}^{K^{(0)}}$ naturally, we investigate the box topology on $|K|$ inherited from $\mathbf{R}^{K^{(0)}}$ that is finer than the metric topology and coarser than the Whitehead (weak) topology.

The polyhedron $|K|$ of a simplicial complex K has typical two topologies, the *weak* (*Whitehead*) *topology* and the *metric topology*. In case K is locally finite, these topologies are identical. In this paper, we introduce a new topology which is finer than the metric topology and coarser than the weak topology in general.

The vertices of a simplicial complex K is denoted by $K^{(0)}$. Let $\mathbf{R}^{K^{(0)}}$ is the linear space of all real-valued functions defined on $K^{(0)}$ with the operations defined coordinate-wise. For each $v \in K^{(0)}$, let $\mathbf{e}_v \in \mathbf{R}^{K^{(0)}}$ be the unit vector defined by $\mathbf{e}_v(v) = 1$ and $\mathbf{e}_v(u) = 0$ if $u \neq v$. By $\mathbf{R}_{\mathrm{f}}^{K^{(0)}}$, we denote the linear subspace of $\mathbf{R}^{K^{(0)}}$ generated by $\mathbf{e}_v$, $v \in K^{(0)}$, i.e.,

$$\mathbf{R}_{\mathrm{f}}^{K^{(0)}} = \{x \in \mathbf{R}^{K^{(0)}} \mid x(v) = 0 \text{ except for finitely many } v \in K^{(0)}\}.$$

Identifying vertices $v \in K^{(0)}$ with $\mathbf{e}_v \in \mathbf{R}^{K^{(0)}}$, we can realize $|K|$ in $\mathbf{R}_{\mathrm{f}}^{K^{(0)}}$, that is, we can regard $|K| \subset \mathbf{R}_{\mathrm{f}}^{K^{(0)}}$. Thus, $|K|$ can inherit various topologies from $\mathbf{R}_{\mathrm{f}}^{K^{(0)}}$.

The linear space $\mathbf{R}_{\mathrm{f}}^{K^{(0)}}$ has the *norm* defined by

$$\|x\| = \sum_{v \in K^{(0)}} |x(v)|,$$

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which is inherited from the Banach space $\ell_1(K^{(0)})$. The metric topology of $|K|$ is induced by this norm. This topology is also induced by the norms

$$\|x\|_\infty = \sup_{v\in K^{(0)}} |x(v)|, \quad \|x\|_2 = \sqrt{\sum_{v\in K^{(0)}} x(v)^2}, \quad \text{etc.}^1$$

The linear space $\mathbf{R}_\mathrm{f}^{K^{(0)}}$ has the *product* (*Tychonoff*) *topology* and it is a locally convex topological linear space with respect to this topology.[2] Although the product topology on $\mathbf{R}_\mathrm{f}^{K^{(0)}}$ is not generated by any norm, the topology of $|K|$ inherited from the product topology coincides with the metric topology of $|K|$.

The weak topology of $|K|$ is no other than the relative (or subspace) topology of the *finite topology* of $\mathbf{R}_\mathrm{f}^{K^{(0)}}$ which is the weak topology determined by the Euclidean topology on each finite-dimensional linear subspace (cf. Appendix One, A.4.2 and B.5 in [2]). However, $\mathbf{R}_\mathrm{f}^{K^{(0)}}$ is not a topological linear space with respect to the finite topology. In fact, the addition is not continuous with respect to this topology [2, Appendix One, A.4.3].

By the way, remind that $\mathbf{R}_\mathrm{f}^{K^{(0)}}$ is a locally convex topological linear space with respect to the *box topology*, where the origin (the null element) $\mathbf{0} \in \mathbf{R}_\mathrm{f}^{K^{(0)}}$ has the open neighborhood basis consisting of the following sets:

$$\mathbf{R}_\mathrm{f}^{K^{(0)}} \cap \prod_{v\in K^{(0)}} (\varepsilon_v, -\varepsilon_v), \quad \varepsilon_v > 0, \quad v \in K^{(0)}.$$

The box topology is important in study of topology of LF spaces (see [3], [4]). The box topology of $\mathbf{R}_\mathrm{f}^{K^{(0)}}$ induces the new topology on $|K|$ which is also called the *box topology*.

By $|K|_\mathrm{m}$, $|K|_\mathrm{w}$ and $|K|_\mathrm{b}$ we denote the spaces $|K|$ with the metric topology, the weak topology and the box topology, respectively. As is easily observed, the identities $\mathrm{id} : |K|_\mathrm{w} \to |K|_\mathrm{b}$ and $\mathrm{id} : |K|_\mathrm{b} \to |K|_\mathrm{m}$ are continuous. In other words,

$$\text{the metric topology} \subset \text{the box topology} \subset \text{the weak topology}.$$

In case K is locally finite, these topologies are equal because the metric topology coincides with the weak topology. It will be shown that if K is locally countable or $\dim K \leq 1$, then the box topology of $|K|$ coincides with the weak topology, that is, $|K|_\mathrm{b} = |K|_\mathrm{w}$ as spaces (Theorem 1). Let J be a well-known 1-dimensional countable simplicial complex such that $|J|_\mathrm{m} \neq |J|_\mathrm{w}$ as spaces.[3] Then, $|J|_\mathrm{b} \neq |J|_\mathrm{m}$

[1] The topology induced by $\|\cdot\|_\infty$ is no other than the *uniform convergence topology*.
[2] The product topology is no other than the *point-wise convergence topology*.
[3] The space $|J|_\mathrm{m}$ is called the *hedgehog*.

as spaces (Corollary 1). We will construct a 2-dimensional simplicial complex B such that $|B|_{\mathrm{b}} \neq |B|_{\mathrm{w}}$ as spaces (Theorem 3).

In case K is a *full simplicial complex*, the space $|K|_{\mathrm{b}}$ is identified with the convex subspace of the locally convex topological linear space $\mathbf{R}_{\mathrm{f}}^{K^{(0)}}$ with the box topology. Then, it follows from the Dugundji Extension Theorem [1] (cf. Section 5.1 of [5]) that $|K|_{\mathrm{b}}$ is an absolute extensor for metrizable spaces. We will prove that for an arbitrary simplicial complex K, the space $|K|_{\mathrm{b}}$ is an absolute neighborhood extensor for metrizable spaces (Theorem 2).

