CHAIN MIXING ENDOMORPHISMS ARE APPROXIMATED BY SUBSHIFTS ON THE CANTOR SET

By
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Abstract. Let $f$ be a chain mixing continuous onto mapping from the Cantor set onto itself. Let $g$ be a homeomorphism on the Cantor set that is topologically conjugate to a subshift. Then, homeomorphisms that are topologically conjugate to $g$ approximate $f$ in the topology of uniform convergence if a trivial necessary condition on the periodic points holds. In particular, if $f$ is a chain mixing continuous onto mapping from the Cantor set onto itself with a fixed point, then homeomorphisms on the Cantor set that are topologically conjugate to a subshift approximate $f$ in the topology of uniform convergence. In addition, homeomorphisms on the Cantor set that are topologically conjugate to a subshift without periodic points approximate any chain mixing continuous onto mappings from the Cantor set onto itself. In particular, let $f$ be a homeomorphism on the Cantor set that is topologically conjugate to a full shift. Let $g$ be a homeomorphism on the Cantor set that is topologically conjugate to a subshift. Then, a sequence of homeomorphisms that is topologically conjugate to $g$ approximates $f$.

1. Introduction

Let $(X,d)$ be a compact metric space. Let $f : X \to X$ be a continuous onto mapping. In this manuscript, the pair $(X,f)$ is called a topological dynamical system. Let $\mathcal{H}^+(X)$ be the set of all topological dynamical systems on $X$. For any $f$ and $g$ in $\mathcal{H}^+(X)$, we define $d(f,g) := \sup_{x \in X} d(f(x),g(x))$. Then, $(\mathcal{H}^+(X),d)$
is a metric space of uniform convergence. \( \mathcal{H}(X) \) denotes the set of all homeomorphisms from \( X \) onto itself. In this manuscript, we mainly consider the case in which \( X \) is homeomorphic to the Cantor set, denoted by C. T. Kimura [3, Theorem 1] and I [4] have shown that the subset of \( \mathcal{H}(C) \) consisting of all expansive homeomorphisms with the pseudo-orbit tracing property is dense in \( \mathcal{H}(C) \). SFT\((C) \) denotes the set of all \( f \in \mathcal{H}(C) \) that is topologically conjugate to some two-sided subshift of finite type. Then, SFT\((C) \) coincides with the set of all expansive \( f \in \mathcal{H}(C) \) with the pseudo-orbit tracing property (P. Walters [5, Theorem 1]). Therefore, SFT\((C) \) is dense in \( \mathcal{H}(C) \). A topological dynamical system \((X, f)\) is said to be topologically mixing if for any pair of non-empty open sets \( U, V \subseteq X \), there exists a non-negative integer \( N \) such that \( f^n(U) \cap V \neq \emptyset \) for all \( n > N \). In [4], it is shown that if \( f \in \mathcal{H}(C) \) is topologically mixing, then there exists a sequence \( \{g_k\}_{k=1,2,...} \) of topologically mixing elements of SFT\((C) \) such that \( g_k \to f \) as \( k \to \infty \). Let \( f \) be a chain mixing element of \( \mathcal{H}^+(C) \) and \( g \), an element of \( \mathcal{H}(C) \) that is topologically conjugate to a two-sided subshift. In this manuscript, we consider the condition in which homeomorphisms that are topologically conjugate to \( g \) approximate \( f \). Let \((X, f)\) be a topological dynamical system and \( \delta > 0 \). A sequence \( \{x_i\}_{i=0,1,...,l} \) of elements of \( X \) is a \( \delta \) chain from \( x_0 \) to \( x_l \) if \( d(f(x_i), x_{i+1}) < \delta \) for all \( i = 0, 1, \ldots, l-1 \). Then, \( l \) is called the length of the chain. A topological dynamical system \((X, f)\) is chain mixing if for every \( \delta > 0 \) and every pair \( x, y \in X \), there exists a positive integer \( N \) such that for all \( n > N \), there exists a \( \delta \) chain from \( x \) to \( y \) of length \( n \). Let \((X, f)\) and \((Y, g)\) be topological dynamical systems. We write \((Y, g) \Rightarrow (X, f)\) if there exists a sequence of homeomorphisms \( \{\psi_k\}_{k=1,2,...} \) from \( Y \) onto \( X \) such that \( \psi_k \circ g \circ \psi_k^{-1} \to f \) as \( k \to \infty \). If \((Y, g) \Rightarrow (X, f)\) and if \( g^n \) has a fixed point for some positive integer \( n \), then \( f^n \) must also have a fixed point. We write \((Y, g) \Rightarrow (X, f)\) if this trivial necessary condition on periodic points holds. We show the following:

**Theorem 1.1.** Let \( X \) be homeomorphic to the Cantor set. Let \((X, f)\) be a chain mixing topological dynamical system. Let \((\Lambda, \sigma)\) be a two-sided subshift such that \( \Lambda \) is homeomorphic to \( C \). Then, the following conditions are equivalent:

1. \((\Lambda, \sigma) \Rightarrow (X, f)\);
2. \((\Lambda, \sigma) \Rightarrow (X, f)\).

