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WEIERSTRASS GAP SEQUENCES AT POINTS OF CURVES ON SOME RATIONAL SURFACES

By

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Abstract. Let $\tilde{C}$ be a non-singular plane curve of degree $d \geqq 8$ with an involution σ over an algebraically closed field of characteristic 0 and $\tilde{P}$ a point of $\tilde{C}$ fixed by σ. Let $\pi : \tilde{C} \to C = \tilde{C}/\langle\sigma\rangle$ be the double covering. We set $P = \pi(\tilde{P})$. When the intersection multiplicity at $\tilde{P}$ of the curve $\tilde{C}$ and the tangent line at $\tilde{P}$ is equal to $d-3$ or $d-4$, we determine the Weierstrass gap sequence at P on C using blowing-ups and blowing-downs of some rational surfaces.

1. Introduction

Let C be a complete non-singular irreducible curve of genus g over an algebraically closed field k of charcteristic 0, which is called a *curve* in this paper. For a pointed curve (C, P) we define

$$H(P) = \{\alpha \in \mathbf{N}_0 \mid \text{there exists } f \in k(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of P* where $\mathbf{N}_0$, $k(C)$ and $(f)_\infty$ denote the additive monoid of non-negative integers, the field of rational functions on C and the divisor of poles of f respectively. Let $\{l_1 < l_2 < \cdots < l_g\}$ be the complement $\mathbf{N}_0 \backslash H(P)$ of $H(P)$ in $\mathbf{N}_0$ where g is the genus of C. The sequence $l_1, l_2, \ldots, l_g$ is called the *Weierstrass gap sequence* at P.

Let $\tilde{C}$ be a plane curve of degree d. Here, we note again that a plane curve is non-singular in this paper. For a point $\tilde{P}$ of $\tilde{C}$ we denote by $T_{\tilde{P}}$ the tangent line

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at $\tilde{P}$ on $\tilde{C}$. Let $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C})$ be the intersection multiplicity of $T_{\tilde{P}}$ and $\tilde{C}$ at $\tilde{P}$. It is not difficult to determine the Weierstrass semigroup $H(\tilde{P})$ if $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d$ or $d-1$ or $d-2$. In these cases each semigroup $H(\tilde{P})$ is uniquely determined. In the case where $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-3$ Coppens and Kato [1] determined the Weierstrass semigroups $H(\tilde{P})$. But when $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-4$ they only gave the candidates of the Weierstrass semigroups $H(\tilde{P})$. In fact, it is an open problem to determine the above Weierstrass semigroups in this case. In this paper we are interested in the following probelm:

Let $\tilde{C}$ be a plane curve of degree d with an involution σ and $\tilde{P}$ its point fixed by σ. Determine the Weierstrass semigroup of $\pi(\tilde{P})$ where $\pi : \tilde{C} \to \tilde{C}/\langle\sigma\rangle$ is the double covering.

This problem is solved for $d \leqq 7$ in [5]. In this paper we will show the following:

MAIN THEOREM. *Let $\tilde{C}$ be a plane curve of degree $d \geqq 8$ with an involution σ and $\tilde{P}$ its point fixed by σ. When $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-3$ or $d-4$, we can determine the Weierstrass semigroup of $\pi(\tilde{P})$ where $\pi : \tilde{C} \to \tilde{C}/\langle\sigma\rangle$ is the double covering.*

In the case $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-3$ we calculate the order sequence of a canonical divisor at $\pi(\tilde{P})$, i.e., the complement $\mathbf{N}_0 \backslash H(\pi(\tilde{P}))$, using divisors consisting of fibers and minimal sections on the Hirzeburch surface $S = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))$ with index two, regarding $C = \tilde{C}/\langle\sigma\rangle$ as a curve on the surface S. In this case the semigroup $H(\pi(\tilde{P}))$ is also uniquely determined even though there are more than one kind of the semigroups $H(\tilde{P})$. We note that this method works well for the cases $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d, d-1, d-2$. Moreover, using the same method we get the complement $\mathbf{N}_0 \backslash H(\pi(\tilde{P}))$ except only one element in the case $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-4$ in Section 2. So, solving our problem is to get the remaining one element. In Section 3, to get the element we blow up rational surfaces whose first one is the Hirzebruch surface S, and construct divisors on some rational surfaces, which we blow down. The unknown order of a canonical divisor at the point $\pi(\tilde{P})$ is calculated using the blowing-down of some divisor to S.

2. Curves on the Hirzebruch Surface with Index Two

We use the following notation throughout this section: Let $\tilde{C} \subset \mathbf{P}^2$ be a plane curve of degree $d \geqq 4$ with an involution σ. Then σ is extended to the automorphism of $\mathbf{P}^2 = \operatorname{Proj} k[x, y, z]$ corresponding to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ by some coordinate transformation of $\mathbf{P}^2$. We also denote this automorphism by σ. The automorphism σ fixes the line defined by the equation $z = 0$ and the point $(0:0:1)$. Consider the quotient map $\tilde{\pi} : \mathbf{P}^2 \to \mathbf{P}^2/\langle\sigma\rangle \cong \mathbf{P}(1,1,2)$ where $\mathbf{P}(1,1,2)$ is the weighted projective space as follows: the coordinates $(x, y, z) \neq (0,0,0)$ and $(\lambda x, \lambda y, \lambda^2 z)$ with $\lambda \in k \backslash \{0\}$ define the same point on $\mathbf{P}(1,1,2)$. Using blowing-up of the morphism $\tilde{\pi}$ at the points $(0:0:1) \in \mathbf{P}^2$ and $\tilde{\pi}((0:0:1)) \in \mathbf{P}(1,1,2)$ we get the commutative diagram

$$\begin{array}{ccccc}
\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1)) & \xrightarrow{\tilde{\eta}} & \mathbf{P}^2 & \supset & \tilde{C} \\
\Big\downarrow{\scriptstyle\tilde{\pi}} & & \Big\downarrow{\scriptstyle\tilde{\pi}} & & \Big\downarrow{\scriptstyle\pi} \\
S = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2)) & \xrightarrow{\eta} & \mathbf{P}(1:1:2) & \supset & C = \tilde{C}/\langle\sigma\rangle
\end{array}$$