The subdivision preserves the weak topology but does not the metric topology. On the other hand, the product preserves the metric topology but does not the weak topology. Like this, these topologies have good and bad points. The box topology is intermediate between the weak topology and the metric topology. Besides, it is inherited from a locally convex topological linear space. Then, we expected for this new topology to inherit both good points from the weak and the metric topologies. But, against this expectation, the box topology is worse than these topologies. It will be shown that even the barycentric subdivision does not preserve the box topology (Theorem 5) and that a simplicial map does not need to be continuous with respect to the box topology (Theorem 7).

As for the continuity of simplicial maps with respect to the box topology, it can be proved that if K is locally countable or locally finite-dimensional, every simplicial map $f : K \to L$ is continuous with respect to the box topologies (Theorem 6). A simplicial map $f : K \to L$ is *proper* if $f^{-1}(v) \cap K^{(0)}$ $(= (f \,|\, |K^{(0)}|)^{-1}(v))$ is finite for each $v \in L^{(0)}$. We will also show that every proper simplicial map $f : K \to L$ is continuous with respect to the box topologies (Theorem 6).

For infinte simplicial complexes, refer to [5, Chapter 3].

1. Comparison between Topologies

By $\mathbf{N}$ and ω, we denote the positive integers and the non-negative integers, respectively. Namely, $\omega = \mathbf{N} \cup \{0\}$. Let K be a simplicial complex. The vertices of a simplex $\sigma \in K$ is denoted by $\sigma^{(0)}$. The boundary of σ is denoted by $\partial\sigma$ and $\operatorname{rint} \sigma = \sigma \setminus \partial\sigma$ is the radial interior of σ. For each $x \in |K|$, let $c_K(x)$ be the smallest simplex of K containing x, which is called the *carrier* of x in K. Let $c_K(x)^{(0)} = \{v_1, \dots, v_n\}$. Then, we can find $t_1, \dots, t_n > 0$ such that $\sum_{i=1}^n t_i = 1$ and $x = \sum_{i=1}^n t_i v_i$. We define

$$\beta_v^K(x) = \begin{cases} t_i & \text{if } v = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $\beta^K : |K| \to \mathbf{R}_{\mathrm{f}}^{K^{(0)}} \subset \ell_1(K^{(0)})$ defined by $\beta^K(x) = (\beta_v^K(x))_{v \in K^{(0)}}$. The metric topology is induced by the following metric ρ_K:

$$\rho_K(x, y) = \sum_{v \in K^{(0)}} |\beta_v^K(x) - \beta_v^K(y)| = \|\beta^K(x) - \beta^K(y)\|.$$

For a subcomplex L of K, $x \in |L|$ if and only if $c_K(x) \in L$.

Regarding $\beta^K(|K|)$ as a subspace of the space $\mathbf{R}^{K^{(0)}}$ with the box topology, we introduce the new topology on $|K|$ and the space $|K|$ with this topology is denoted by $|K|_{\mathrm{b}}$. For each $x \in |K|$, $U \subset |K|$ is a neighborhood of x in $|K|_{\mathrm{b}}$ if and only if there are $\varepsilon_v > 0$, $v \in K^{(0)}$, such that

$$\{y \in |K| \mid \forall v \in K^{(0)},\ |\beta_v^K(x) - \beta_v^K(y)| < \varepsilon_v\} \subset U.$$

Proposition 1. *Let K be a simplicial complex and L a subcomplex of K. Then, $|L|_{\mathrm{b}}$ is a closed subspace of $|K|_{\mathrm{b}}$.*

Proof. We can identify

$$\mathbf{R}^{L^{(0)}} = \{x \in \mathbf{R}^{K^{(0)}} \mid \forall v \in K^{(0)} \setminus L^{(0)},\ x(v) = 0\} \subset \mathbf{R}^{K^{(0)}}.$$

Since the box topology of $\mathbf{R}^{L^{(0)}}$ is the relative topology of the box topology of $\mathbf{R}^{K^{(0)}}$, the topology of $|L|_{\mathrm{b}}$ is also the relative topology of the box topology of $\mathbf{R}^{K^{(0)}}$. Then, it follows that $|L|_{\mathrm{b}}$ is a subspace of $|K|_{\mathrm{b}}$. The closedness of $|L|$ in $|K|_{\mathrm{b}}$ is obvious because $|L|$ is closed in $|K|_{\mathrm{m}}$. □

The star $\mathrm{St}(\sigma, K)$ of $\sigma \in K$ is the subcomplex of K consisting of all faces of simplexes of K containing σ as a face. The link $\mathrm{Lk}(\sigma, K)$ of $\sigma \in K$ is the subcomplex of the star $\mathrm{St}(\sigma, K)$ consisting of simplexes of $\mathrm{St}(\sigma, K)$ missing σ. For simplicity, we write $\mathrm{St}(x, K) = \mathrm{St}(c_K(x), K)$ for a point $x \in |K|$, which is called the *star* at x in K. For each vertex $v \in K^{(0)}$, $(\beta_v^K)^{-1}((0, 1])$ is denoted by $O(v, K)$ and called the *open star* at v in K. Note that $O(v, K) = |\mathrm{St}(v, K)| \setminus |\mathrm{Lk}(v, K)|$. The *open star* at a point $x \in |K|$ is defined as $O(x, K) = \bigcap_{v \in c_K(x)^{(0)}} O(v, K)$, which is open in $|K|_{\mathrm{m}}$, and hence open in both $|K|_{\mathrm{b}}$ and $|K|_{\mathrm{w}}$. Since $O(x, K) \subset \mathrm{St}(x, K)$, the star $\mathrm{St}(x, K)$ is a neighborhood of x in $|K|_{\mathrm{m}}$, $|K|_{\mathrm{b}}$, and $|K|_{\mathrm{w}}$.