**Corollary 1.2.** Let \( X \) be homeomorphic to the Cantor set. Let \((X, f)\) be a chain mixing topological dynamical system with a fixed point. Let \((\Lambda, \sigma)\) be a two-sided subshift such that \( \Lambda \) is homeomorphic to \( C \). Then, \((\Lambda, \sigma) \Rightarrow (X, f)\).
Corollary 1.3. Let $X$ be homeomorphic to the Cantor set. Let $(X, f)$ be a chain mixing topological dynamical system. Let $(\Lambda, \sigma)$ be a two-sided subshift such that $\Lambda$ is homeomorphic to $C$ without periodic points. Then, $(\Lambda, \sigma) \Rightarrow (X, f)$.

Corollary 1.4. Let $n > 1$ be an integer. Let $(S_n^1, s)$ be the two-sided full shift of $n$ symbols. Let $(L, s)$ be a two-sided subshift such that $L$ is homeomorphic to $C$. Then, $(L, s) \Rightarrow (S_n^1, s)$.

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2. Preliminaries

Let $\mathbb{Z}$ denote the set of all integers; $\mathbb{N}$, the set of all nonnegative integers; and $\mathbb{Z}_+$, the set of all positive integers. Let $(X, d)$ be a compact metric space. A topological dynamical system $(X, f)$ is topologically conjugate to a topological dynamical system $(Y, g)$ if there exists a homeomorphism $c: Y \to X$ such that $f \circ c = c \circ g$. Such a homeomorphism is called a topological conjugacy.

Lemma 2.1. Let $(X, f)$ be a topological dynamical system. Let $(Y_k, g_k)$ $(k = 1, 2, \ldots)$ be a sequence of topological dynamical systems. Suppose there exists a sequence of homeomorphisms $c_k: Y_k \to X$ such that $c_k \circ g_k \circ c_k^{-1} \to f$ as $k \to \infty$. Let $(Z, h)$ be a topological dynamical system such that $(Z, h) \Rightarrow (Y_k, g_k)$ for all $k = 1, 2, \ldots$. Then, $(Z, h) \Rightarrow (X, f)$.

Proof. Let $\varepsilon > 0$. Then, there exists $N \in \mathbb{Z}_+$ such that $d(\psi_k \circ g_k \circ \psi_k^{-1}, f) < \varepsilon/2$ for all $k > N$. Assume $k > N$. Let $\delta > 0$ be such that if $d(y, y') < \delta$, then $d(\psi_k(y), \psi_k(y')) < \varepsilon/2$. Because $(Z, h) \Rightarrow (Y_k, g_k)$, there exists a homeomorphism $\psi': Z \to Y_k$ such that $d(\psi' \circ h \circ \psi'^{-1}, g_k) < \delta$. Therefore, we find that

$$d((\psi_k \circ \psi') \circ h \circ (\psi_k \circ \psi')^{-1}, f) < d(\psi_k \circ (\psi' \circ h \circ \psi'^{-1}) \circ \psi_k^{-1}, \psi_k \circ g_k \circ \psi_k^{-1})$$

$$+ d(\psi_k \circ g_k \circ \psi_k^{-1}, f) < \varepsilon.$$  

For a topological dynamical system $(X, f)$, we define

$$\text{Per}(X, f) := \{ n \in \mathbb{Z}_+ | f^n(x) = x \text{ for some } x \in X \}.$$
Let \((X, f)\) and \((Y, g)\) be topological dynamical systems. Suppose that \((Y, g) \supseteq (X, f)\). Then, for each \(n \in \mathbb{Z}_+\), \((Y, g^n) \supseteq (X, f^n)\). Consider a sequence of homeomorphisms \(\{\psi_k\}_{k=1,2,\ldots}\) from \(Y\) onto \(X\) such that \(\psi_k \circ g \circ \psi_k^{-1} \to f\) as \(k \to \infty\). Then, for each \(n \in \mathbb{Z}_+\), the fixed points of \(\psi_k \circ g^n \circ \psi_k^{-1}\) approach some of the fixed points of \(f^n\). Thus, we obtain \(\operatorname{Per}(Y, g) \subseteq \operatorname{Per}(X, f)\). We write \((Y, g) \overset{\operatorname{per}}{\to} (X, f)\) if \(\operatorname{Per}(Y, g) \subseteq \operatorname{Per}(X, f)\). Thus, we obtain the following:

\textbf{Lemma 2.2.} Let \((X, f)\) and \((Y, g)\) be topological dynamical systems. If \((Y, g) \supseteq (X, f)\), then \((Y, g) \overset{\operatorname{per}}{\to} (X, f)\).