Then we regard $\tilde{C}/\langle\sigma\rangle$ (resp. $\tilde{C}$) as a subscheme of $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ (resp. $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$) by identifying $\tilde{C}/\langle\sigma\rangle$ (resp. C) with its strict transform of the blowing-up of $\mathbf{P}(1:1:2)$ (resp. $\mathbf{P}^2$) at $\tilde{\pi}((0:0:1))$ (resp. $(0:0:1)$). Hence, the double covering $\pi : \tilde{C} \to C$ becomes the restriction of the morphim from $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ to $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$, which is also denoted by $\tilde{\pi}$. Let $\tilde{\rho} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbf{P}^1$ and $\rho : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2)) \to \mathbf{P}^1$ be structure morphisms. Let F and $\tilde{F}$ be fibers of ρ and $\tilde{\rho}$ respectiely. For any point $P \in \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ (resp. $\tilde{P} \in \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ we denote by F_P (resp. $\tilde{F}_{\tilde{P}}$) the fiber containing P (resp. $\tilde{P}$). Moreover, we denote by $\tilde{E}_0$ and E_0 minimal sections of $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ and $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ respectively. Then we have $\tilde{\pi}^*(F) = \tilde{F}$ and $\tilde{\pi}^* E_0 = 2\tilde{E}_0$. Let H be the divisor on $\mathbf{P}(1:1:2)$ defined by the set $\{x : y : 0\}$. We identify the inverse image of H to $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ with H, because H does not contain $(0:0:1)$. Then the branched locus of $\tilde{\pi} : \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ is $H + E_0$, which is linearly equivalent to $2(E_0 + F)$. Hence, we may describe $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ as $\mathbf{P}(\mathcal{O}_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} \oplus \mathcal{O}_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))}(-E_0 - F))$. If we regard C as a subscheme of $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$, then by [3] we get $C \sim eE_0 + 2eF$ if $d = 2e$ and $C \sim eE_0 + (2e+1)F$ if $d = 2e+1$ where the symbol $\sim$ means a linear equivalence.

Let $\tilde{P}$ be a point of $\tilde{C}$ with $\sigma(\tilde{P}) = \tilde{P}$, $T_{\tilde{P}}$ the tangent line at $\tilde{P}$ on $\tilde{C}$ and $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C})$ the intersection multiplicity between $T_{\tilde{P}}$ and $\tilde{C}$ at $\tilde{P}$. We set $t = I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C})$ and assume that $t \geqq 4$. Moreover, we set $P = \pi(\tilde{P})$. Let L be the line defined by the set $\{x : y : 0\}$. Then we have $\sigma(T_{\tilde{P}}) = T_{\tilde{P}}$ and $\sigma(L) = L$. We will show that $\sigma(T_{\tilde{P}}) \neq L$. In fact, the divisor $(H + E_0)|_C$ on $C \subset \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ is the branch locus of the morphism $\pi : \tilde{C} \to C = \tilde{C}/\langle\sigma\rangle$, so it must be reduced. But, if $T_{\tilde{P}} = L$ held, then $(H + E_0)|_C = (t/2)P + \cdots \geqq 2P$, which implies that the branch locus is not reduced. This is a contradiction. Hence, $T_{\tilde{P}}$ should be the line through the point $(0:0:1)$ and a point $A \in L$. Let K_S be a canonical divisor on $S = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$. Then $K_S \sim -2E_0 - 4F$. Hence, a canonical divisor K_C on C is linearly equivalent to $((e-2)E_0 + (2e-4)F)|_C$ and $((e-2)E_0 + (2e-3)F)|_C$ if $d = 2e$ and $d = 2e+1$ respectively.

LEMMA 2.1. *If the degree d of the plane curve $\tilde{C}$ is even, then so is the intersection multiplicity t between $T_{\tilde{P}}$ and $\tilde{C}$ at $\tilde{P}$.*

PROOF. Since we have $T_{\tilde{P}} \neq L$, the tangent line $T_{\tilde{P}}$ is the line through the point $(0:0:1)$ and a point $A \in L$. Let $\tilde{T}_{\tilde{P}}$ be the total transform $\tilde{\eta}^{-1}(T_{\tilde{P}})$ of $T_{\tilde{P}}$, Then we have $\tilde{T}_{\tilde{P}} = \tilde{E}_0 + \tilde{F}_A$ where $\tilde{F}_A$ is the fiber containing A. Here, we also denote the point $\tilde{\eta}^{-1}(A)$ by A. Moreover, we get

$$(\tilde{E}_0, \tilde{\eta}^{-1}(\tilde{C})) = (\tilde{E}_0, 2e\tilde{E}_o + d\tilde{F}) = 0 \ \text{(resp. 1)} \quad \text{if } d = 2e \ \text{(resp. } 2e+1).$$

Now we have

$$t = I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = I_{\tilde{P}}(\tilde{F}_A, \tilde{\eta}^{-1}(\tilde{C})) + I_{\tilde{P}}(\tilde{E}_0, \tilde{\eta}^{-1}(\tilde{C})).$$

Hence, if $\tilde{P} \notin \tilde{E}_0$, i.e., $\tilde{P} \neq (0:0:1)$, then we get $I_{\tilde{P}}(\tilde{F}_A, \tilde{\eta}^{-1}(\tilde{C})) = t$. If $\tilde{P} \in \tilde{E}_0$, then we have $(\tilde{E}_0, \tilde{\eta}^{-1}(\tilde{C})) > 0$, which implies that $d = 2e+1$. This is not our case. Therefore, if $d = 2e$, then $\tilde{P} \notin \tilde{E}_0$. We note that $\tilde{\pi} \circ \tilde{\eta}(\tilde{F}_A) = T_P$ and $\tilde{\pi} \circ \tilde{\eta} \circ \tilde{\eta}^{-1}(\tilde{C}) = C$. Hence, we get $I_P(T_P, C) = t/2$. Thus, if d is even, then the multiplicity t between $T_{\tilde{P}}$ and $\tilde{C}$ at $\tilde{P}$ should be even. □

First, we treat the case where t is even. In this case, we have $\tilde{P} \notin \tilde{E}_0$. In fact, let $\tilde{P} \in \tilde{E}_0$. Then we get $(\tilde{E}_0, \tilde{\eta}^{-1}(\tilde{C})) = 1$, which implies that $I_{\tilde{P}}(\tilde{F}_A, \tilde{\eta}^{-1}(\tilde{C})) = t-1$ where $\tilde{F}_A$ is as in the proof of Lemma 2.1. Hence, we have $I_P(T_P, C) = (t-1)/2$. Thus, t should be odd.

We will calculate the order sequence of K_C at P when $t = I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-3$ which is even.