Lemma 1. *For each $x \in |K|$, $U \subset |K|$ is a neighborhood of x in $|K|_{\mathrm{b}}$ if and only if there are $\varepsilon_v > 0$, $v \in \mathrm{St}(x, K)^{(0)}$, such that*

$$\{y \in |K| \mid \forall v \in \mathrm{St}(x, K)^{(0)}, |\beta_v^K(x) - \beta_v^K(y)| < \varepsilon_v\} \subset U.$$

Proof. The "if" part is trivial. To prove the "only if" part, let $U \subset |K|$ be a neighborhood of x in $|K|_{\mathrm{b}}$. By the definition of the box topology, there are $\varepsilon_v > 0$, $v \in K^{(0)}$, such that

$$\{ y \in |K| \mid \forall v \in K^{(0)}, |\beta_v^K(x) - \beta_v^K(y)| < \varepsilon_v \} \subset U.$$

For each $v \in c_K(x)^{(0)}$, replacing $\varepsilon_v > 0$, we may assume that $\varepsilon_v < \beta_v^K(x)$. Let

$$V = \{ y \in |K| \mid \forall v \in \mathrm{St}(x, K)^{(0)}, |\beta_v^K(x) - \beta_v^K(y)| < \varepsilon_v \}.$$

For each $y \in V$, $\beta_v^K(y) > 0$ for every $v \in c_K(x)^{(0)}$. This means that $c_K(x) \le c_K(y)$, hence $c_K(y) \in \mathrm{St}(x, K)$. Therefore, $\beta_v^K(y) = 0 < \varepsilon_v$ for every $v \in K^{(0)} \setminus \mathrm{St}(x, K)^{(0)}$. Then, it follows that

$$V = \{ y \in |K| \mid \forall v \in K^{(0)}, |\beta_v^K(x) - \beta_v^K(y)| < \varepsilon_v \} \subset U.$$

This completes the proof. □

Theorem 1. *Let K be a simplicial complex. If K is* (1) *locally countable or* (2) $\dim K \le 1$, *then* $|K|_{\mathrm{b}} = |K|_{\mathrm{w}}$ *as spaces.*

Proof. (1): Due to Lemma 1, it suffices to show the claim in the case K is countable. In this case, $|K|_{\mathrm{b}}$ can be regarded as a subspace of the space $\mathbf{R}^{\mathbf{N}}$ with the box topology, where $|K| \subset \mathbf{R}_{\mathrm{f}}^{\mathbf{N}}$. As is well-known, the subspace $\mathbf{R}_{\mathrm{f}}^{\mathbf{N}}$ of the space $\mathbf{R}^{\mathbf{N}}$ with the box topology is the direct limit of the tower $\mathbf{R}^1 \subset \mathbf{R}^2 \subset \mathbf{R}^3 \subset \cdots$ of Euclidean spaces, that is, its topology is the weak topology with respect to this tower. Then, $|K|_{\mathrm{b}}$ has the weak topology with respect to $|K| \cap \mathbf{R}^n$, $n \in \mathbf{N}$. Since each simplex of K is contained in some $|K| \cap \mathbf{R}^n$, it follows that $|K|_{\mathrm{b}}$ has the weak topology with respect to K, that is, $|K|_{\mathrm{b}} = |K|_{\mathrm{w}}$.

(2): Let $x \in |K|$. In case $x = v_0 \in K^{(0)}$, each $y \in |\mathrm{St}(v_0, K)| \setminus \{v_0\}$ is contained in some 1-simplex $\langle v_0, v \rangle \in K$ $(= K^{(1)})$, where note that

$$0 \le \beta_{v_0}^K(v_0) - \beta_{v_0}^K(y) = 1 - \beta_{v_0}^K(y) = \beta_v^K(y).$$

It follows from Lemma 1 that $U \subset |K|$ is a neighborhood of x in $|K|_{\mathrm{b}}$ if and only if there are $\delta_v > 0$, $v \in \mathrm{Lk}(v_0, K)^{(0)}$, such that

$$\{ y \in |\mathrm{St}(v_0, K)| \mid \forall v \in \mathrm{Lk}(v_0, K)^{(0)}, \ \beta_v^K(y) < \delta_v \} \subset U.$$

When $c_K(x)$ is a 1-simplex $\langle v_0, v_1 \rangle \in K$, we can write $x = (1 - t)v_0 + tv_1$ for some $0 < t < 1$. Since $|\mathrm{St}(x, K)| = c_K(x)$, it follows from Lemma 1 that $U \subset |K|$ is a neighborhood of x in $|K|_b$ if and only if there is some $0 < \delta < \min\{t, 1-t\}$ such that

$$\{(1-s)v_0 + sv_1 \,|\, 0 < t - \delta < s < t + \delta\} \subset U.$$

Therefore, $U \subset |K|$ is a neighborhood of x in $|K|_b$ if and only if U is neighborhood of x in $|K|_w$. Thus, we have the result. □

The following 1-dimensional countable simplicial complex J is a well-known example of a simplicial complex such that $|J|_w \neq |J|_m$:

$$J = \{v_0, v_n, \langle v_0, v_n \rangle \,|\, n \in \mathbf{N}\},$$

where $v_n \neq v_m$ if $n \neq m \in \omega$. Since $|J|_b = |J|_w$ by Theorem 1, it follows that $|J|_b \neq |J|_m$. Note that if a simplicial complex K is not locally finite, then K contains a subcomplex which is simplicially isomorphic to J. Thus, by Proposition 1, we have the following:

Corollary 1. *If a simplicial complex K is not locally finite, then $|K|_b \neq |K|_m$ as spaces.* □

2. Absolute Neighborhood Extensors for Metrizable Spaces

As mentioned in Introduction, for a full simplicial complex K, the space $|K|_b$ is an absolute extensor for metrizable spaces. This fact extends as follows:

Theorem 2. *For every simplicial complex K, $|K|_b$ is an absolute neighborhood extensor for metrizable spaces.*

To prove this theorem, we need the following proposition:

Proposition 2. *Let K be a simplicial complex and X a metrizable space. Then, $f : X \to |K|_b$ is continuous if and only if $f : X \to |K|_w$ is continuous.*