Let \(C\) be the Cantor set in the interval \([0, 1]\). A compact metrizable totally disconnected perfect space is homeomorphic to \(C\). Therefore, any non-empty open and closed subset of \(C\) is homeomorphic to \(C\). Let \(V = \{v_1, v_2, \ldots, v_n\}\) be a finite set of \(n > 0\) symbols with discrete topology. Let \(\Sigma(V) := V^\mathbb{Z}\) with the product topology. Then, \(\Sigma(V)\) is a compact metrizable totally disconnected perfect space, and hence, it is homeomorphic to \(C\). We define a homeomorphism \(\sigma : \Sigma(V) \to \Sigma(V)\) as follows:

\[\sigma(x)_i = x_{i+1}\quad\text{for all } i \in \mathbb{Z}.\]

The pair \((\Sigma(V), \sigma)\) is called a \textit{two-sided full shift} of \(n\) symbols. If a closed set \(\Lambda \subseteq \Sigma(V)\) is invariant under \(\sigma\), i.e. \(\sigma(\Lambda) = \Lambda\), then \((\Lambda, \sigma|_{\Lambda})\) is called a \textit{two-sided subshift}. In this manuscript, \(\sigma|_{\Lambda}\) is abbreviated to \(\sigma\). A directed graph \(G\) is a pair \((V, E)\) of a finite set \(V\) of vertices and a set of directed edges \(E \subseteq V \times V\). Let \(G = (V, E)\) be a directed graph. \(\Sigma(G)\) denotes the two-sided subshift defined as follows:

\[\Sigma(G) := \{x \in V^\mathbb{Z} \mid (x_i, x_{i+1}) \in E\ \text{for all } i \in \mathbb{Z}\}.\]

A two-sided subshift is said to be of \textit{finite type} if it is topologically conjugate to \((\Sigma(G), \sigma)\) for some directed graph \(G\). Throughout this manuscript, unless otherwise stated, we assume that all the vertices appear in some element of \(\Sigma(G)\), i.e. all the vertices of \(G\) have both at least one indegree and at least one outdegree. We define a set of words of length \(k\) in \(\Sigma(G)\) as follows:

\[W(k, G) := \{w_0w_1 \cdots w_{k-1} \in V^{[0,1,\ldots,k-1]} \mid (w_i, w_{i+1}) \in E\ \text{for all } i = 0, 1, \ldots, k-2\}.\]

For a word \(w = a_0a_1 \cdots a_{k-1}\) of length \(k\) and an integer \(m\), we define a subset \(C_m(w) \subseteq \Sigma(G)\) as follows:

\[C_m(w) = \{x \in \Sigma(G) \mid x_{m+i} = a_i\ \text{for all } i = 0, 1, \ldots, k-1\}.\]
Chain mixing endomorphisms are approximated

Such a set is called a cylinder. Because \( C_m(w) \) is an open and closed subset of \( \Sigma(G) \), if \( \Sigma(G) \) is homeomorphic to \( C \) and if \( C_m(w) \) is not empty, then \( C_m(w) \) is also homeomorphic to \( C \). A word \( a_0a_1\cdots a_{k-1} \in W(k, G) \) is also called a path of length \( k - 1 \) from \( a_0 \) to \( a_{k-1} \) in \( G \). Let \( x \) be an element of some two-sided subshift. Let \( i \leq j \) be integers. Then, a word \( x_i \cdots x_j \) is also called a segment of length \( j - i + 1 \).

**Lemma 2.3** (Lemma 1.3 of R. Bowen [1]). Let \( G = (V, E) \) be a directed graph. Suppose that every vertex of \( V \) has both at least one outdegree and at least one indegree. Then, \( \Sigma(G) \) is topologically mixing if and only if there exists an \( N \in \mathbb{Z}_+ \) such that for any pair of vertices \( u \) and \( v \) of \( V \), there exists a path from \( u \) to \( v \) of length \( n > N \).