PROPOSITION 2.2. *Let d be odd, i.e., $d = 2e+1$. Let $t = d-3$. Then the gap sequence at P is $2i(e-1)+1, \ldots, 2i(e-1)+e-i-1, (2i+1)(e-1)+1, \ldots, (2i+1)(e-1)+e-i-1$ $(0 \leqq i \leqq e-2)$.*

PROOF. Since t is even, we have $\tilde{P} \notin \tilde{E}_0$. Thus, we get $I_P(T_P, C) = t/2$, i.e., there exists a fiber F_P of ρ such that $I_P(F_P, C) = t/2$. Let F be a fiber of ρ such that $I_P(F, C) = 0$. H is an irreducible curve containing P because of $\tilde{P} \notin \tilde{E}_0$. Now we have $I_P(H, F) = I_P(E_0 + 2F, F) = 1$. Hence, H and F intersect transversally. Let F_P be the fiber of ρ. We note that

$$K_C \sim (K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + C)|_C \sim ((e-2)E_0 + (2e-3)F)|_C \sim ((e-2)H + F)|_C.$$

Moreover, since we have

$$\deg K_C = ((e-2)E_0 + (2e-3)F).(eE_0 + (2e+1)F) = 2e(e-1) - 2,$$

the genus of C is $e(e-1)$.

We will see $H(P)$ using the divisors F, $H \sim E_0 + 2F$ and F_P on $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$. We note that $I_P(H, C) = 1$, because $H \ni P$ and H is not the tangent line at P on C.

We consider the following divisors which are linearly equivalent to $K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + C \sim (e-2)E_0 + (2e-3)F$:

$$2iF_P + jH + (e-2-j)E_0 + (2e-3-2i-2j)F,$$

$$(2i+1)F_P + jH + (e-2-j)E_0 + (2e-4-2i-2j)F$$

for $0 \leqq i \leqq e-2$ and $0 \leqq j \leqq e-i-1$. Let $t = d-3 = 2e-2$, i.e., $t/2 = e-1$. Then we have $I_P(F_P, C) = e-1$. Hence, it is shown that the gap sequence at P is $2i(e-1)+1, \dots, 2i(e-1)+e-i-1$, $(2i+1)(e-1)+1, \dots, (2i+1)(e-1)+e-i-1$ $(0 \leqq i \leqq e-2)$. □

We give the known results on the gap sequence at P when $t = d$, which is also proved by our method, because these results will be used in the next section.

Remark 2.3. Let $d = 2e$ be even and $t = d$. Then the gap sequence at P is

$$2ie+1, \dots, 2ie+e-i-1 \quad (0 \leqq i \leqq e-2)$$

and

$$(2i+1)e+1, \dots, (2i+1)e+e-i-2 \quad (0 \leqq i \leqq e-3).$$

Hence, the Weierstrass semigroup $H(P)$ is generated by e and $2e-1$.

Remark 2.4. Let $d = 2e+1$ be odd and $t = d$. Then the gap sequence at P is

$$2ie+i+1, 2ie+i+2, \dots, 2ie+e-1 \quad (0 \leqq i \leqq e-2),$$

$$(2i+1)e+i+1, (2i+1)e+i+2, \dots, (2i+1)e+e-1 \quad (0 \leqq i \leqq e-2).$$

Hence, the Weierstrass semigroup $H(P)$ is generated by e and $2e+1$.

In the case where $t = I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d - 4$ we can determine the gaps at P except one gap.

LEMMA 2.5. *Let $d = 2e$ be even and $t = d - 4$. The set of gaps at P contains the set*

$$G = \{1, 2, \dots, e-2\} \cup \left(\bigcup_{i=0}^{e-3} \{(2i+1)(e-2)+1, \dots, (2i+1)(e-2)+e-i-2\}\right)$$
$$\cup \left(\bigcup_{i=1}^{e-2} \{2i(e-2)+1, \dots, 2i(e-2)+e-i-1\}\right).$$

Hence, only one gap at P is not determined.

PROOF. Let F be a fiber of ρ such that $I_P(F, C) = 0$. In view of $\tilde{P} \notin \tilde{E}_0$ we get $I_P(H, C) = 1$. Moreover, we have the fiber F_P of ρ such that $I_P(F_P, C) = e - 2$. We consider the following divisors which are linearly equivalent to $K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + C \sim (e-2)E_0 + (2e-4)F$:

$$2iF_P + jH + (e-2-j)E_0 + (2e-4-2i-2j)F$$

for $0 \leqq i \leqq e-2$ and $0 \leqq j \leqq e-i-2$ and

$$(2i+1)F_P + jH + (e-2-j)E_0 + (2e-5-2i-2j)F$$

for $0 \leqq i \leqq e-3$ and $0 \leqq j \leqq e-i-3$. Using the above divisors we can get the gaps at P except one gap, because the orders of $(e-2)H$ and $(e-2)E_0 + (2e-5)F + F_P$ at P are the same, which is $e-2$. $\square$

LEMMA 2.6. *Let $d = 2e$ and $t = d - 4$. Let $a < e - 2$ and $l \geqq 0$. The order sequence of $|aE_0 + (2a+l)F|$ at P is constructed by a fiber $F \not\ni P$, $E_0 \not\ni P$, $H \sim E_0 + 2F$ and the fiber F_P with $P \in H$, $I_P(H, E) = 1$ and $I_P(E, F_P) = e - 2$.*

PROOF. We consider the following divisors which are linearly equivalent to $aE_0 + (2a+l)F$:

$$jH + (a-j)E_0 + (l + 2(a-j) - i)F + iF_P \quad (0 \leqq i \leqq l, 0 \leqq j \leqq a)$$

and

$$(a-j)E_0 + jH + (2a - 2i + 1 - 2j)F + (l + 2i - 1)F_P \quad (1 \leqq i \leqq a, 0 \leqq j \leqq a - i)$$
$$(a-j)E_0 + jH + (2a - 2i - 2j)F + (l + 2i)F_P \quad (1 \leqq i \leqq a, 0 \leqq j \leqq a - i).$$

The above divisors determine $(a+1)(l+1) + a(a+1)$ distinct orders at P. On the other hand we have

$$
\begin{aligned}
h^0(aE_0 + (2a+l)F) &= h^0(S^a(\mathcal{O} \oplus \mathcal{O}(-2)) \otimes \mathcal{O}(2a+l)) \\
&= h^0((\mathcal{O} \oplus \mathcal{O}(-2) \oplus \cdots \oplus \mathcal{O}(-2a)) \otimes \mathcal{O}(2a+l)) \\
&= h^0(\mathcal{O}(2a+l) \oplus \mathcal{O}(2a+l-2) \oplus \cdots \oplus \mathcal{O}(2a+l-2a)) \\
&= (a+1)(l+1) + a(a+1). \qquad \square
\end{aligned}
$$

Lemma 2.7. *Let $d = 2e+1$ and $t = d-4$. The set of gaps at P contains*

$$\{1, 2, \dots, e-2\} \cup \left(\bigcup_{i=0}^{e-2} \{(2i+1)(e-2)+i+1, \dots, (2i+1)(e-2)+e-1\}\right)$$
$$\cup \left(\bigcup_{i=1}^{e-2} \{2i(e-2)+i+1, \dots, 2i(e-2)+e-1\}\right).$$

Hence, only one gap at P is not determined.

