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RANDOM GRAPHS WITH A RANDOM BIJECTION

By

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Abstract. We show that the theory of random graphs with a bijection between the binary Cartesian product of the universe and the universe has a model companion which is complete, simple, and unsupersimple.

1. Introduction

Let T be an L-theory and σ a new unary function symbol. The theory $T \cup \{\sigma$ is an automorphism$\}$, which is usually written as T_σ, is of particular interest in model theory. In particular, it is an interesting problem to determine whether or not T_σ has a model companion.

If T is the theory of algebraically closed field, it is known that T_σ has a model companion, usually called $ACFA$. The theory $ACFA$ was used in Hrushovski's proof of the Manin-Mumford conjecture (see [5]). In contrast to this, Kikyo [6] showed that if T is the theory of random graphs then T_σ has no model companion.

In this paper, we discuss a somewhat related problem when f is a binary function symbol. Let

$$T_f = T \cup \{f \text{ is a bijection between } M^2 \text{ and } M\},$$

where M is the universe of the structure. In this paper, we treat the case when T is the theory of random graphs, and show that T_f has a model companion. We also show that the model companion is complete and simple (in the sense of Shelah). Further we will show that the model companion is not supersimple.

The present work is related to other authors' works including [3], [7], and [2]. In [3], Chatzidakis and Piliay proved the following: Suppose that L is a language

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containing a unary predicate symbol S and T is an L-theory which has quantifier elimination. For a new unary predicate symbol P, let T_P be the theory $T \cup \{\forall x(P(x) \to S(x))\}$. Then if T eliminates the quantifier $\exists^\infty$, then T_P has a model companion. To get a new theory, they added a new predicate symbol to a simple theory and we add a new function symbol to the theory of random graphs.

In [7], Tsuboi proved under certain assumptions that for two simple theories T_1 and T_2 (in disjoint languages), one can find a simple theory extending $T_1 \cup T_2$. If we know there is a simple (model complete) theory T_1 extending the theory $\{f$ is a bijection between M^2 and $M\}$, then by taking T_2 as the theory of random graphs, we can apply [7] to get a simple theory extending T_f. So the main task is to show the existence of T_1 described above. But, for self-containedness of the present paper, without applying [7], we directly show the existence of a model companion of T_f.

In [2], Casanovas and Kim showed an example of a supersimple nonlow theory. Their structure is divided into two sorts, the sort for points and the sort for sets. One of the motivation of the present paper is to know how structure is obtained by combining two sorts of their structure into one sort.

Basic definitions and facts are reviewed in Section 2.

In Section 3, we start our construction. For simplicity, we discuss only the case f is a binary function symbol in this paper. Without significant changes, everything in this paper can be generalized to the case f is an n-ary function symbol. For a technical reason, we do not consider the function f itself, but will consider the graph of f. For this purpose we add a ternary relation symbol R_f to the graph language. In this section, we also introduce the notion of good pairs. Using this notion, we give the exact set T of axioms for model companions.

In Section 4, we prove that T is simple, but not supersimple.

Notation. For sets A and B, we often write AB to denote the union $A \cup B$.

2. Preliminaries

In this paper, L is a countable language and T is an L-theory. We do not assume T to be complete, unless otherwise stated. We use $x, y, \dots$ for denoting variables. Finite tuples of variables are denoted by $\bar{x}, \bar{y}, \dots$. Formulas are denoted by $\varphi, \psi, \dots$.

Definition 2.1. Let T be an L-theory. A model M of T is said to be an existentially closed model of T if for any quantifier-free L-formula $\varphi(\bar{x}, \bar{y})$, and finite tuple $\bar{a}$ of M, if $N \models \exists \bar{x}\varphi(\bar{x}, \bar{a})$ for some $N \supset M$ which is a model of T, then $M \models \exists \bar{x}\varphi(\bar{x}, \bar{a})$.

For example, an algebraically closed field is an existentially closed model of the theory of fields.

Fact 2.2 ([4] Theorem 8.2.1). *Suppose that T is an $\forall\exists$-theory. Then there exists an existentially closed model of T.*

Definition 2.3. Let T and T' be two L-theories.

(1) T is said to be model complete if every embedding between models of T is elementary.
(2) T' is said to be a model companion of T if the following conditions are satisfied:
    (a) every model of T can be embedded in some model of T';
    (b) every model of T' can be embedded in some model of T;
    (c) T' is model complete.