**Proof.** See Lemma 1.3 of R. Bowen [1].

Let \( f : X \to X \) be a mapping and \( \mathcal{U} \), a covering of \( X \). For the sake of conciseness, we define a directed graph \( G_{f, \mathcal{U}} = (V_{f, \mathcal{U}}, E_{f, \mathcal{U}}) \) as follows:

\[
V_{f, \mathcal{U}} = \mathcal{U} \quad \text{and} \quad (a_0, a_1) \in E_{f, \mathcal{U}} \quad \text{if} \quad f(a_0) \cap a_1 \neq \emptyset.
\]

Note that if \( \emptyset \notin \mathcal{U} \), then all the vertices have at least one outdegree. In addition, if \( f \) is an onto mapping, then all the vertices have at least one indegree. Let \( (X, d) \) be a compact metric space and \( K \subset X \). The diameter of \( K \) is defined by \( \text{diam}(K) := \sup\{d(x, y) \mid x, y \in K\} \). For a finite covering \( \mathcal{U} \) of \( X \), we define \( \text{mesh}(\mathcal{U}) := \max\{\text{diam}(U) \mid U \in \mathcal{U}\} \).

**Lemma 2.4.** Let \( (X, d) \) be a compact metric space and \( f : X \to X \), a continuous mapping. Then, for any \( \varepsilon > 0 \), there exists \( \delta = \delta(f, \varepsilon) > 0 \) such that

\[
\delta < \frac{\varepsilon}{2};
\]

if \( d(x, y) \leq \delta \), then \( d(f(x), f(y)) < \frac{\varepsilon}{2} \) for all \( x, y \in X \).

**Proof.** This lemma directly follows from the uniform continuity of \( f \). 

For two directed graphs \( G = (V, E) \) and \( G' = (V', E') \), \( G \) is said to be a subgraph of \( G' \) if \( V \subseteq V' \) and \( E \subseteq E' \).
Lemma 2.5. Let \((X,d)\) be a compact metric space, \(f:X \to X\) be a continuous mapping, and \(\varepsilon > 0\). Let \(\delta = \delta(f, \varepsilon)\) be as in lemma 2.4 and \(\mathcal{U}\), a finite covering of \(X\) such that \(\operatorname{mesh}(\mathcal{U}) < \delta\). Let \(g:X \to X\) be a mapping such that \(G_{g,\mathcal{U}}\) is a subgraph of \(G_{f,\mathcal{U}}\). Then, \(d(f,g) < \varepsilon\).

Proof. Let \(x \in X\). Then, \(x \in U\) and \(g(x) \in U'\) for some \(U, U' \in \mathcal{U}\). Because \(G_{g,\mathcal{U}}\) is a subgraph of \(G_{f,\mathcal{U}}\), there exists a \(y \in U\) such that \(f(y) \in U'\). Therefore, from lemma 2.4, it follows that

\[
d(f(x),g(x)) \leq d(f(x),f(y)) + d(f(y),g(x)) < \frac{\varepsilon}{2} + \operatorname{diam}(U') < \varepsilon.
\]

From this lemma, we obtain the following:

Lemma 2.6. Let \((X,d)\) be a compact metric space; \(f:X \to X\), a continuous mapping; and \(\{\mathcal{U}_k\}_{k=1,2,\ldots}\), a sequence of finite coverings of \(X\) such that \(\operatorname{mesh}(\mathcal{U}_k) \to 0\) as \(k \to \infty\). Let \(\{g_k\}_{k=1,2,\ldots}\) be a sequence of mappings from \(X\) to \(X\) such that \(G_{g_k,\mathcal{U}_k}\) is a subgraph of \(G_{f,\mathcal{U}_k}\) for all \(k\). Then, \(g_k \to f\) as \(k \to \infty\).

A covering \(\mathcal{U}\) of \(X\) is called a partition if \(U \cap U' = \emptyset\) for all \(U, U' \in \mathcal{U}\), where \(U \neq U'\). The Cantor set has a partition by open and closed subsets of an arbitrarily small mesh.

Lemma 2.7. Let \(G = (V,E)\) be a directed graph. Suppose that every vertex of \(G\) has both at least one outdegree and at least one indegree. Suppose that \(\Sigma(G)\) is topologically mixing and that \(\Sigma(G)\) is not a single point. Then, \(\Sigma(G)\) is homeomorphic to \(C\).

Proof. Suppose that \(\Sigma(G)\) is topologically mixing. Then, by lemma 2.3, there exists an \(N \in \mathbb{Z}_+\) such that for any pair \(u\) and \(v\) of vertices of \(G\), there exists a path from \(u\) to \(v\) of length \(n\) for all \(n > N\). Then, it is easy to check that every point \(x \in \Sigma(G)\) is not isolated. Hence, \(\Sigma(G)\) is homeomorphic to \(C\).