Proof. Let F be a fiber of ρ such that $I_P(F, C) = 0$. We have $I_P(H, C) = 0$ because of $\tilde{P} \in \tilde{E}_0$. Moreover, we get $I_P(C, E_0) = 1$ because of $(C, E_0) = 1$ and $\tilde{P} \in \tilde{E}_0$. Hence we have the fiber F_P of ρ satisfying $I_P(F_P, C) = (t-1)/2 = (d-5)/2 = e-2$. We consider the following divisors which are linearly equivalent to $K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + C \sim (e-2)E_0 + (2e-3)F$:

$$2iF_P + jE_0 + (e-2-j)H + (2j-2i+1)F,$$
$$(2i+1)F_P + jE_0 + (e-2-j)H + (2j-2i)F$$

for $0 \leqq i \leqq e-2$ and $i \leqq j \leqq e-2$. Using the above divisors we can get the gaps at P except one gap, because the orders of $(e-2)E_0 + (2e-3)F$ and $(e-2)H + F_P$ at P are the same, which is $e-2$. $\square$

Lemma 2.8. *Let $d = 2e+1$ and $t = d-4$. Let $a < e-2$, $l \geqq 0$ or $a = e-2$, $l = 0$. The order sequence of $|aE_0 + (2a+l)F|$ at P is constructed by a fiber $F \not\ni P$, $E_0 \ni P$, $H \sim E_0 + 2F$ and the fiber F_P with $I_P(E_0, E) = 1$, $P \notin H$ and $I_P(E, F_P) = e-2$.*

Proof. By the same method as in the proof of Lemma 2.6 we get the result. $\square$

3. Blowing-Up and Blowing-Down of Divisors on Rational Surfaces

In this section we will treat the case where $\tilde{C}$ is a plane curve of degree $d \geqq 8$ with an involution σ and a fixed point $\tilde{P}$ by σ with $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-4$.

LEMMA 3.1. *Let* $\pi : \tilde{C} \to C$ *be a double covering with a ramification point* $\tilde{P}$ *where* $\tilde{C}$ *is a plane curve of degree* $d \geqq 8$. *We regard* C *as a closed subscheme of* $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$. *We assume that* $t = I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d - 4$.

i) *Let* $d = 2e$ *be even. Then the one more order of the intersection at* P *between* $K_{\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-2))} + C$ *and* C *belongs to the following set*:

$$\begin{aligned} Z_1 = {} & (\textstyle\bigcup_{i=1}^{e-2}\{(2i+1)(e-2)+e-i-2, \dots, (2i+1)(e-2)+e-3\}) \\ & \cup (\textstyle\bigcup_{i=2}^{e-1}\{2i(e-2)+e-i-1, \dots, 2i(e-2)+e-3\}) \\ & \cup \{(2e-1)(e-2), (2e-1)(e-2)+1, \dots, 2e(e-2)\}. \end{aligned}$$

ii) *Let* $d = 2e+1$ *be odd. Then the one more order of the intersection at* P *between* $K_{\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-2))} + C$ *and* C *belongs to the following set*:

$$\begin{aligned} Z_2 = {} & (\textstyle\bigcup_{i=2}^{e-2}\{2i(e-2)+1, \dots, 2i(e-2)+i-1\}) \\ & \cup (\textstyle\bigcup_{i=2}^{e-2}\{(2i+1)(e-2)+1, \dots, (2i+1)(e-2)+i-1\}) \\ & \cup \{2(e-1)(e-2)+1, 2(e-1)(e-2)+2, \dots, 2(e+1)(e-2)+2\}. \end{aligned}$$

PROOF. i) The genus of the curve C is $(e-1)^2$. Hence the order at P is less than or equal to $2(e-1)^2 - 2 = 2e(e-2)$. By Lemma 2.5 we get the result.

ii) The genus of the curve C is $e(e-1)$. Hence the order at P is less than or equal to $2e(e-1) - 2 = 2(e+1)(e-2) + 2$. By Lemma 2.7 we get the result. □

PROPOSITION 3.2. *Let* P *be any point of* $S = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$. *Let* F_P *be the fiber containing* P. *For any non-negative integers* $n \geqq 1$, $m \geqq 2n$ *and* $j \leqq n$ *there exists a non-singular curve* A *on* $S = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ *such that* $A \sim nE_0 + mF_P$ *and* $I_P(A, F_P) = n - j$ *(resp.* $2n - j$*) if* $P \notin E_0$ *(resp.* $P \in E_0$*).*

PROOF. Let e_1 be the exceptional divisor of the blowing-up of $\pi_1 : S_1 \to S$ at P and $F_P - e_1$ the proper transform of F_P for π_1. Let P_1 be the intersection between e_1 and $F_P - e_1$. Let e_2 be the exceptional divisor of the blowing-up of $\pi_2 : S_2 \to S_1$ at P_1 and $F_P - e_1 - e_2$ the proper transform of $F_P - e_1$ for π_2. Let P_2 be the intersection between e_2 and $F_P - e_1 - e_2$. For any $3 \leqq i \leqq n - j$ let e_i be the exceptional divisor of the blowing-up of $\pi_i : S_i \to S_{i-1}$ at P_{i-1} and $F_P - e_1 - e_2 - \cdots - e_i$ the proper transform of $F_P - e_1 - \cdots - e_{i-1}$ for π_i. Let P_i be the intersection between e_i and $F_P - e_1 - \cdots - e_i$.