Fact 2.4 ([4] Theorem 8.3.6). *Let T be an L-theory. T has a model companion if and only if the class of existentially closed models of T is axiomatizable in a first-order theory and, in that case, the model companion of T is its axiomatization.*

Until the end of this section, we assume the following: T is a countable complete theory; We work in a big saturated model $\mathscr{M}$ of T; $A, B, \ldots$ denote subsets of $\mathscr{M}$ whose cardinalities are strictly less than that of $\mathscr{M}$; $a, b, \ldots$ denote elements of $\mathscr{M}$; $\bar{a}, \bar{b}, \ldots$ denote finite tuples of elements of $\mathscr{M}$; For $\bar{a}$ and A, $\mathrm{tp}(\bar{a}/A)$ denote the set of formulas with parameters in A which are realized by $\bar{a}$ in $\mathscr{M}$.

Definition 2.5. Let $\varphi(\bar{x}, \bar{y})$ be an L-formula.

(1) We say that $\varphi(\bar{x}, \bar{b})$ divides over A if there is an indiscernible sequence $(\bar{b}_i)_{i<\omega}$ over A with $\bar{b}_0 = \bar{b}$ such that $\bigcup_{i<\omega}\{\varphi(\bar{x}, \bar{b}_i)\}$ is inconsistent.
(2) We say that a partial type $\Gamma(\bar{x})$ divides over A if there is a formula $\varphi(\bar{x})$ such that $\Gamma(\bar{x}) \vdash \varphi(\bar{x})$ and $\varphi(\bar{x})$ divides over A.

Definition 2.6. We say that T is simple if T has local character of dividing, that is, for any finite tuple $\bar{a}$ and any set B, there is a countable subset A of B such that $\mathrm{tp}(\bar{a}/B)$ does not divide over A.

DEFINITION 2.7. We say that T is supersimple if for any finite tuple $\bar{a}$ and any set B, there is a finite subset A of B such that $\mathrm{tp}(\bar{a}/B)$ does not divide over A.

3. Construction

Let $L = \{R, R_f\}$ be a relational language, where R is a binary relation symbol and R_f is a ternary relation symbol. We always assume that for any L-structure A, R^A is a graph relation, and R_f^A is the graph of a partial injective function from A^2 to A.

Let T_f be the following (incomplete) theory:

- the universe, say M, is a random graph;
- R_f^M is the graph of a bijection between M^2 and M.

DEFINITION 3.1. Let $A \subset B$ be an extension of L-structures. We say that (A, B) is a good pair if for any $a, b, c \in B$,

- if $a, b \in A$ and $B \models R_f(a, b, c)$, then $c \in A$ and
- if $c \in A$ and $B \models R_f(a, b, c)$, then $a, b \in A$.

Intuitively, if (A, B) is a good pair, then A is closed under f and f^{-1} in B, where f is the (partial) function which maps (a, b) to c if $R_f(a, b, c)$ holds in B for any $a, b, c \in B$.

DEFINITION 3.2. Suppose $\bar{a} = (a_0, \ldots, a_{m-1})$ and $\bar{b} = (b_0, \ldots, b_{n-1})$ are two finite tuples. Let $\varphi_{\bar{a}, \bar{b}}(\bar{x}, \bar{y})$ denote the conjunction of the L-diagram of $\bar{a}^\frown\bar{b}$, where $\bar{x}$ is a tuple of variables for $\bar{a}$ and $\bar{y}$ is a tuple of variables for $\bar{b}$. The formula $\varphi_{\bar{a}, \bar{a}}(\bar{x}, \emptyset)$ will be denoted by $\psi_{\bar{a}}(\bar{x})$.

REMARK 3.3. For simplicity, let us use the notation $\varphi_{A,B}(X, Y)$ for sets A and B of elements and sets X and Y of variables to denote the formula $\varphi_{\bar{a}, \bar{b}}(\bar{x}, \bar{y})$ where $\bar{a}$ (resp. $\bar{b}$, $\bar{x}$, $\bar{y}$) is a tuple which is an enumeration of all elements of A (resp. B, X, Y).