3. Proof of the Main Result

In this section, we prove certain lemmas and propositions in order to prove the main result. For a mapping \(\pi: Y \to X\) and a covering \(\mathcal{U}\) of \(X\), the covering
\{\pi^{-1}(U) \mid U \in \mathcal{U}\} is denoted by \(\pi^{-1}(\mathcal{U})\). For any mapping \(g : Y \to Y\), we define a directed graph \(G_{g,\pi,\mathcal{U}} = (V, E)\) as follows:

\[V = \mathcal{U};\]

\[E = \{(a_0, a_1) \in \mathcal{U} \times \mathcal{U} \mid \pi(g(\pi^{-1}(a_0))) \cap a_1 \neq \emptyset\}.\]

A vertex \(a\) in \(G_{g,\pi,\mathcal{U}}\) has at least one outdegree if \(\pi^{-1}(a) \neq \emptyset\).

**Lemma 3.1.** Let \(X\) and \(Y\) be homeomorphic to \(C\). Let \(f : X \to X\) be a continuous mapping; \(g : Y \to Y\), a mapping; and \(\mathcal{U}_k\), a sequence of finite partitions of \(X\) by non-empty open and closed subsets such that \(\text{mesh}(\mathcal{U}_k) \to 0\) as \(k \to \infty\). Suppose that there exists a sequence \(\pi_k\) \((k = 1, 2, \ldots)\) of continuous mappings from \(Y\) to \(X\) such that \(\pi_k(Y) \cap U \neq \emptyset\) for all \(U \in \mathcal{U}_k\) and that the directed graph \(G_{g,\pi_k,\mathcal{U}_k}\) is a subgraph of \(G_{f,\mathcal{U}_k}\) for all \(k \in \mathbb{Z}_+\). Then, there exists a sequence \(\psi_k\) \((k = 1, 2, \ldots)\) of homeomorphisms from \(Y\) onto \(X\) such that \(\psi_k \circ g \circ \psi_k^{-1} \to f\) as \(k \to \infty\).

**Proof.** Let \(k \in \mathbb{Z}_+\) be fixed. By assumption, for each \(U \in \mathcal{U}_k\), \(\pi_k^{-1}(U)\) is a non-empty open and closed subset of \(Y\). Therefore, there exists a homeomorphism \(\psi_k : Y \to X\) such that \(\psi_k(\pi_k^{-1}(U)) = U\) for each \(U \in \mathcal{U}_k\). By construction, \(G_{\psi_k \circ g \circ \psi_k^{-1}, \mathcal{U}_k} = G_{g,\pi_k,\mathcal{U}_k}\). By assumption, \(G_{\psi_k \circ g \circ \psi_k^{-1}, \mathcal{U}_k}\) is a subgraph of \(G_{f,\mathcal{U}_k}\). Therefore, the conclusion follows from lemma 2.6.

**Lemma 3.2.** Let \(X\) and \(Y\) be homeomorphic to \(C\). Let \(f : X \to X\) be a continuous mapping; \(g : Y \to Y\), a mapping; and \(\mathcal{U}_k\), a sequence of finite partitions of \(X\) by non-empty open and closed subsets such that \(\text{mesh}(\mathcal{U}_k) \to 0\) as \(k \to \infty\). Suppose that there exists a sequence of continuous mappings \(\pi_k : Y \to X\) such that \(\pi_k \circ g = f \circ \pi_k\) and that \(\pi_k(Y) \cap U \neq \emptyset\) for all \(U \in \mathcal{U}_k\). Then, there exists a sequence \(\psi_k\) \((k = 1, 2, \ldots)\) of homeomorphisms from \(Y\) onto \(X\) such that \(\psi_k \circ g \circ \psi_k^{-1} \to f\).

**Proof.** Let \(k \in \mathbb{Z}_+\) be fixed. By assumption, for each \(U \in \mathcal{U}_k\), \(\pi_k^{-1}(U)\) is a non-empty open and closed subset of \(Y\). Therefore, there exists a homeomorphism \(\psi_k : Y \to X\) such that \(\psi_k(\pi_k^{-1}(U)) = U\) for each \(U \in \mathcal{U}_k\). Because \(\pi_k(g(\pi_k^{-1}(U))) = f(\pi_k(\pi_k^{-1}(U))) = f(U)\), \(G_{g,\pi_k,\mathcal{U}_k}\) is a subgraph of \(G_{f,\mathcal{U}_k}\). Therefore, the conclusion follows from lemma 3.1.