We will show that there exists a non-singular curve A_{n-j} which is linearly equivalent to $nE_0 + mF - e_1 - \cdots - e_{n-j}$. First, in view of $m \geqq 2n$ we see that

$$(nE_0 + mF - e_1 - \cdots - e_{n-j})^2 = -2n^2 + 2nm - (n-j) \geqq 2n^2 - n + j$$
$$= n(2n-1) + j > 0.$$

Next, we will show that $|nE_0 + mF - e_1 - \cdots - e_{n-j}|$ is base-point-free. In view of $m \geqq 2n$ it suffices to show that $|nE_0 + 2nF - e_1 - \cdots - e_{n-j}|$ is base-point-free. Since we have

$$nE_0 + 2nF - e_1 - \cdots - e_{n-j} \sim (n-1)E_0 + (2n-1)F + (E_0 + F - e_1 - \cdots - e_{n-j})$$

and $|(n-1)E_0 + (2n-1)F|$ is base-point-free, the base locus of $|nE_0 + 2nF - e_1 - \cdots - e_{n-j}|$ is contained in $E_0 + (F - e_1 - \cdots - e_{n-j})$. We will show that $F - e_1 - \cdots - e_{n-j}$ is not a base locus of $|nE_0 + 2nF - e_1 - \cdots - e_{n-j}|$. We have a long exact sequence

$$0 \to \mathcal{O}(nE_0 + (2n-1)F) \to \mathcal{O}(nE_0 + 2nF - e_1 - \cdots - e_{n-j})$$
$$\to \mathcal{O}_{F-e_1-\cdots-e_{n-j}}(nE_0 + 2nF - e_1 - \cdots - e_{n-j}) \to 0.$$

Since we have

$$(nE_0 + 2nF - e_1 - \cdots - e_{n-j}, F - e_1 - \cdots - e_{n-j}) = n - (n-j) = j,$$

we get

$$0 \to \mathcal{O}(nE_0 + (2n-1)F) \to \mathcal{O}(nE_0 + 2nF - e_1 - \cdots - e_{n-j}) \to \mathcal{O}_{\mathbf{P}^1}(j) \to 0.$$

Since $(n+1)E_0 + (2n+3)F$ is ample, by Kodaira's Vanishing Theorem, we get

$$H^1(\mathcal{O}((n-1)E_0 + (2n-1)F)) = H^1(\mathcal{O}(K_{\mathbf{P}(\mathcal{O}\oplus\mathcal{O}(-2))} + (n+1)E_0 + (2n+3)F)) = 0.$$

Moreover, using the exact sequence

$$0 \to \mathcal{O}((n-1)E_0 + (2n-1)F) \to \mathcal{O}(nE_0 + (2n-1)F) \to \mathcal{O}_{E_0}(-1) \cong \mathcal{O}_{\mathbf{P}^1}(-1) \to 0$$

we get $H^1(\mathcal{O}(nE_0 + (2n-1)F)) = 0$. Thus, the map

$$H^0(\mathcal{O}(nE_0 + 2nF - e_1 - \cdots - e_{n-j})) \to H^0(\mathcal{O}_{\mathbf{P}^1}(j))$$

is surjective. Since for any $x \in F - e_1 - \cdots - e_{n-j} \cong \mathbf{P}^1$ there is some $s \in H^0(\mathcal{O}_{\mathbf{P}^1}(j))$ with $s(x) \neq 0$, we get some $\bar{s} \in H^0(\mathcal{O}(nE_0 + 2nF - e_1 - \cdots - e_{n-j}))$

such that $\bar{s}(x) \neq 0$. Hence, $F - e_1 - \cdots - e_{n-j}$ is not contained in the base locus of the linear system $|nE_0 + 2nF - e_1 - \cdots - e_{n-j}|$. We will show that E_0 is not a base locus of $|nE_0 + 2nF - e_1 - \cdots - e_{n-j}|$. We have an exact sequence

$$0 \to \mathcal{O}((n-1)E_0 + 2nF - e_1 - \cdots - e_{n-j}) \to \mathcal{O}(nE_0 + 2nF - e_1 - \cdots - e_{n-j})$$
$$\to \mathcal{O}_{E_0}(nE_0 + 2nF - e_1 - \cdots - e_{n-j}) \cong \mathcal{O}_{\mathbf{P}^1} \to 0,$$

because of

$$(nE_0 + 2nF - e_1 - \cdots - e_{n-j}, E_0) = -2n + 2n = 0.$$

Since $(n+1)E_0 + (2n+3)F$ is ample, by Kodaira's Vanishing Theorem we get

$$H^1(\mathcal{O}((n-1)E_0 + (2n-1)F)) = H^1(K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + (n+1)E_0 + (2n+3)F) = 0.$$

Moreover, we have an exact sequence

$$0 \to \mathcal{O}((n-1)E_0 + (2n-1)F) \to \mathcal{O}((n-1)E_0 + 2nF - e_1 - \cdots - e_{n-j})$$
$$\to \mathcal{O}_{F - e_1 - \cdots - e_{n-j}}((n-1)E_0 + 2nF - e_1 - \cdots - e_{n-j}) \cong \mathcal{O}_{\mathbf{P}^1}(j-1) \to 0.$$

Hence, we obtain

$$H^1((n-1)E_0 + 2nF - e_1 - \cdots - e_{n-j}) = 0,$$

which implies that the map

$$H^0(\mathcal{O}(nE_0 + 2nF - e_1 - \cdots - e_{n-j})) \to H^0(\mathcal{O}_{\mathbf{P}^1})$$

is surjective. Hence, for any $x \in E_0$ we have $\bar{s} \in H^0(\mathcal{O}(nE_0 + 2nF - e_1 - \cdots - e_{n-j}))$ such that $\bar{s}(x) \neq 0$. Therefore, E_0 is not contained in the base locus of $|nE_0 + 2nF - e_1 - \cdots - e_{n-j}|$. Thus, the linear system $|nE_0 + 2nF - e_1 - \cdots - e_{n-j}|$ is base-point-free. Hence, the linear system $|nE_0 + mF - e_1 - \cdots - e_{n-j}|$ is base-point-free, because of $nE_0 + mF - e_1 - \cdots - e_{n-j} = nE_0 + 2nF - e_1 - \cdots - e_{n-j} + (m-2n)F$. Let $\varphi : S_{n-j} \to \mathbf{P}^{\dim|nE_0 + mF - e_1 - \cdots - e_{n-j}|}$ be the rational map defined by the base-point-free linear system $|nE_0 + mF - e_1 - \cdots - e_{n-j}|$. Since $(nE_0 + mF - e_1 - \cdots - e_{n-j})^2 > 0$, we have $\dim \varphi(S_{n-j}) = 2$. In fact, we assume $\dim \varphi(S_{n-j}) \leqq 1$. Let H and H' be two general hyperplanes in $\mathbf{P}^{\dim|nE_0 + mF - e_1 - \cdots - e_{n-j}|}$. Then we would have $\varphi(S_{n-j}) \cap H \cap H' = \emptyset$. By the way we have $\varphi^* H \sim nE_0 + mF - e_1 - \cdots - e_{n-j}$. Since $(nE_0 + mF - e_1 - \cdots - e_{n-j})^2 > 0$, we get $\varphi(S_{n-j}) \cap H \cap H' \neq \emptyset$. This is a contradiction. By Theorems of Bertini (for example see Theorems 7.18 and 7.19 in [4]) there exists a non-singular irreducible curve A_{n-j} which is linearly equivalent to $nE_0 + mF - e_1 - \cdots - e_{n-j}$. Let A be the non-singular curve which we get by the succession of the blowing-downs of the above non-singular irreducible curve A_{n-j} to S. We get

