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GEOMETRIC CLASSIFICATION OF QUADRATIC ALGEBRAS IN TWO VARIABLES

By

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Abstract. In this paper, we classify quadratic algebras in two variables at two levels: (1) up to isomorphism of graded algebras, (2) up to graded Morita equivalence. In general, it is difficult to classify algebras by looking at generators and relations, so we take a geometric approach, namely, using point schemes defined by Artin, Tate and Van den Bergh, to complete the classification.

1. Introduction

Thoughout this paper, we fix an algebraically closed field k, and denote by $k\langle X\rangle = k\langle x_1, \dots, x_n\rangle$ the free algebra in n variables x_i of degree 1 over k. Let $R \subseteq k\langle X\rangle_2$ be a subvector space and we denote by (R) the ideal in $k\langle X\rangle$ generated by R. Then the algebra $k\langle X\rangle/(R)$ is called a quadratic algebra. The main purpose of this paper is to classify quadratic algebras in two variables, that is, $k\langle x, y\rangle/(f_1, \dots, f_r)$ where $f_1, \dots, f_r$ are linearly independent and $\deg f_i = 2$ for all i, at two levels:

(1) up to isomorphism of graded algebras, and
(2) up to equivalence of graded module categories (graded Morita equivalence).

Classification of quadratic algebras in two variables is divided into five cases by the number of relations, namely, $r = 0, 1, 2, 3, 4$. Clearly, $k\langle x, y\rangle$ is the only quadratic algebra in the case $r = 0$, and $k\langle x, y\rangle/(x^2, xy, yx, y^2)$ is the only quadratic algebra in the case $r = 4$. In the case $r = 1$, the classification is well-known. Furthermore algebras in the case $r = 3$ are classified using quadratic dual of the case $r = 1$. In this paper, we mainly describe methods and results in the case $r = 2$.

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Let A be a graded k-algebra. Then we denote by GrMod A the category of graded right A-modules and graded right A-module homomorphisms of degree 0.

Now, given two quadratic algebras in n variables A and B, we have questions to ask: $A \cong B$ as graded algebras? GrMod $A \cong$ GrMod B? It is difficult to answer these questions by looking at generators and relations of A and B. Therefore we take a geometric approach to these questions, using point schemes defined by Artin, Tate and Van den Bergh [2]. If $\Gamma_A, \Gamma_B \subset \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ are the point schemes of A, B, then the following holds (Theorem 2.2, Theorem 2.7):

$$\begin{array}{ccc} A \cong B & \Longrightarrow & \exists \sigma \in \mathrm{Aut}_k \, \mathbf{P}^{n-1} \text{ such that } \Gamma_A \xrightarrow[\sim]{\sigma \times \sigma} \Gamma_B \\ \Downarrow & & \Downarrow \\ \mathrm{GrMod}\, A \cong \mathrm{GrMod}\, B & \Longrightarrow & \exists \sigma, \tau \in \mathrm{Aut}_k \, \mathbf{P}^{n-1} \text{ such that } \Gamma_A \xrightarrow[\sim]{\sigma \times \tau} \Gamma_B \end{array}$$

Thus we can classify quadratic algebras by making use of the point schemes. In the case of two variables, this geometric approrch is quite effective. In fact, we complete the classification in the case $r = 2$ mainly using geometric method.

2. Point Schemes and Twsisting Systems

First we study the notion of the point scheme.

DEFINITION 2.1 [2]. Let $A = k\langle X \rangle / I$ be a graded k-algebra generated in degree 1. Then we define

$$\Gamma_i := \mathscr{V}(I_i) = \{(p_1, \dots, p_i) \in (\mathbf{P}^{n-1})^{\times i} \mid f(p_1, \dots, p_i) = 0 \text{ for all } f \in I_i\}$$

where $i \geq 1$. For $j \geq i$, if $pr_i^j : \Gamma_j \to \Gamma_i$ is the restriction of the projection $(\mathbf{P}^{n-1})^{\times j} \to (\mathbf{P}^{n-1})^{\times i}$ to first i coordinates, then $\{\Gamma_i, pr_i^j\}$ is an inverse system of schemes. The point scheme of A is defined by the inverse limit

$$\Gamma := \varprojlim \Gamma_i.$$

However for the purpose of this paper, we define the point scheme of A by

$$\Gamma_A := \Gamma_2$$

by abuse of language, because we consider quadratic algebras only.