Proof. The "if" part is obvious. To show the "only if" part, assume that $f : X \to |K|_w$ is not continuous at $x_0 \in X$, that is, $f(x_0)$ has a neighborhood V in $|K|_w$ such that $f(U) \not\subset V$ for any neighborhood U of x_0 in X. Then, we can find a sequence $x_1, x_2, \ldots \in X$ such that $x_0 = \lim_{n\to\infty} x_n$ and $f(x_n) \notin V$ for every $n \in \mathbf{N}$. Let

$$L = \{\sigma \in K \,|\, \exists n \in \mathbf{N} \text{ such that } \sigma \leq c_K(f(x_n))\}.$$

Then, L is a countable subcomplex of K and $\{f(x_n) \mid n \in \omega\} \subset |L|$. Hence, $f(x_0) \neq \lim_{n\to\infty} f(x_n)$ in $|L|_{\mathrm{w}}$. On the other hand, the countability of L implies $|L|_{\mathrm{w}} = |L|_{\mathrm{b}}$ by Theorem 1(1). Consequently, $f : X \to |K|_{\mathrm{b}}$ is not continuous. □

Remark 1. Every metric space is a k-space and the above proposition holds even if X is a k-space. Indeed, when X is a k-space, assume that $f : X \to |K|_{\mathrm{b}}$ is continuous. For each compact set C in X, $f(C)$ is compact in $|K|_{\mathrm{m}}$. Hence, there is a countable subcomplex L of K such that $f(C) \subset |L|$. In fact, $f(C)$ has a countable dense subset D. The desired subcomplex $L \subset K$ can be defined as a subcomplex of K consisting of all faces of $c_K(y)$, $y \in D$. According to Theorem 1 and Proposition 1, $|L|_{\mathrm{w}} = |L|_{\mathrm{b}}$ is a subspace $|K|_{\mathrm{b}}$. Then, $f|C : C \to |L|_{\mathrm{w}}$ is continuous, so $f|C : C \to |K|_{\mathrm{w}}$ is also continuous because $|L|_{\mathrm{w}}$ is a subspace of $|K|_{\mathrm{w}}$. Since X is a k-space, it follows that $f : X \to |K|_{\mathrm{w}}$ is continuous.

Now, we can prove Theorem 2.

Proof of Theorem 2. Let X be a metrizable space and $f : A \to |K|_{\mathrm{b}}$ a map from a closed set A in X. By Proposition 2, $f : A \to |K|_{\mathrm{w}}$ is also continuous. Since $|K|_{\mathrm{w}}$ is an absolute neighborhood extensor for metrizable spaces, there exists a neighborhood U of A in X with a continuous extension $\tilde{f} : U \to |K|_{\mathrm{w}}$ of f. Then, $\tilde{f} : U \to |K|_{\mathrm{b}}$ is also continuous. Thus, $f : A \to |K|_{\mathrm{b}}$ extends over a neighborhood of A in X. □

3. The Simplicial Complex B

We define a simplicial complex B as follows:

$$B^{(0)} = \{v_n \mid n \in \omega\} \cup \{v_\lambda \mid \lambda \in \mathbf{N}^{\mathbf{N}}\},$$

$$B = \{\sigma \mid \exists n \in \mathbf{N}, \exists \lambda \in \mathbf{N}^{\mathbf{N}} \text{ such that } \sigma \leq \langle v_0, v_n, v_\lambda \rangle\}.$$

Then, $\operatorname{card} B^{(0)} = 2^{\aleph_0}$ and $\dim B = 2$. For each $n \in \mathbf{N}$ and $\lambda \in \mathbf{N}^{\mathbf{N}}$, let

$$a_{n,\lambda} = \left(1 - \frac{2}{\lambda(n)+1}\right)v_0 + \frac{1}{\lambda(n)+1}v_n + \frac{1}{\lambda(n)+1}v_\lambda \in \langle v_0, v_n, v_\lambda \rangle.$$

Theorem 3. *The box topology on $|B|$ is different from the weak topology, that is, $|B|_{\mathrm{b}} \neq |B|_{\mathrm{w}}$ as spaces.*

Proof. Let $A = \{a_{n,\lambda} \mid n \in \mathbf{N}, \lambda \in \mathbf{N}^{\mathbf{N}}\} \subset |B|$. For each $n \in \mathbf{N}$ and $\lambda \in \mathbf{N}^{\mathbf{N}}$, A meets the simplex $\langle v_0, v_n, v_\lambda \rangle$ at the point $a_{n,\lambda}$. Hence, A is closed in $|B|_{\mathrm{w}}$. On the other hand, A is not closed in $|B|_{\mathrm{b}}$. On the contrary, assume that A is closed in $|B|_{\mathrm{b}}$, that is, $|B| \setminus A$ is open in $|B|_{\mathrm{b}}$. Since $v_0 \in |B| \setminus A$, we can find $\varepsilon_n > 0$, $n \in \omega$ and $\varepsilon_\lambda > 0$, $\lambda \in \mathbf{N}^{\mathbf{N}}$, such that

$$(\beta^B_{v_0})^{-1}((1-\varepsilon_0, 1]) \cap \bigcap_{n \in \mathbf{N}} (\beta^B_{v_n})^{-1}([0, \varepsilon_n)) \cap \bigcap_{\lambda \in \mathbf{N}^{\mathbf{N}}} (\beta^B_{v_\lambda})^{-1}([0, \varepsilon_\lambda)) \subset |B| \setminus A.$$

Let $\lambda_0 \in \mathbf{N}^{\mathbf{N}}$ be the map defined by

$$\lambda_0(n) = \max\left\{n, \left[\frac{2}{\varepsilon_0}\right], \left[\frac{1}{\varepsilon_n}\right]\right\} \quad \text{for each } n \in \mathbf{N},$$

where $[t] \in \mathbf{Z}$ is the largest integer such that $[t] \le t$, hence $[t]+1$ is the smallest integer such that $t < [t]+1$. Choose $n_0 \in \mathbf{N}$ so that $1/n_0 < \varepsilon_{\lambda_0}$. Then, it follows that

$$\beta^B_{v_{\lambda_0}}(a_{n_0, \lambda_0}) = \frac{1}{\lambda_0(n_0)+1} < \frac{1}{n_0} < \varepsilon_{\lambda_0},$$

$$\beta^B_{v_0}(a_{n_0, \lambda_0}) = 1 - \frac{2}{\lambda_0(n_0)+1} > 1 - \varepsilon_0 \quad \text{and}$$

$$\beta^B_{v_{n_0}}(a_{n_0, \lambda_0}) = \frac{1}{\lambda_0(n_0)+1} < \varepsilon_{n_0}.$$