DEFINITION 3.4. Define the theory T as follows:

$$T := T_f \cup \{\forall X(\psi_A(X) \to \exists Y \varphi_{A,B}(X, Y)) \mid (A, B) \text{ is a good pair}\}.$$

LEMMA 3.5. *For any model M of T_f, M is an existentially closed model of T_f if and only if M is a model of T.*

PROOF. (only if): Suppose that M is an existentially closed model of T_f, (A, B) is a good pair, and there is $C \subset M$ such that $M \models \psi_A(C)$. Take an sufficiently saturated random graph (N, R^N) extending (M, R^M), the reduct of M to the language $\{R\}$. Then by the saturation of N, we can get a subset D' of $N \backslash M$ such that CD' is isomorphic to AB as graphs. Let σ denote this isomorphism.

We expand (N, R^N) to an L-structure (N, R^N, R_f^N) as follows. Let

- $R_f^N \cap M^3 = R_f^M$ and
- $N \models R_f^N(a, b, c)$ if and only if $B \models R_f^B(\sigma(a), \sigma(b), \sigma(c))$ for each $a, b, c \in D'$.

Then, R_f^N is the graph of an injection f_1 from a subset of N^2. Take any extension f_2 of f_1 which is a bijection between N^2 and N and define $N \models R_f^N(a, b, c)$ if $f_2(a, b) = c$ for any $a, b, c \in N$. Then we have $M \subset N \models T_f$ and $N \models \varphi_{A,B}(C, D')$. Since M is an existentially closed model of T_f and $C \subset M$, we have

$$M \models \exists Y \varphi_{A,B}(C, Y).$$

(if): Let M be a model of T and N a model of T_f extending M. Suppose that $N \models \varphi(A, B)$, where φ is a quantifier-free L-formula, A is a finite subset of M, and B is a finite subset of $N \backslash M$. Then (A, B) is a good pair, since R_f^M is the graph of a bijection between M^2 and M. So we can take a subset C of M such that $AC \cong AB$ as L-structures. Because φ is quantifier-free and $N \models \varphi(A, B)$, we have

$$M \models \varphi(A, C).$$

Therefore, M is an existentially closed model of T_f. □

COROLLARY 3.6. *T is a model companion of T_f.*

PROOF. By the above lemma and Fact 2.4. □

For $M \models T$ and $A \subset M$, let $\mathrm{cl}_M(A)$ denote the smallest subset B of M which satisfies

(1) $A \subset B$,

(2) for any $a, b, c \in M$, if $M \models R_f^M(a, b, c)$ and $a, b \in B$, then $c \in B$, and

(3) for any $a, b, c \in M$, if $M \models R_f^M(a, b, c)$ and $c \in B$, then $a, b \in B$.

If there is no confusion, the subscripts M in $\mathrm{cl}_M(A)$ will be omitted for simplicity. We say that A is closed (in M) if $\mathrm{cl}(A) = A$.

Remark 3.7. Suppose that M is a model of T. For $a \in M$ and $1 \leq n < \omega$, we define sets $S_n(a)$ as follows:

(1) $S_0(a) = \{a\}$

(2) $S_1(a) = \{b_0, b_1\}$, where $M \models R_f(b_0, b_1, a)$.

(3) $S_{n+1}(a) = \bigcup \{S_1(b) \mid b \in S_n(a)\}$.

Then for $A \subset M$ and $b \in M$, we have

$$b \in \mathrm{cl}(A) \Leftrightarrow \left(\bigcup_{n<\omega} S_n(b)\right) \cap \left(\bigcup_{a \in A} \bigcup_{n<\omega} S_n(a)\right) \neq \emptyset.$$

Proposition 3.8. *T is complete.*

Proof. Let M and N be two ω_1-saturated models of T. Suppose that a countable closed subset A of M and a partial L-isomorphism σ from A to $\sigma(A) \subset N$ are given. Take any $b \in M \setminus A$ and put $B = \mathrm{cl}(bA)$. We want to extend σ to a partial isomorphism from B. Let $p(X, Y)$ be the quantifier-free type of B, where X is a set of variables for A, and Y is a set of variables for $B \setminus A$. Take any finite subset A_0 of A and any finite subset B_0 of $B \setminus A$. Note that, by closedness of A, $(A_0, A_0 B_0)$ is a good pair. So $\varphi_{A_0, B_0}(\sigma(A_0), Y_0)$ is satisfiable in N, where Y_0 is a subset of Y corresponding to B_0. Then by ω_1-saturation of N, the type $p(\sigma(A), Y)$ is satisfiable in N. Therefore, by a back-and-forth argument, we can show that M and N are elementarily equivalent. □

4. Simplicity

In this section, we prove that T is simple but not supersimple. Again, we work in a big saturated model $\mathscr{M}$ of T.