Let \(\Lambda\) be a two-sided subshift and \(x \in \Lambda\). Then, for \(k < l\), a word \(x_kx_{k+1}\cdots x_l\) is said to be \(j\)-periodic if \(k \leq i < i + j \leq l\) implies \(x_i = x_{i+j}\).
Lemma 3.3 (Krieger’s Marker Lemma, (2.2) of M. Boyle [2]). Let $(\Lambda, \sigma)$ be a two-sided subshift. Given $k > N > 1$, there exists a closed and open set $F$ such that

1. the sets $\sigma^l(F)$, $0 \leq l < N$, are disjoint,
2. if $x \in \Lambda$ and $x_{-k} \cdots x_k$ is not a $j$-periodic word for any $j < N$, then

$$x \in \bigcup_{-N < l < N} \sigma^l(F).$$

Proof. See M. Boyle [2, (2.2)].

The next lemma is essentially a part of the proof of the extension lemma given in M. Boyle [2, (2.4)]. The proof essentially follows that of the extension lemma.

Lemma 3.4. Let $(\Lambda, \sigma)$ be a two-sided subshift and $(\Sigma, \sigma)$, a mixing two-sided subshift of finite type. Let $W$ be a finite set of words that appear in some elements of $\Sigma$. Suppose that $\Lambda$ is not a finite set of periodic points and that $(\Lambda, \sigma) \not\per (\Sigma, \sigma)$. Then, there exists a continuous shift-commuting mapping $\pi : \Lambda \to \Sigma$ such that there exists an element $x \in \pi(\Lambda)$ in which all words of $W$ appear as segments of $x$.

Proof. $\Sigma$ is isomorphic to $\Sigma(G)$ for some directed graph $G = (V, E)$. Therefore, without loss of generality, we assume that $\Sigma = \Sigma(G)$. Because $(\Sigma(G), \sigma)$ is a mixing subshift of finite type, there exists an $n > 0$ such that for every pair of elements $v, v' \in V$ and every $m \geq n$, there exists a word of the form $v \cdots v'$ of length $m$. In addition, there exists an element $\tilde{x} \in \Sigma(G)$ such that $\tilde{x}$ contains all words of $W$ as segments. Let $w_0$ be a segment of $\tilde{x}$ that contains all words of $W$. Let $n_0$ be the length of the word $w_0$. Let $N = 2n + n_0$. If $v, v' \in V$ and $m \geq N$, then there exists a word of the form $v \cdots w_0 \cdots v'$ of length $m \geq N$. Let $k > 2N$. Using Krieger’s marker lemma, there exists a closed and open subset $F \subset \Lambda$ such that the following conditions hold:

1. the sets $\sigma^l(F)$, $0 \leq l < N$, are disjoint;
2. if $x \in \Lambda$ and $x \notin \bigcup_{-N < l < N} \sigma^l(F)$, then $x_{-k} \cdots x_k$ is a $j$-periodic word for some $j < N$;
3. the number $k$ is large enough to ensure that if $j$ is less than $N$ and a $j$-periodic word of length $2k + 1$ occurs in some element of $\Lambda$, then that word defines a $j$-periodic orbit which actually occurs in $\Lambda$. 

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The existence of \( k \) follows from the compactness of \( \Lambda \). Let \( x \in \Lambda \). If \( \sigma^j(x) \in F \), then we mark \( x \) at position \( i \). There exists a large number \( L > 0 \) such that whether or not \( \sigma^j(x) \in F \) is determined only by the \( 2L + 1 \) block \( x_{i-L} \cdots x_{i+L} \). If \( x \) is marked at position \( i \), then \( x \) is unmarked for position \( l \) with \( i < l < i + N \). Suppose that \( x_i \cdots x_{l'} \) is a segment of \( x \) such that \( x \) is marked at \( i \) and \( l' \) and that \( x \) is unmarked at \( l \) for all \( i < l < l' \). Then, \( l' - i \geq N \). If \( x \in \bigcup_{-N < i < N} \sigma^j(F) \), then \( x \) is marked at some \( i \) where \( -N < i < N \). Suppose that \( x_{-N + 1} \cdots x_{N - 1} \) is an unmarked segment. Then, \( x \notin \bigcup_{-N < i < N} \sigma^j(F) \), and according to condition (2) \( x_{-k} \cdots x_k \) is a \( j \)-periodic word for some \( j < N \). Suppose that \( x_i \cdots x_{l'} \) is an unmarked segment of length at least \( 2N - 1 \), i.e. \( l' - i \geq 2N - 2 \). Then, for each \( l \) with \( i + N - 1 \leq l \leq l' - N + 1 \), \( x_{l-k} \cdots x_{l+k} \) is a \( j \)-periodic word for some \( j < N \). Therefore, it is easy to check that \( x_{i + N - k - 1} \cdots x_{i + N + k} \) is a \( j \)-periodic word for some \( j < N \). In this proof, we call a maximal unmarked segment an interval. Let \( x \in \Lambda \). Let \( \ldots x_i \) be a left infinite interval. Then, it is \( j \)-periodic for some \( j < N \). Similarly, a right infinite interval \( x_i \ldots \) is \( j \)-periodic for some \( j < N \). If \( x \) itself is an interval, then it is a periodic point with period \( j < N \). If an interval is finite, then it has a length of at least \( N - 1 \). We call intervals of length less than \( 2N - 1 \) as short intervals. We call intervals of length greater than or equal to \( 2N - 1 \) as long intervals. If \( x \) has a long interval \( x_i \cdots x_{l'} \), then \( x_{i + N - k} \cdots x_{l' - N + 1 + k} \) is \( j \)-periodic for some \( j < N \). We have to construct a shift-commuting mapping \( \phi : \Lambda \to \Sigma \). Let \( V' \) be the set of symbols of \( \Lambda \). Let \( \Phi : V' \to V \) be an arbitrary mapping. Let \( x \in \Lambda \). Suppose that \( x \) is marked at \( i \). Then, we let \( (\phi(x))_l \) be \( \Phi(x_l) \).