Moreover, $\beta^B_{v_n}(a_{n_0, \lambda_0}) = 0 < \varepsilon_n$ for each $n \in \mathbf{N} \setminus \{n_0\}$ and $\beta^B_{v_\lambda}(a_{n_0, \lambda_0}) = 0 < \varepsilon_\lambda$ for each $\lambda \in \mathbf{N}^{\mathbf{N}} \setminus \{\lambda_0\}$. Therefore, $a_{n_0, \lambda_0} \in |B| \setminus A$, which is a contradiction. □

According to Proposition 1, if L is a subcomplex of K then $|L|_{\mathrm{b}}$ is a subspace of $|K|_{\mathrm{b}}$. Then, we have the following corollary:

COROLLARY 2. *If a simplicial complex K contains a subcomplex which is simplicially isomorphic to B defined as above, then $|K|_{\mathrm{b}} \neq |K|_{\mathrm{w}}$ as spaces.* □

4. The Box Topology of Subdivisions

Any simplicial subdivision of a simplicial complex K preserves the weak topology and the barycentric subdivision of K preserves the metric topology. We consider this with respect the box topology. In case K is locally countable or $\dim K \le 1$, any simplicial subdivision K' of K is also locally countable or $\dim K' \le 1$, so $|K|_{\mathrm{b}} = |K|_{\mathrm{w}} = |K'|_{\mathrm{w}} = |K'|_{\mathrm{b}}$ by Theorem 1. Other cases are discussed here.

For each $x \in |K|$, $c_{K'}(x) \subset c_K(x)$, hence $\beta_v^K(v') = 0$ for $v' \in c_{K'}(x)^{(0)}$ and $v \in K^{(0)} \setminus c_K(x)^{(0)}$. Then, we have

$$x = \sum_{v' \in c_{K'}(x)^{(0)}} \beta_{v'}^{K'}(x) v' = \sum_{v \in c_K(x)^{(0)}} \left(\sum_{v' \in c_{K'}(x)^{(0)}} \beta_{v'}^{K'}(x) \beta_v^K(v') \right) v,$$

which implies that

$$(*) \qquad \sum_{v' \in c_{K'}(x)^{(0)}} \beta_{v'}^{K'}(x) \beta_v^K(v') = \beta_v^K(x) \quad \text{for each } v \in K^{(0)}.$$

For each $\sigma \in K$, choose a point $u(\sigma) \in \operatorname{rint} \sigma$ $(= \sigma \setminus \partial\sigma)$. Then, for $\sigma_1 < \cdots < \sigma_k \in K$, $u(\sigma_1), \ldots, u(\sigma_k)$ are vertices of the $(k-1)$-simplex in σ_k. A *derived subdivision* K' of a simplicial complex K is a subdivision consisting of such all simplexes. The barycentric subdivision $\operatorname{Sd} K$ of K is a typical derived subdivision. For each $x \in |K|$, there are $\sigma_1 < \cdots < \sigma_k \in K$ such that $c_{K'}(x) = \{u(\sigma_i) \mid i = 1, \ldots, k\}$ and

$$x = \sum_{i=1}^{k} \beta_{u(\sigma_i)}^{K'}(x) u(\sigma_i) = \sum_{v \in \sigma_k^{(0)}} \left(\sum_{i=1}^{k} \beta_{u(\sigma_i)}^{K'}(x) \beta_v^K(u(\sigma_i)) \right) v,$$

where $\beta_v^K(u(\sigma_i)) > 0$ if and only if $v \in \sigma_i^{(0)}$. Then, $c_K(x) = \sigma_k$ and

$$\beta_v^K(x) = \sum_{i=1}^{k} \beta_{u(\sigma_i)}^{K'}(x) \beta_v^K(u(\sigma_i)) \quad \text{for each } v \in \sigma_k^{(0)},$$

where $v \notin \sigma_i^{(0)}$ implies $\beta_v^K(u(\sigma_i)) = 0$.

Theorem 4. *Let K' be a simplicial subdivision of a simplicial complex K. If* (1) *K is locally finite-dimensional or* (2) *K' is a derived subdivision of K, then* $\mathrm{id} : |K'|_{\mathrm{b}} \to |K|_{\mathrm{b}}$ *is continuous.*

Proof. To prove the continuity at a point $x \in |K'| = |K|$, let $\varepsilon_v > 0$ be given for each $v \in K^{(0)}$.

(1): For each $v' \in K'^{(0)}$, we define $\delta_{v'} > 0$ as follows:

$$\delta_{v'} = \begin{cases} \min\left\{ \dfrac{1}{2} \beta_{v'}^{K'}(x), \ \dfrac{\min\{\varepsilon_v \mid v \in c_K(v')^{(0)}\}}{\dim \operatorname{St}(c_K(x), K) + 1} \right\} & \text{if } v' \in c_{K'}(x)^{(0)}, \\[2ex] \dfrac{\min\{\varepsilon_v \mid v \in c_K(v')^{(0)}\}}{\dim \operatorname{St}(c_K(x), K) + 1} & \text{if } v' \notin c_{K'}(x)^{(0)}, \end{cases}$$

where $\dim \mathrm{St}(c_K(x), K) < \infty$ because K is locally finite-dimensional. Assume that $y \in |K'| = |K|$ and $|\beta_{v'}^{K'}(x) - \beta_{v'}^{K'}(y)| < \delta_{v'}$ for every $v' \in K'^{(0)}$. We show that $|\beta_v^K(x) - \beta_v^K(y)| < \varepsilon_v$ for every $v \in K^{(0)}$.