$$\begin{aligned} I_P(A, F_P) &= I_P(nE_0 + mF - e_1 - \cdots - e_{n-j}, F_P) \\ &= I_P(nE_0 + mF, F_P) - \sum_{i=1}^{n-j} I_P(e_i, F_P) \\ &= I_P(nE_0, F_P) - \sum_{i=1}^{n-j} I_P(e_i, (F_P - e_i) + e_i). \end{aligned}$$

If $P \notin E_0$, then we have $I_P(A, F_P) = -\sum_{i=1}^{n-j} I_P(e_i, e_i) = n - j$. If $P \in E_0$, then we have $I_P(A, F_P) = n - \sum_{i=1}^{n-j} I_P(e_i, e_i) = 2n - j$. □

Proposition 3.3. *Let C be a non-singular curve on $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ with $C \sim eE_0 + 2eF$ and $I_P(C, F_P) = e - 2$. Let A be an effective divisor which is linearly equivalent to*

$$K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + C \sim (e-2)E_0 + 2(e-2)F.$$

Let $P \in C \backslash E_0$. We assume that $I_P(A, C) = m$ for some m which belongs to the set Z_1 in Lemma 3.1 i). *Then A is irreducible and non-singular at P. Moreover, we have $I_P(A, F_P) = e - 2$.*

Proof. We assume that A were not irreducible. Let $A = A_1 + \cdots + A_r$ where A_i's are the irreducible components of A. Then $A_i \sim a_iE_0 + b_iF$ with $a_i = 1$, $b_i = 0$ or $a_i = 0$, $b_i = 1$ or $e - 2 > a_i > 0$ and $b_i \geqq 2a_i$. We set $I_P(A_i, C) = m_i$. By Lemma 2.6 m_i's are constructed by a fiber $F \not\ni P$, $E_0 \not\ni P$, $H \sim E_0 + 2F$ with $P \in H$ and $I_P(H, C) = 1$ and a fiber F_P with $I_P(C, F_P) = e - 2$. Hence, for any $m \in Z_1$ we could not have $I_P(A, C) = m$ by the proof of Lemma 2.5. This is a contradiction. Thus, A is irreducible. Since $I_P(A, C) = m > e - 2$, i.e., $A - e_1 - \cdots - e_k \geqq 0$ with $k = e - 1$ and $I_P(F_P, C) = e - 2$, we get $I_P(A, F_P) = e - 2$. Hence, A is non-singular at P. In fact, let m_1 be the multiplicity of A at $P_1 = P$. Let P_{j+1} be the intersection between $A - e_1 - \cdots - e_j$ and e_j for $1 \leqq j \leqq e - 3$ and m_{j+1} the multiplicity of A at P_{j+1}. Then we have $m_1 + m_2 + \cdots + m_{e-2} \leqq I_P(A, F_P) = e - 2$, which implies that $m_1 = m_2 = \cdots = m_{e-2} = 1$ because of $m_j \geqq 1$ for all j. □

THEOREM 3.4. *Let $d = 2e$ be even. Let m be in the set Z_1 in Lemma* 3.1 i). *If $2e(e-2) - m$ belongs to the numerical semigroup $\langle e-2, 2(e-2)-1 \rangle$ generated by $e-2$ and $2(e-2)-1$, then there exists a double covering $\pi : \tilde{C} \to C$ with a ramification point $\tilde{P}$ where $\tilde{C}$ is a plane curve of degree d with $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d - 4$ such that m is an order at $P = \pi(\tilde{P})$, i.e., $m+1$ is a gap at P.*

PROOF. Let $P \in S = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ with $P \notin E_0$. In Proposition 3.2 we set $n = e-2$, $m = 2(e-2)$ and $j = 0$. Then there exists a non-singular curve A on S such that $A \sim (e-2)E_0 + 2(e-2)F$ and $I_P(A, F_P) = e-2$. We set $P_1 = P$. Let $\pi_1 : S_1 \to S$ be the blowing-up at P_1. Let e_1 be the exceptional divisor and P_2 the intersection between $A - e_1$ and e_1, which is also the intersection between $F_P - e_1$ and e_1. For any $2 \leqq i \leqq e-3$ let $\pi_i : S_i \to S_{i-1}$ be the blowing-up at P_i. Let e_i be the exceptional divisor and P_{i+1} the intersection between $A - e_1 - \cdots - e_i$ and e_i, which is also the intersection between $F_P - e_1 - \cdots - e_i$ and e_i. Let $m \geqq e-1$. For any $k = e-2 \leqq j \leqq m-1$ let $\pi_j : S_j \to S_{j-1}$ be the blowing-up at P_j. Let e_j be the exceptional divisor and P_{j+1} the intersection between $A - e_1 - \cdots - e_j$ and e_j. In this case we have $P_{k+1} \notin F_P - e_1 - \cdots - e_k$. Let m be in the set Z_1. We want to show that there exists a non-singular curve C on $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ with $C \sim eE_0 + 2eF$ satisfying the following:

$$P_1 \in C, \quad P_2 \in C - e_1, \quad P_3 \in C - e_1 - e_2, \quad P_m \in C - e_1 - \cdots - e_{m-1},$$
$$P_{m+1} \in C - e_1 - \cdots - e_m \quad \text{and} \quad P_{m+1} \notin A - e_1 - \cdots - e_m,$$

which implies that $I_P(C, A) = m$. By Bertini's Theorem it suffices to show that the linear system $|eE_0 + 2eF - e_1 - \cdots - e_m|$ is base point free. Now, we want to find a necessary and sufficient condition such that $|eE_0 + 2eF - e_1 - \cdots - e_m|$ is base point free. We have