THEOREM 2.2 (cf. [5]). *Let $A = k\langle X \rangle / (R)$, $B = k\langle X \rangle / (S)$ be quadratic algebras, and $\Gamma_A, \Gamma_B \subset \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ the point schemes. Then graded k-algebra homo-*

morphism $\sigma : B \to A$ induces a morphism ${}^t\sigma \times {}^t\sigma : \Gamma_A \to \Gamma_B$. In particular if $A \cong B$ as graded k-algebras, then $\Gamma_A \cong \Gamma_B$.

Proof. Let $\sigma|_{B_1}$ be the restriction map of σ to B_1, and $\tilde{\sigma} : k\langle X\rangle \to k\langle X\rangle$ the natural extended map of $\sigma|_{B_1}$. Since $\tilde{\sigma}(f)$ is in R for any $f \in S$, we get

$$f({}^t(\sigma|_{B_1})(p), {}^t(\sigma|_{B_1})(q)) = \tilde{\sigma}(f)(p,q) = 0$$

for any $(p,q) \in \Gamma_A$. It follows that $({}^t(\sigma|_{B_1})(p), {}^t(\sigma|_{B_1})(q)) \in \Gamma_B$. Thus we can define ${}^t\sigma \times {}^t\sigma : \Gamma_A \to \Gamma_B$ by $(p,q) \mapsto ({}^t(\sigma|_{B_1})(p), {}^t(\sigma|_{B_1})(q))$. The assignments $A \mapsto \Gamma_A$, $\sigma \mapsto {}^t\sigma \times {}^t\sigma$ define a contravariant functor from the category of quadratic algebras in n variables to the category of schemes, so $\Gamma_A \cong \Gamma_B$ is induced by $A \cong B$. □

By using this theorem, we can check whether two quadratic algebras are not isomorphic.

Zhang [8] found the necessary and sufficient condition for graded Morita equivalence, introducing the notion of a twisting sysyem.

Definition 2.3 [8]. Let A be a graded k-algebra. Let $\theta = \{\theta_i \mid i \in \mathbf{Z}\}$ be a set of graded k-linear automorphisms of A. Then θ is called a twisting system of A if

$$\theta_l(a\theta_p(b)) = \theta_l(a)\theta_{l+p}(b)$$

for all $p, q, l \in \mathbf{Z}$ and all $a \in A_p$, $b \in A_q$. Given a twisting system of A, say θ, the twist of A by θ, denoted by A^θ, is defined to be A as a graded k-vector space with a new multiplication $*$ by

$$a * b = a\theta_p(b) \quad (a \in A_p, b \in A_q).$$

If M is a graded right A-module, then the twist of M by θ, denoted by M^θ, is defined to be M as a graded k-vector space with a new action $*$ defined by

$$m * a = m\theta_p(a) \quad (m \in M_p, a \in A_q).$$

Remark 2.4. If A is a graded k-algebra, and $\varphi \in \mathrm{Aut}_k A$ is a graded k-algebra automorphism, then the set $\{\theta_i := \varphi^i\}$ is a twisting system.

Theorem 2.5 [8, Theorem 3.5]. *If A and B are graded k-algebras generated in degree 1, then* $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, B$ *if and only if B is isomorphic to a twist of A by a twisting system.*

LEMMA 2.6 [8, Proposition 2.8]. *Let A, B be graded k-algebras generated in degree* 1. *Then B is isomorphic to a twist of A if and only if there exists a set of graded k-linear isomorphisms $\{\phi_i\}$ from B to A which satisfies*

$$\phi_l(ab) = \phi_l(a)\phi_{l+p}(b) \tag{2-1}$$

for all $p, q, l \in \mathbf{Z}$ and all $a \in B_p$, $b \in B_q$.

THEOREM 2.7 (cf. [5]). *Let $A = k\langle X\rangle/(R)$, $B = k\langle X\rangle/(S)$ be quadratic algebras, and $\Gamma_A, \Gamma_B \subset \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ the point schemes. If* $\operatorname{GrMod} A \cong \operatorname{GrMod} B$, *then there exist $\sigma, \tau \in \operatorname{Aut}_k \mathbf{P}^{n-1}$ which restrict to an isomorhism $\sigma \times \tau : \Gamma_A \to \Gamma_B$.*