First note that $\beta_{v'}^{K'}(y) > \beta_{v'}^{K'}(x)/2 > 0$ for each $v' \in c_{K'}(x)^{(0)}$ because $\delta_{v'} \leq \beta_{v'}^{K'}(x)/2$. Then, $c_{K'}(x) \leq c_{K'}(y)$, which implies $c_{K'}(y) \in \mathrm{St}(c_{K'}(x), K')$. Since $|\mathrm{St}(c_{K'}(x), K')| \subset |\mathrm{St}(c_K(x), K)|$, it follows that $\dim c_{K'}(y) \leq \dim \mathrm{St}(c_K(x), K)$. Thus, we have

$$v' \in c_{K'}(y)^{(0)},\ v \in c_K(v')^{(0)} \Rightarrow \delta_{v'} \leq \frac{\varepsilon_v}{\dim c_{K'}(y) + 1}.$$

It should be noted that $v \in c_K(v')^{(0)}$ if and only if $v' \in O(v, K)$ $(= (\beta_v^K)^{-1}((0,1]))$. Moreover, $\beta_{v'}^{K'}(x) = 0$ if $v' \in c_{K'}(y)^{(0)} \setminus c_{K'}(x)^{(0)}$, and $\beta_v^K(v') = 0$ if $v \notin c_K(v')^{(0)}$. For each $v \in c_K(y)^{(0)}$, it follows from (*) that

$$\begin{aligned}
|\beta_v^K(x) - \beta_v^K(y)| &= \left| \sum_{v' \in c_{K'}(x)^{(0)}} \beta_{v'}^{K'}(x)\beta_v^K(v') - \sum_{v' \in c_{K'}(y)^{(0)}} \beta_{v'}^{K'}(y)\beta_v^K(v') \right| \\
&\leq \sum_{v' \in c_{K'}(y)^{(0)}} |\beta_{v'}^{K'}(x) - \beta_{v'}^{K'}(y)|\ \beta_v^K(v') \\
&< \sum_{v' \in c_{K'}(y)^{(0)}} \delta_{v'}\beta_v^K(v') = \sum_{v' \in c_{K'}(y)^{(0)} \cap O(v,K)} \delta_{v'}\beta_v^K(v') \\
&\leq \sum_{v' \in c_{K'}(y)^{(0)} \cap O(v,K)} \frac{\varepsilon_v}{\dim c_{K'}(y) + 1} \\
&\leq \sum_{v' \in c_{K'}(y)^{(0)}} \frac{\varepsilon_v}{\dim c_{K'}(y) + 1} = \varepsilon_v.
\end{aligned}$$

Note that $\beta_v^K(x) = \beta_v^K(y) = 0$ if $v \in K^{(0)} \setminus c_K(y)^{(0)}$. Therefore, $|\beta_v^K(x) - \beta_v^K(y)| < \varepsilon_v$ for every $v \in K^{(0)}$.

(2): Let $K'^{(0)} = \{u(\sigma) \mid \sigma \in K\}$, where $u(\sigma) \in \mathrm{rint}\,\sigma$. For each $\sigma \in K$, we define

$$\delta_{u(\sigma)} = \begin{cases} \min\left\{\frac{1}{2}\beta_{u(\sigma)}^{K'}(x),\ 2^{-(\dim\sigma+1)}\min\{\varepsilon_v \mid v \in \sigma^{(0)}\}\right\} & \text{if } u(\sigma) \in c_{K'}(x)^{(0)}, \\ 2^{-(\dim\sigma+1)}\min\{\varepsilon_v \mid v \in \sigma^{(0)}\} & \text{if } u(\sigma) \notin c_{K'}(x)^{(0)}. \end{cases}$$

Assume that $y \in |K|$ and $|\beta_{u(\sigma)}^{K'}(x) - \beta_{u(\sigma)}^{K'}(y)| < \delta_{u(\sigma)}$ for every $\sigma \in K$. Similar to the above case (1), $c_{K'}(x) \leq c_{K'}(y)$. Now, we have $\sigma_1 < \cdots < \sigma_k \in K$ such that $c_{K'}(y)^{(0)} = \{u(\sigma_1), \ldots, u(\sigma_k)\}$, where it should be noted that $\dim \sigma_i + 1 \geq i$.

For each $v \in \sigma_k^{(0)}$, let $v \in \sigma_j^{(0)} \setminus \sigma_{j-1}^{(0)}$. Then, $\beta_v^K(u(\sigma_i)) = 0$ for $i < j$, and $v \in \sigma_i^{(0)}$ for $i \geq j$. It follows that

$$
\begin{aligned}
|\beta_v^K(x) - \beta_v^K(y)| &= \left| \sum_{i=1}^{k} \beta_{u(\sigma_i)}^{K'}(x)\beta_v^K(u(\sigma_i)) - \sum_{i=1}^{k} \beta_{u(\sigma_i)}^{K'}(y)\beta_v^K(u(\sigma_i)) \right| \\
&\leq \sum_{i=1}^{k} |\beta_{u(\sigma_i)}^{K'}(x) - \beta_{u(\sigma_i)}^{K'}(y)| \ \beta_v^K(u(\sigma_i)) \\
&< \sum_{i=j}^{k} \delta_{u(\sigma_i)} \leq \sum_{i=j}^{k} 2^{-(\dim \sigma_i + 1)} \varepsilon_v \\
&\leq \sum_{i=j}^{k} 2^{-i} \varepsilon_v < \varepsilon_v.
\end{aligned}
$$

Note that $\beta_v^K(x) = \beta_v^K(y) = 0$ if $v \in K^{(0)} \setminus \sigma_k^{(0)}$. Thus, $|\beta_v^K(x) - \beta_v^K(y)| < \varepsilon_v$ for every $v \in K^{(0)}$. This completes the proof. □

The following theorem shows that even the barycentric subdivision of a simplicial complex need not preserve the box topology.