We map periodic points of period \( j < N \) to periodic points of \( \Sigma \). Then, we construct a coding of \( \phi(x) \) in three parts. For any \( (v, v', l) \in V \times V \times \{N - 1, N, N + 1, \ldots, 2N - 2\} \), we choose a word \( \Psi(v, v', l) \) in \( G \) of length \( l \) such that the word of the form \( v \Psi(v, v', l) v' \) is a path in \( G \).

(A) Coding for short interval: Let \( x_i \cdots x_{l'} \) be a short interval. Then, \( x \) is marked at \( i - 1 \) and \( l' + 1 \). We have already defined a code for position \( i - 1 \) and \( l' + 1 \) as \( \Phi(x_{i - 1}) \) and \( \Phi(x_{l' + 1}) \), respectively. The coding for \( \{i, i + 1, i + 2, \ldots, l'\} \) is defined by the path \( \Psi(\Phi(x_{i - 1}), \Phi(x_{l' + 1})), l' - i + 1 \).

(B) Coding for periodic segment: For an infinite or a long interval, there exists a corresponding periodic point of \( \Lambda \). The periodic points of \( \Lambda \) are already mapped to periodic points of \( \Sigma \). Therefore, an infinite or a long periodic segment can be mapped to a naturally corresponding periodic segment.

(C) Coding for transition part: To consider a transition segment, let \( x_i \cdots x_{l'} \) be a long interval. Then, \( x_{l + N - 1} \) has already been mapped to \( \Phi(x_{l - 1}) \) and \( x_{l + N - 1} \) is mapped according to periodic points. Assume \( x_{l + l + N - 1} \) is mapped to \( v_0 \). The segment \( x_{l - 1} \cdots x_{l + N - 1} \) has length \( N + 1 \). We map the segment \( x_i \cdots x_{l + N - 2} \) to
from mixing, there exists an \( S \) finite type such that \( L \) such that \( w \) that involves \( \text{take transition segment or at least one short interval. In the above coding, we can} \)

\[ \Psi(\Phi(x_{i-1}), v_0, N - 1). \]

In the same manner, the transition coding of right hand side of a long interval is defined. In the same manner, the transition coding of the left or the right infinite interval is defined. It is easy to check that there exists a large number \( L' > 0 \) such that the coding of \( (\phi(x))_i \) is determined only by the block \( x_{i-L'} \cdots x_{i+L'} \). Therefore, \( \phi : \Lambda \to \Sigma \) is continuous. Because \( \Lambda \) is not a set of finite periodic points, there exists an \( x \in \Lambda \) such that \( x \) contains at least one transition segment or at least one short interval. In the above coding, we can take \( \Psi \) such that a short interval or a transition segment is mapped to a word that involves \( w_0 \).

\[ \square \]

**Proposition 3.5.** Let \((\Sigma, \sigma)\) be a topologically mixing two-sided subshift of finite type such that \( \Sigma \) is homeomorphic to \( C \). Let \((\Lambda, \sigma)\) be a two-sided subshift such that \( \Lambda \) is homeomorphic to \( C \).

Then, \((\Lambda, \sigma) \twoheadrightarrow (\Sigma, \sigma)\) if and only if \((\Lambda, \sigma) \xrightarrow{\text{per}} (\Sigma, \sigma)\).