$$eE_0 + 2eF - e_1 - \cdots - e_m \sim (A - e_1 - \cdots - e_m) + 2(E_0 + 2F).$$

Since $E_0 + 2F$ is base point free, the linear system $|eE_0 + 2eF - e_1 - \cdots - e_m|$ is base point free if and only if the linear system $|eE_0 + 2eF - e_1 - \cdots - e_m|$ has no base point on $A - e_1 - \cdots - e_m$. Hence, it is sufficient to prove that the linear system

$$|\mathcal{O}_{A-e_1-\cdots-e_m}(eE_0 + 2eF - e_1 - \cdots - e_m)|$$

is base point free. We have

$$(eE_0 + 2eF)|_A \sim 2eF_P|_A = 2e(e-2)P,$$

because of $(E_0, A) = (E_0, (e-2)E_0 + 2(e-2)F) = 0$, $(A, F) = e-2$ and $I_P(A, F_P) = e-2$. Thus, we get

$$(eE_0 + 2eF - e_1 - \cdots - e_m)|_{A-e_1-\cdots-e_m}$$
$$\sim (eE_0 + 2eF)|_{A-e_1-\cdots-e_m} - (e_1 + \cdots + e_m)|_{A-e_1-\cdots-e_m},$$

which is linearly equivalent to $(2e(e-2)-m)P$ on A. Hence, $|eE_0 + 2eF - e_1 - \cdots - e_m|$ is base point free if and only if so on A is $(2e(e-2)-m)P$, i.e., $2e(e-2)-m$ is a non-gap of P on A. Since we have $A \sim (e-2)E_0 + 2(e-2)F$ and $I_P(A, F_P) = e-2$, by Remark 2.3 we get the desired non-singular curve C on $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$, which we regards as a closed subscheme of $\mathbf{P}(1:1:2)$. Let $\tilde{C}$ be the fiber product $\mathbf{P}^2 \times_{\mathbf{P}(1:1:2)} C$ of $\mathbf{P}^2$ and C over $\mathbf{P}(1:1:2)$. Then the projection $p_2 : \tilde{C} \to C$ is the desired double covering with a ramification point $\tilde{P}$ over P. Since $I_P(F_P, C) = e-2$, we have $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-4$. □

Corollary 3.5. (1) *Let C be a non-singular curve on $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ with $C \sim eE_0 + 2eF$ and $I_P(C, F_P) = e-2$ with $e \geqq 3$. Let $P \in C \backslash E_0$. Let A be an effective divisor which is linearly equivalent to*

$$K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + C \sim (e-2)E_0 + 2(e-2)F.$$

We assume that $I_P(A, C) = m$ for some m which belongs to the set Z_1 in Lemma 3.1 i). *Then $2e(e-2)-m$ belongs to the numerical semigroup $\langle e-2, 2(e-2)-1\rangle$.*

(2) *Assume that m belongs to the set Z_1 in Lemma* 3.1 i). *The following are equivalent*:

i) *There exists a double covering $\pi : \tilde{C} \to C$ with a ramification point $\tilde{P}$ where $\tilde{C}$ is a plane curve of degree d such that $m+1$ is a gap at $P = \pi(\tilde{P})$ on C*

ii) $2e(e-2) - m \in \langle e-2, 2e-5\rangle$.

Proof. (1) By Proposition 3.3 A is irreducible and non-singular at P. Moreover, we have $I_P(A, F_P) = e-2$. Since $C \sim eE_0 + 2eF$ is base-point-free, so is $(eE_0 + 2eF)|_A$. Hence, $(eE_0 + 2eF - e_1 - \cdots - e_m)|_{A-e_1-\cdots-e_m} \sim (2e(e-2)-m)P$ is base-point-free on A. By Remark 2.3 we get $2e(e-2)-m \in \langle e-2, 2(e-2)-1\rangle$.

(2) ii) follows from i) by (1). Moreover, ii) implies i) by Theorem 3.4. □

EXAMPLE 3.1. Let $d = 8$, hence $e = 4$, and $t = d - 4 = 4$, i.e., there exists a double covering $\pi : \tilde{C} \to C$ with a branch point P where $\tilde{C}$ is a plane curve of degree 8. Then the genus of C is $(e-1)^2 = 9$. By Lemma 2.5 the set of gaps at P contains $\{1, 2, 3, 4, 5, 6, 7, 9\}$. By Lemma 3.1 i) the remaining gap $m+1$ at P belongs to the set $\{8, 10, 11, 12, 13, 14, 15, 16, 17\}$. By Corollary 3.5 (2) 16 is deleted, because $\langle e-2, 2e-5\rangle = \langle 2, 3\rangle \not\ni 16 - 15$. That is to say, the gap sequence at P is 1, 2, 3, 4, 5, 6, 7, 9, γ where $\gamma = m+1$ is in the set $\{8, 10, 11, 12, 13, 14, 15, 17\}$. Conversely, there exists a double covering $\eta : \tilde{E} \to E$ with a branch point Q whose gaps form the above set.

Using Lemmas 2.7 and 2.8 we get the following in a similar way to the proof of Proposition 3.3:

PROPOSITION 3.6. *Let C be a non-singular curve on $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ with $C \sim eE_0 + (2e+1)F$ and $I_P(C, F_P) = e-2$. Let A be an effective divisor which is linearly equivalent to*

$$K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + C \sim (e-2)E_0 + (2e-3)F.$$

Let $P \in C \cap E_0$. We assume that $I_P(A, C) = m$ for some m which belongs to the set Z_2 in Lemma 3.1 ii)*. Then A is irreducible and non-singular at P. Moreover, we have $I_P(A, F_P) = e-2$.*

THEOREM 3.7. *Let d be odd. We set $d = 2e+1$. Let m be in the set Z_2 in Lemma* 3.1 ii)*. If $2(e^2 - e - 1) - m$ belongs to the numerical semigroup $\langle e-2, 2e-3\rangle$ generated by $e-2$ and $2e-3$, then there exists a double covering $\pi : \tilde{C} \to C$ with a ramification point $\tilde{P}$ where $\tilde{C}$ is a plane curve of degree d with $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d-4$ such that m is an order at $P = \pi(\tilde{P})$, i.e., $m+1$ is a gap at P.*