Theorem 5. *The barycentric subdivision of the simplicial complex B does not preserve the box topology, that is,* $|\mathrm{Sd}\, B|_{\mathrm{b}} \neq |B|_{\mathrm{b}}$ *as spaces.*

Proof. We prove that $\mathrm{id} : |B|_{\mathrm{b}} \to |\mathrm{Sd}\, B|_{\mathrm{b}}$ is not continuous at v_0. (Note that $\mathrm{id} : |\mathrm{Sd}\, B|_{\mathrm{b}} \to |B|_{\mathrm{b}}$ is continuous by Theorem 4(2).) For each $n \in \mathbf{N}$ and $\lambda \in \mathbf{N}^{\mathbf{N}}$, let $\sigma_{n,\lambda} = \langle v_0, v_n, v_\lambda \rangle$, with $w_{n,\lambda}$ the barycenter of $\sigma_{n,\lambda}$, and define $\varepsilon_{w_{n,\lambda}} = 2/\lambda(n) > 0$. For any $\delta_v > 0$, $v \in B^{(0)}$, we can find $n_0 \in \mathbf{N}$, $\lambda_0 \in \mathbf{N}^{\mathbf{N}}$ and $x \in \sigma_{n_0,\lambda_0}$ such that

$$
|\beta_v^B(v_0) - \beta_v^B(x)| < \delta_v \quad \text{for every } v \in B^{(0)} \text{ but}
$$
$$
|\beta_{w_{n_0,\lambda_0}}^{\mathrm{Sd}\, B}(v_0) - \beta_{w_{n_0,\lambda_0}}^{\mathrm{Sd}\, B}(x)| \geq \varepsilon_{w_{n_0},\lambda_0},
$$

which means that $\mathrm{id} : |B|_{\mathrm{b}} \to |\mathrm{Sd}\, B|_{\mathrm{b}}$ is not continuous at v_0. In fact, we define $\lambda_0 \in \mathbf{N}^{\mathbf{N}}$ by

$$
\lambda_0(n) = \max\left\{3, n, \left[\frac{1}{\delta_{v_n}}\right] + 1, \left[\frac{2}{\delta_{v_0}}\right] + 1\right\},
$$

and choose $n_0 \in \mathbf{N}$ so that $n_0^{-1} < \delta_{v_{\lambda_0}}$. Let

$$x = \left(1 - \frac{2}{\lambda_0(n_0)}\right)v_0 + \frac{1}{\lambda_0(n_0)}v_{n_0} + \frac{1}{\lambda_0(n_0)}v_{\lambda_0}$$
$$= \left(1 - \frac{3}{\lambda_0(n_0)}\right)v_0 + \frac{3}{\lambda_0(n_0)}\left(\frac{1}{3}v_0 + \frac{1}{3}v_{n_0} + \frac{1}{3}v_{\lambda_0}\right)$$
$$= \left(1 - \frac{3}{\lambda_0(n_0)}\right)v_0 + \frac{3}{\lambda_0(n_0)}w_{n_0,\lambda_0}.$$

Then, $|\beta_v^B(v_0) - \beta_v^B(x)| < \delta_v$ for every $v \in B^{(0)}$ because

$$1 - \beta_{v_0}^B(x) = \frac{2}{\lambda_0(n_0)} < \delta_{v_0}, \quad \beta_{v_{n_0}}^B(x) = \frac{1}{\lambda_0(n_0)} < \delta_{v_{n_0}},$$

$$\beta_{v_{\lambda_0}}^B(x) = \frac{1}{\lambda_0(n_0)} \leq \frac{1}{n_0} < \delta_{v_{\lambda_0}} \text{ and}$$

$$\beta_v^B(x) = 0 < \delta_v \quad \text{for } v \neq v_0, v_{n_0}, v_{\lambda_0}.$$

On the other hand,

$$|\beta_{w_{n_0,\lambda_0}}^{\mathrm{Sd}\,B}(v_0) - \beta_{w_{n_0,\lambda_0}}^{\mathrm{Sd}\,B}(x)| = \beta_{w_{n_0,\lambda_0}}^{\mathrm{Sd}\,B}(x) = \frac{3}{\lambda_0(n_0)} > \frac{2}{\lambda_0(n_0)} = \varepsilon_{w_{n_0,\lambda_0}}.$$

This completes the proof. □

5. The Continuity of Simplicial Maps

Every simplicial map is continuous with respect to both of the metric topology and the weak topology. Concerning the box topology, we have the following result:

THEOREM 6. *Let K and L be simplicial complexes and $f : K \to L$ a simplicial map. In the following cases, $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is continuous.*
(1) *K is locally countable,*
(2) *K is locally finite-dimensional,*
(3) *f is proper.*

In the above (3), a simplicial map $f : K \to L$ is *proper* if $f^{-1}(v) \cap K^{(0)}$ $(= (f \,|\, |K^{(0)}|)^{-1}(v))$ is finite for each $v \in L^{(0)}$.

PROOF. (1): Note that $f|\sigma$ is continuous for every $\sigma \in K$ because both simplexes σ and $f(\sigma)$ have the Euclidean topology. Because of the local countability of K, $|K|_{\mathrm{b}} = |K|_{\mathrm{w}}$ as spaces. Then, it follows that f is continuous.

(2): We show the continuity of f at $x = \sum_{v \in K^{(0)}} \beta_v^K(x)v \in |K|$. Let $k = \dim \mathrm{St}(x, K) < \infty$ (the local finite-dimensionality of K). Given $\varepsilon_u > 0$, $u \in \mathrm{St}(f(x), L)^{(0)}$, we define

$$\delta_v = \begin{cases} \min\left\{\frac{1}{2}\beta_v^K(x),\ \frac{\varepsilon_{f(v)}}{k+1}\right\} & \text{if } v \in c_K(x)^{(0)}, \\ \frac{\varepsilon_{f(v)}}{k+1} & \text{if } v \in \mathrm{St}(x, K)^{(0)} \setminus c_K(x)^{(0)}. \end{cases}$$

Take $x' \in |\mathrm{St}(x, K)|$ such that $|\beta_v^K(x) - \beta_v^K(x')| < \delta_v$ for each $v \in \mathrm{St}(x, K)^{(0)}$. Since $|\beta_v^K(x) - \beta_v^K(x')| < \beta_v^K(x)/2$, it follows that $\beta_v^K(x') > \beta_v^K(x)/2$, which implies $c_K(x) \le c_K(x')$. Because $c_K(x') \in \mathrm{St}(x, K)$, we have $\mathrm{card}\, c_K(x')^{(0)} \le k+1$. For each $u \in \mathrm{St}(f(x), L)^{(0)}$,

$$\begin{aligned} |\beta_u^L(f(x)) - \beta_u^L(f(x'))| &= \left| \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \beta_v^K(x) - \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \beta_v^K(x') \right| \\ &\le \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} |\beta_v^K(x) - \beta_v^K(x')| \\ &\le \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \delta_v \le \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \frac{\varepsilon_{f(v)}}{k+1} \le \varepsilon_u. \end{aligned}$$

This shows that $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is continuous.