PROOF. Suppose $\operatorname{GrMod} A \cong \operatorname{GrMod} B$. Then $A^\theta \cong B$ for some twisting system θ by Theorem 2.5. Moreover there exists a set of k-linear isomorphisms $\{\phi_i : B \to A\}$ which satisfies (2-1) by Lemma 2.6. Let $\sigma_i := \phi_i|_{B_1} : B_1 \to A_1$ for any i. Since for any $f := \sum_{l,m} \alpha_{lm} x_l x_m \in S$, $f_i := \sum_{l,m} \alpha_{lm}\sigma_i(x_l)\sigma_{i+1}(x_m)$ is in R, so we get

$$f({}^t\sigma_i(p), {}^t\sigma_{i+1}(q)) = f_i(p, q) = 0$$

for any $(p, q) \in \Gamma_A$. It follows that $({}^t\sigma_i(p), {}^t\sigma_{i+1}(q)) \in \Gamma_B$. Thus we can define ${}^t\sigma_i \times {}^t\sigma_{i+1} : \Gamma_A \to \Gamma_B$ by $(p, q) \mapsto ({}^t\sigma_i(p), {}^t\sigma_{i+1}(q))$. Since ${}^t\sigma_i \times {}^t\sigma_{i+1}$ is an automorphism of $\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$, ${}^t\sigma_i \times {}^t\sigma_{i+1}|_{\Gamma_A}$ is injective. Similarly, for any $(p', q') \in \Gamma_B$, we can check that $(({}^t\sigma_i)^{-1}(p'), ({}^t\sigma_{i+1})^{-1}(q')) \in \Gamma_A$ and

$${}^t\sigma_i \times {}^t\sigma_{i+1}(({}^t\sigma_i)^{-1}(p'), ({}^t\sigma_{i+1})^{-1}(q')) = (p', q'),$$

so ${}^t\sigma_i \times {}^t\sigma_{i+1}|_{\Gamma_A}$ is surjective. □

By using this theorem, we can check whether two quadratic algebras are not graded Morita equivalent. Now we define the Hilbert series.

DEFINITION 2.8. Let V be a locally finite graded k-vector space. Then we define the Hilbert series of V by

$$H_V(t) = \sum_{i=-\infty}^{\infty} (\dim V_i) t^i \in \mathbf{Z}[[t, t^{-1}]].$$

THEOREM 2.9 (cf. [8]). *Let A, B be graded k-algebras generated in degree* 1, *and $H_A(t)$, $H_B(t)$ the Hilbert series. If* $\operatorname{GrMod} A \cong \operatorname{GrMod} B$, *then $H_A(t) = H_B(t)$.*

Proof. Suppose $\mathrm{GrMod}\,A \cong \mathrm{GrMod}\,B$. Then $A^\theta \cong B$ for some twisting system θ by Theorem 2.5. It means that $A \cong B$ as graded k-vector spaces, so $H_A(t) = H_B(t)$. $\square$

It follows that if $H_A(t) \neq H_B(t)$, then $\mathrm{GrMod}\,A \not\cong \mathrm{GrMod}\,B$, in particular, $A \not\cong B$.

3. The Case $r = 1$

In this section, we recall the classification in the case $r = 1$. We define an equivalence relation on $M_2(k)$ by $M \approx M'$ if M equals M' up to non-zero scalar multiplication. If we associate to $f = \alpha x^2 + \beta xy + \gamma yx + \delta y^2$ the matrix $M_f := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then it is well-known $k\langle x, y\rangle/(f) \cong k\langle x, y\rangle/(g)$ if and only if there exists $P \in GL_2(k)$ such that $M_{f'} = PM_f{}^tP$. This method is used in classifying (commutative) quadratic forms. However, in noncommutative case, this method is not effective. Since we can not assume M_f is symmetric, so instead, we associated to $f = \alpha x^2 + \beta xy + \gamma yx + \delta y^2$ the matrix

$$M_f := \begin{pmatrix} \beta & -\alpha \\ \delta & -\gamma \end{pmatrix}$$

in this paper. Then the following holds.

Lemma 3.1. *Let $A = k\langle x, y\rangle/(f_1, \dots, f_r)$, $B = k\langle x, y\rangle/(g_1, \dots, g_r)$ be quadratic algebras. If there exists $P \in GL_2(k)$ such that $M_{g_i} \approx PM_{f_i}P^{-1}$ for all $1 \leq i \leq r$, then $A \cong B$. In particular, if $A = k\langle x, y\rangle/(f)$, $B = k\langle x, y\rangle/(g)$, then there exists $P \in GL_2(k)$ such that $M_g \approx PM_fP^{-1}$ if and only if $A \cong B$.*

Proof. This follows from the definition of M_f and calculations. $\square$

For any $f = \alpha x^2 + \beta xy + \gamma yx + \delta y^2$,

$$PM_fP^{-1} \approx \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leftrightarrow x^2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leftrightarrow xy \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \leftrightarrow -x^2 + xy - yx \\ \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \leftrightarrow xy - \lambda yx \quad (\lambda \neq 0) \end{cases} \tag{3-1}$$

for some invertible matrix $P \in GL_2(k)$, where

$$\exists P \in GL_2(k) \quad \text{such that } \begin{pmatrix} 1 & 0 \\ 0 & \lambda' \end{pmatrix} \approx P\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}P^{-1} \Leftrightarrow \lambda' = \lambda^{\pm 1}.$$

by the classification of the "Jordan" canonical form for $M_2(k)$ up to non-zero scalar multiplication. We get the classification of $k\langle x, y\rangle/(f)$ by Lemma 3.1.