PROOF. Let $P \in S = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ with $P \in E_0$. In Proposition 3.2 we set $n = e-2$, $m = 2e-3$ and $j = e-2$. Then there exists a non-singular curve A on S such that $A \sim (e-2)E_0 + (2e-3)F$ and $I_P(A, F_P) = e-2$. We set $P_1 = P$. Let $\pi_1 : S_1 \to S$ be the blowing-up at P_1. Let e_1 be the exceptional divisor and P_2 the intersection between $A - e_1$ and e_1, i.e., the intersection between $F_P - e_1$ and e_1. For any $2 \leqq i \leqq e-3$ let $\pi_i : S_i \to S_{i-1}$ be the blowing-up at P_i. Let e_i be the exceptional divisor and P_{i+1} the intersection between $A - e_1 - \cdots - e_i$ and e_i, i.e., the intersection between $F_P - e_1 - \cdots - e_i$ and e_i. Let $m \geqq e-1$. For any $k = e-2 \leqq j \leqq m-1$ let $\pi_j : S_j \to S_{j-1}$ be the blowing-up at P_j. Let e_j be the exceptional divisor and P_{j+1} the intersection between $A - e_1 - \cdots - e_j$ and e_j. In this case we have $P_{k+1} \notin F_P - e_1 - \cdots - e_k$. Let m be in the set Z_2. We want to show that there exists a non-singular curve C with $C \sim eE_0 + (2e+1)F$ satisfying the following:

$$P_1 \in C, P_2 \in C - e_1, P_3 \in C - e_1 - e_2, \ldots, P_m \in C - e_1 - \cdots - e_{m-1},$$

$$P_{m+1} \in C - e_1 - \cdots - e_m \quad \text{and} \quad P_{m+1} \notin A - e_1 - \cdots - e_m,$$

which implies that $I_P(C, A) = m$. By Bertini's Theorem it suffices to show that the linear system $|eE_0 + (2e+1)F - e_1 - \cdots - e_m|$ is base point free. Then there is $C \sim eE_0 + (2e+1)F$ such that $P_{m+1} \notin C - e_1 - \cdots - e_m$, because $|C - e_1 - \cdots - e_m|$ is base point free. Now, we want to find a necessary and sufficient condition such that $|eE_0 + (2e+1)F - e_1 - \cdots - e_m|$ is base point free. We have

$$eE_0 + (2e+1)F - e_1 - \cdots - e_m \sim (A - e_1 - \cdots - e_m) + 2(E_0 + 2F).$$

Since $E_0 + 2F$ is base point free, the linear system $|eE_0 + (2e+1)F - e_1 - \cdots - e_m|$ is base point free if and only if the linear system $|eE_0 + (2e+1)F - e_1 - \cdots - e_m|$ has no base point on $A - e_1 - \cdots - e_m$. Hence, it is sufficient to prove that the linear system

$$|\mathcal{O}_{A-e_1-\cdots-e_m}(eE_0 + (2e+1)F - e_1 - \cdots - e_m)|$$

is base point free. We have

$$\begin{aligned}(eE_0 + (2e+1)F)|_A \sim e(E_0, A) + (2e+1)F_P|_A &= ((2e+1)(e-2) + e)P \\ &= 2(e^2 - e - 1)P,\end{aligned}$$

because of $E_0 \ni P$, $(E_0, A) = (E_0, (e-2)E_0 + (2e-3)F) = 1$, $(A, F) = e-2$ and $I_P(A, F_P) = e - 2$. Thus, we get

$$\begin{aligned}&(eE_0 + (2e+1)F - e_1 - \cdots - e_m)|_{A-e_1-\cdots-e_m} \\ &\quad \sim (eE_0 + (2e+1)F_P)|_{A-e_1-\cdots-e_m} - (e_1 + \cdots + e_m)|_{A-e_1-\cdots-e_m},\end{aligned}$$

which is linearly equivalent to $(2(e^2 - e - 1) - m)P$ on A. Hence, $|eE_0 + (2e+1)F - e_1 - \cdots - e_m|$ is base point free if and only if so on A is $(2(e^2 - e - 1) - m)P$, i.e., $2(e^2 - e - 1) - m$ is a non-gap of P on A. Since we have $A \sim (e-2)E_0 + (2e-3)F$ and $I_P(A, F_P) = e - 2$, by Remark 2.4 we get the desired non-singular curve C. We can get a double cover $\tilde{C}$ of C with a ramification point $\tilde{P}$ over P such that $I_{\tilde{P}}(T_{\tilde{P}}, \tilde{C}) = d - 4$ as in the proof of Theorem 3.4 because of $P \in E_0$. $\square$

By Proposition 3.6 and Theorem 3.7 we get the following:

COROLLARY 3.8. (1) *Let C be a non-singular curve on $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ with $C \sim eE_0 + (2e+1)F$ and $I_P(C, F_P) = e-2$. Let $P \in C \cap E_0$.*

Let A be an effective divisor which is linearly equivalent to

$$K_{\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))} + C \sim (e-2)E_0 + (2e-3)F.$$

We assume that $I_P(A, C) = m$ for some m which belongs to the set Z_2 in Lemma 3.1 ii). *Then $2(e^2 - e - 1) - m$ belongs to the numerical semigroup $\langle e-2, 2e-3 \rangle$.*

(2) *Assume that m belongs to the set Z_2 in Lemma* 3.1 ii). *The following are equivalent*:

i) *There exists a double covering $\pi : \tilde{C} \to C$ with a ramification point $\tilde{P}$ where $\tilde{C}$ is a plane curve of degree d such that $m+1$ is a gap at $P = \pi(\tilde{P})$ on C*

ii) $2(e^2 - e - 1) - m \in \langle e-2, 2e-3 \rangle$.

EXAMPLE 3.2. Let $d = 9$, hence $e = 4$ and $t = d - 4$, i.e., there exists a double covering $\pi : \tilde{C} \to C$ with a branch point P where $\tilde{C}$ is a plane curve of degree 9. Then the genus of C is $e(e-1) = 12$. By Lemma 2.7 the set of gaps at P contains $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13\}$. By Lemma 3.1 ii) the remaining gap $m+1$ at P belongs to the set $\{10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. By Corollary 3.8 (2) 20 and 22 are deleted, because $\langle e-2, 2e-3 \rangle = \langle 2, 5 \rangle \not\ni 22 - 19, 22 - 21$. That is to say, the gap sequence at P is 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, γ where $\gamma = m+1$ is in the set $\{10, 12, 14, 15, 16, 17, 18, 19, 21, 23\}$. Conversely, there exists a double covering $\eta : \tilde{E} \to E$ with a branch point Q whose gaps form the above set.

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