(3): As in (2), we show the continuity of f at $x = \sum_{v \in K^{(0)}} \beta_v^K(x)v \in |K|$. For each $u \in L^{(0)}$, let $k_u = \mathrm{card}\, f^{-1}(u) \cap K^{(0)} \in \omega$ (the properness of f). Given $\varepsilon_u > 0$, $u \in \mathrm{St}(f(x), L)^{(0)}$, we define

$$\delta_v = \begin{cases} \min\left\{\frac{1}{2}\beta_v^K(x),\ \frac{\varepsilon_{f(v)}}{k_{f(v)}}\right\} & \text{if } v \in c_K(x)^{(0)}, \\ \frac{\varepsilon_{f(v)}}{k_{f(v)}} & \text{if } v \in \mathrm{St}(x, K)^{(0)} \setminus c_K(x)^{(0)}. \end{cases}$$

Take $x' \in |\mathrm{St}(x, K)|$ such that $|\beta_v^K(x) - \beta_v^K(x')| < \delta_v$ for each $v \in \mathrm{St}(x, K)^{(0)}$. As the case (2), $c_K(x) \le c_K(x') \in \mathrm{St}(x, K)$. Then, for each $u \in \mathrm{St}(f(x), L)^{(0)}$,

$$\begin{aligned} |\beta_u^L(f(x)) - \beta_u^L(f(x'))| &\le \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} |\beta_v^K(x) - \beta_v^K(x')| \\ &\le \sum_{v \in c_K(x')^{(0)} \cap f^{-1}(u)} \delta_v \le \sum_{v \in K^{(0)} \cap f^{-1}(u)} \frac{\varepsilon_{f(v)}}{k_{f(v)}} \le \varepsilon_u. \end{aligned}$$

Therefore, $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is continuous. □

Finally, we show the following:

THEOREM 7. *Let K and L be simplicial complexes such that K contains an uncountable full simplicial complex and L contains an infinite full simplicial complex. Then, there exists a simplicial map $f : K \to L$ such that $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is not continuous.*

PROOF. Without loss of generality, we may assume that L itself is a countable infinite full simplicial complex with $L^{(0)} = \{u_n \mid n \in \omega\}$, where $u_n \neq u_{n'}$ if $n \neq n'$. Let F be an uncountable full simplicial complex contained in K. Take a vertex $v_0 \in F^{(0)}$ and let

$$F^{(0)} \setminus \{v_0\} = \{v_{n,\lambda} \mid (n,\lambda) \in \mathbf{N} \times \Lambda\},$$

where Λ is uncountable and $v_{n,\lambda} \neq v_{n',\lambda'}$ if $(n,\lambda) \neq (n',\lambda')$. We define a simplicial map $f : K \to L$ by

$$f(v) = \begin{cases} u_n & \text{if } v = v_{n,\lambda},\ (n,\lambda) \in \mathbf{N} \times \Lambda, \\ u_0 & \text{if } v = v_0 \text{ or } v \in K^{(0)} \setminus F^{(0)}. \end{cases}$$

Then, $f : |K|_{\mathrm{b}} \to |L|_{\mathrm{b}}$ is not continuous at v_0. Indeed, $V_0 = \bigcap_{n \in \mathbf{N}} (\beta^L_{u_n})^{-1}([0, 2^{-n}))$ is an open neighborhood of $u_0 = f(v_0)$ in $|L|_{\mathrm{b}}$. For each neighborhood U of v_0 in $|K|_{\mathrm{b}}$, we have $\delta_v > 0$, $v \in K^{(0)}$, such that

$$(\beta^K_{v_0})^{-1}((1 - \delta_{v_0}, 1]) \cap \bigcap_{v \in K^{(0)} \setminus \{v_0\}} (\beta^K_v)^{-1}([0, \delta_v)) \subset U.$$

Choose an $n_0 \in \mathbf{N}$ so that $2^{-n_0} < \delta_{v_0}$. Since Λ is uncountable, we have $n_1 > n_0$ such that $\delta_{v_{n_0,\lambda_i}} > 2^{-n_1}$ for infinitely many distinct $\lambda_i \in \Lambda$, $i \in \mathbf{N}$. Let

$$p = (1 - 2^{-n_0}) v_0 + \sum_{i=1}^{2^{n_1 - n_0}} 2^{-n_1} v_{n_0,\lambda_i} \in |F| \subset |K|.$$

Then, it follows that

$$\beta^K_{v_0}(p) = 1 - 2^{-n_0} > 1 - \delta_{v_0},$$

$$\beta^K_{v_{n_0,\lambda_i}}(p) = 2^{-n_1} < \delta_{v_{n_0},\lambda_i} \quad \text{for each } i = 1, 2, \dots, 2^{n_1 - n_0} \text{ and}$$

$$\beta^K_v(p) = 0 < \delta_v \quad \text{if } v \neq v_0, v_{n_0,\lambda_1}, v_{n_0,\lambda_2}, \dots, v_{n_0,\lambda_{2^{n_1 - n_0}}},$$

which means that $p \in U$. On the other hand,

$$f(p) = (1 - 2^{-n_0}) u_0 + 2^{n_1 - n_0} 2^{-n_1} u_{n_0} = (1 - 2^{-n_0}) u_0 + 2^{-n_0} u_{n_0}.$$

Then, $\beta^L_{u_{n_0}}(f(p)) = 2^{-n_0}$, which implies that $f(p) \notin V_0$. Hence, f is not continuous at v_0. □

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