Lemma 4.1. *Suppose that A and B are two closed sets. If A and B have the same quantifier-free types, then* $\mathrm{tp}(A) = \mathrm{tp}(B)$.

Proof. By a similar back-and-forth argument as in Proposition 3.8. □

Lemma 4.2. *For any set A,* $\mathrm{cl}(A) = \mathrm{acl}(A)$.

Proof. By Remark 3.7, we have $\mathrm{cl}(A) \subset \mathrm{acl}(A)$. To show the other direction, assume that $A = \mathrm{cl}(A)$ and $b \notin \mathrm{cl}(A)$. Put $B = \mathrm{cl}(bA) \setminus A$. It is enough to show that there are infinitely many subsets B_i $(i < \omega)$ of $\mathscr{M}$ such that $\mathrm{tp}(B_i'/A) = \mathrm{tp}(B/A)$.

Define a new L-structure N as follows:

(1) the universe of N is $\bigcup_{i<\omega} A'B'_i$,

(2) $B'_i \cap B'_j = \emptyset$ for each $i < j < \omega$,

(3) $A'B_i \cong AB$,

(4) $R^N = \bigcup_{i<\omega} R^{A'B'_i}$, and $R_f^N = \bigcup_{i<\omega} R_f^{A'B'_i}$.

Because A is closed, N is well-defined and (A', N) is a good pair. So identifying A and A', we get an embedding σ from N into $\mathscr{M}$ over A. For each $i < \omega$, because $A\sigma(B'_i)$ is closed, by Lemma 4.1, we have $\text{tp}(\sigma(B'_i)/A) = \text{tp}(B/A)$. □

Lemma 4.3. *Suppose that A and B are two closed sets. Then for any indiscernible sequence $I = (B_i)_{i<\omega}$ with $B_0 = B$ over $A \cap B$, there is a subset A' of M such that* $\text{tp}(A'B_i) = \text{tp}(AB)$ *for each* $i < \omega$.

Proof. For simplicity, we assume $A \cap B = \emptyset$. Notice that B_i's form a Δ-system. So, if C is the intersection of B_i's then B_i's are pairwise disjoint over C. Put $D = \text{cl}(AC) \setminus C$ and $E = \text{cl}(BD) \setminus BD$. It is enough to show that there are subsets D' and E_i $(i < \omega)$ of $\mathscr{M}$ such that $\text{tp}(B_iD'E_i) = \text{tp}(BDE)$.

Define a new L-structure N as follows:

- the universe of N is $(\bigcup_{i<\omega} B'_i)D'(\bigcup_{i<\omega} E_i)$, (put $I' = \bigcup_{i<\omega} B'_i$,)
- $I' \cong I$,
- $E'_i \cap E'_j = \emptyset$ for each $i < j < \omega$,
- $B'_iD'E_i \cong BDE$ for each $i < \omega$,
- $R^N = R^{I'} \cup \bigcup_{i<\omega} R^{B_iD'E_i}$, and $R_f^N = \bigcup_{i<\omega} R_f^{B_iD'E_i}$.

Then (I', N) is a good pair. So identifying I' and I, we get an embedding σ from N into $\mathscr{M}$. Then for each $i < \omega$, because $B_i\sigma(D'E_i)$ is closed, by Lemma 4.1 we have $\text{tp}(B_i\sigma(D'E_i)) = \text{tp}(BDE)$. □

Lemma 4.4. *Suppose A and B are two closed sets, $A \subset B$, and $\bar{a}$ is a finite tuple. Then the following are equivalent*:

(1) $\text{tp}(\bar{a}/B)$ *divides over* A

(2) $\text{cl}(\bar{a}A) \cap B \neq A$

Proof. $(1 \to 2)$: Suppose that $\text{cl}(\bar{a}A) \cap B = A$. Then by lemma 4.3, $\text{tp}(\bar{a}/B)$ does not divide over A.