**Proof.** If \((\Lambda, \sigma) \twoheadrightarrow (\Sigma, \sigma)\), then by lemma 2.2, we obtain \((\Lambda, \sigma) \xrightarrow{\text{per}} (\Sigma, \sigma)\). Suppose that \((\Lambda, \sigma) \xrightarrow{\text{per}} (\Sigma, \sigma)\). Without loss of generality, we can assume that \( \Sigma = \Sigma(G) \) for some directed graph \( G = (V, E) \). We assume that every vertex of \( V \) has both at least one outdegree and at least one indegree. Let \( k \in \mathbb{Z}_+ \). Because \((\Sigma, \sigma)\) is topologically mixing, by lemma 3.4, there exists a continuous shift-commuting mapping \( \pi_k : \Lambda \twoheadrightarrow \Sigma \) and \( x \in \pi_k(\Lambda) \) such that \( x \) contains all words of length \( 2k + 1 \) of \( \Sigma \). Let \( \mathcal{U}_k = \{ C_{-k}(w) \mid w \in W(2k + 1, G) \} \). Then, \( \pi_k(\Lambda) \cap U \neq \emptyset \) for all \( U \in \mathcal{U}_k \). Because \( k \) is arbitrary, by lemma 3.2, we conclude that \((\Lambda, \sigma) \twoheadrightarrow (\Sigma(G), \sigma)\).

\[ \square \]

**Proof of Theorem 1.1**

**Proof.** If \((\Lambda, \sigma) \twoheadrightarrow (X, f)\), then by lemma 2.2, we obtain \((\Lambda, \sigma) \xrightarrow{\text{per}} (X, f)\). Let \((\Lambda, \sigma) \xrightarrow{\text{per}} (X, f)\) hold. Consider a sequence \( \{ \mathcal{U}_k \}_{k=1,2,\ldots} \) of partitions of \( X \) by non-empty open and closed subsets such that \( \text{mesh}(\mathcal{U}_k) \to 0 \) as \( k \to \infty \). Assume \( k \in \mathbb{Z}_+ \). Let \( G_k = G_{f, \mathcal{U}_k} \). Let \( \delta > 0 \) be such that any \( x, x' \in X \) with \( d(x, x') < \delta \) are contained in the same element of \( \mathcal{U}_k \). Let \( \{ x_0, x_1 \} \) be a \( \delta \) chain. Let \( U, U' \in \mathcal{U}_k \) be such that \( x_0 \in U \) and that \( x_1 \in U' \). Then, \( f(U) \cap U' \neq \emptyset \). Therefore, \((U, U')\) is an edge of \( G_k \). Let \( U, V \in \mathcal{U}_k \). Let \( x \in U \) and \( y \in V \). Because \( f \) is chain mixing, there exists an \( N > 0 \) such that for every \( n > N \), there exists a \( \delta \) chain from \( x \) to \( y \) of length \( n \). Therefore, for every \( n > N \), there exists a path in \( G_k \) from \( U \) to \( V \) of length \( n \). From Lemma 2.3, \((\Sigma(G_k), \sigma)\) is topologically mixing. By lemma 2.7, \( \Sigma(G_k) \) is homeomorphic to \( C \). Therefore, there exists a homeo-
morphism $\psi_k : \Sigma(G_k) \to X$ such that for any vertex $u$ of $G_k$, $\psi_k(C_0(u)) = u$. By construction, we obtain $G_{\psi_k \circ \sigma \circ \psi_k^{-1}, \psi_k} = G_{f, \psi_k}$. Because $\text{mesh}(\psi_k) \to 0$ as $k \to \infty$, by lemma 2.6, we find that $\psi_k \circ \sigma \circ \psi_k^{-1} \to f$ as $k \to \infty$. On the other hand, it is easy to verify that $\text{Per}(X, f) \subseteq \text{Per}(\Sigma(G_k), \sigma)$. By assumption, we obtain $\text{Per}(\Lambda, \sigma) \subseteq \text{Per}(\Sigma(G_k), \sigma)$. From proposition 3.5, we obtain $(\Lambda, \sigma) \Rightarrow (\Sigma(G_k), \sigma)$. Therefore, by lemma 2.1, we obtain $(\Lambda, \sigma) \Rightarrow (X, f)$.

**Proof of Corollary 1.2**

**Proof.** If a topological dynamical system $(X, f)$ has a fixed point $x_0$, then $\text{Per}(X, f) = \mathbb{Z}_+$. Therefore, the proof is a direct consequence of theorem 1.1.

**Proof of Corollary 1.3**

**Proof.** Let $(\Lambda, \sigma)$ be a two-sided subshift without periodic points. Then, $\text{Per}(\Lambda, \sigma) = \emptyset$. Therefore, from theorem 1.1, the conclusion follows.

**Proof of Corollary 1.4**

**Proof.** A two-sided full shift is chain mixing and has a fixed point. Therefore, the conclusion is a direct consequence of corollary 1.2.

**References**


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