$(2 \to 1)$: By Lemma 4.2. □

Theorem 4.5. *T is simple.*

Proof. Take a finite tuple $\bar{a}$ and a set A. Without loss of generality, we may assume that A is closed. By the above lemma, $\mathrm{tp}(\bar{a}/A)$ does not divide over $\mathrm{cl}(\bar{a}) \cap A$. Because $\mathrm{cl}(\bar{a})$ is countable, we have local character of dividing. $\square$

Proposition 4.6. *T is not supersimple.*

Proof.

Claim 4.6.1. *We can take a binary tree $A = (a_\eta : \eta \in 2^{<\omega})$ of distinct elements such that $R_f(a_{\eta^\frown 0}, a_{\eta^\frown 1}, a_\eta)$ holds for each $\eta \in 2^{<\omega}$.*

Proof. Let $A' = \{a'_\eta \mid \eta \in 2^{<\omega}\}$ be an L-structure having only required relations. Because $(\emptyset, A')$ is a good pair, we can embed A' in $\mathscr{M}$. $\square$

For each $1 \le n < \omega$, let b_n be the element $a_{(0,\dots,0,1)}$, where $(0,\dots,0)$ has the length n. Put $B = \{b_n \mid 1 \le n < \omega\}$. It is enough to prove that $\mathrm{tp}(a_{(0)}/B)$ divides over $\{b_1,\dots,b_n\}$ for each $1 \le n < \omega$. Take any $1 \le n < \omega$. Let $\varphi(x,y)$ be the formula

$$\exists x_1 \exists y_1 \cdots \exists x_{n-1} \exists y_{n-1} \exists x_n R_f(x_n, y, x_{n-1}) \wedge \bigwedge_{i=2}^{n-1} R_f(x_i, y_i, x_{i-1}) \wedge R_f(x_1, y_1, x).$$

Let f be the function from $\mathscr{M}^2$ to $\mathscr{M}$ which maps (a,b) to c if $\mathscr{M} \models R_f(a,b,c)$ for each $a,b,c \in \mathscr{M}$. Then $\mathscr{M} \models \varphi(a,b)$ if there are elements $a_1, b_1, \dots, a_{n-1}, b_{n-1}, a_n$ such that $f(a_i, b_i) = a_{i-1}$ for each $i = 1,\dots,n$, where $a_0 = a$ and $b_n = b$. Note that $\varphi(x, b_n) \in \mathrm{tp}(a_0/B)$.

Claim 4.6.2. *$\varphi(x, b_n)$ divides over $\{b_1,\dots,b_{n-1}\}$.*

Proof. By Remark 3.7 and the choice of A, we have $b_n \notin \mathrm{cl}(b_1,\dots,b_{n-1})$. We will show that there are subsets C and D_i $(i < \omega)$ such that

- $\mathrm{tp}(C, D_i) = \mathrm{tp}(\mathrm{cl}(b_1,\dots,b_{n-1}), (\mathrm{cl}(b_1,\dots,b_n) \setminus \mathrm{cl}(b_1,\dots,b_{n-1})))$,
- $D_i \cap D_j = \emptyset$ for $i < j < \omega$.

Define a new L-structure N as follows:

- the universe of N is $C \cup \bigcup_{i<\omega} D_i$,
- $D_i \cap D_j = \emptyset$ for $i < j < \omega$,
- $C \cong \mathrm{cl}(b_1 \cdots b_{n-1})$,
- $CD_i \cong \mathrm{cl}(b_1 \cdots b_n)$ for each $i < \omega$,
- $R^N = \bigcup_{i<\omega} R^{CD_i}$, and $R_f^N = \bigcup_{i<\omega} R_f^{CD_i}$.

Then, (CD_0, N) is a good pair. So identifying CD_0 and $\mathrm{cl}(b_1 \cdots b_n)$, we can embed N in $\mathscr{M}$.

Take any $i < \omega$. Because CD_i is closed, by Lemma 4.1, $\mathrm{tp}(D_i/C) = \mathrm{tp}(D_0/C)$. Take an automorphism σ of $\mathscr{M}$ over C mapping D_0 to D_i. Clearly, the formula $\varphi(x, b_n) \wedge \varphi(x, \sigma(b_n))$ is inconsistent. $\square$

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