Theorem 3.2. *Every quadratic algebra of the form $k\langle x, y\rangle/(f)$ is isomorphic to exactly one of the following*:

$$\begin{aligned}
&k\langle x, y\rangle/(x^2),\\
&k\langle x, y\rangle/(xy),\\
&k\langle x, y\rangle/(-x^2 + xy - yx) \ =: k_J[x, y],\\
&k\langle x, y\rangle/(xy - \lambda yx) \ =: k_\lambda[x, y] \quad (\lambda \neq 0)
\end{aligned}$$

where

$$k_\lambda[x, y] \cong k_{\lambda'}[x, y] \Leftrightarrow \lambda' = \lambda^{\pm 1}.$$

Let $A = k\langle x, y\rangle/(x^2)$, $B = k\langle x, y\rangle/(xy)$, $C = k_J[x, y]$, $D = k_\lambda[x, y]$. If we define $\varphi \in \mathrm{Aut}_k\, k[x, y]$ by $\varphi(x) = x$, $\varphi(y) = x + y$, then $\theta = \{\varphi^i\}$ is the twisting system such that $C \cong k[x, y]^\theta$. If we define $\psi \in \mathrm{Aut}_k\, k[x, y]$ by $\psi(x) = x$, $\psi(y) = \lambda y$, then $\eta = \{\psi^i\}$ is the twisting system such that $D \cong k[x, y]^\eta$. It follows that $\mathrm{GrMod}\, C \cong \mathrm{GrMod}\, k[x, y] \cong \mathrm{GrMod}\, D$. The point scheme of each algebra is

$$\Gamma_A = (0,1) \times \mathbf{P}^1 \cup \mathbf{P}^1 \times (0,1), \quad \Gamma_B = (0,1) \times \mathbf{P}^1 \cup \mathbf{P}^1 \times (1,0),$$
$$\Gamma_C = (p,q) \times (p, p+q), \qquad \Gamma_D = (p,q) \times (p, \lambda q),$$

so $\mathrm{GrMod}\, A \not\cong \mathrm{GrMod}\, C$ and $\mathrm{GrMod}\, B \not\cong \mathrm{GrMod}\, C$ by Theorem 2.7. Moreover, the Hilbert series of each algebra is

$$H_A(t) = 1 + 2t + 3t^2 + 5t^3 + 8t^4 + 13t^5 \cdots = (1 + t)/(1 - t - t^2),$$
$$H_B(t) = H_C(t) = H_D(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 \cdots = 1/(1 - t)^2,$$

so $\mathrm{GrMod}\, A \not\cong \mathrm{GrMod}\, B$ by Theorem 2.9. Hence we get the following classification.

Theorem 3.3. *Every quadratic algebra of the form $k\langle x, y\rangle/(f)$ is graded Morita equivalent to exactly one of the following*:

$$\begin{aligned}
&k\langle x, y\rangle/(x^2),\\
&k\langle x, y\rangle/(xy),\\
&k\langle x, y\rangle/(xy - yx) \cong k[x, y].
\end{aligned}$$

Let A be a graded k-algebra and M, N graded right A-modules. Then A is said to be connected if $A_i = 0$ for all $i < 0$, and $A_0 = k$. For each $d \in \mathbf{Z}$, the shift of M, denoted by $M(d)$, is a graded A-module for which $M(d)_i = M_{i+d}$. We define $\underline{\mathrm{Ext}}^i_A(M,N) = \bigoplus_{d \in \mathbf{Z}} \mathrm{Ext}^i_A(M, N(d))$.

We can see that $k_J[x,y]$ and $k_\lambda[x,y]$ have good homological properties analogous to the polynomial algebras.

Definition 3.4 [1]. Let A be a connected graded k-algebra. Then A is called a d-dimensional Artin-Schelter regular (AS-regular, for short) algebra if

- gldim $A = d < \infty$,
- GKdim $A := \limsup_{n\to\infty} \log(\dim_k \sum_{i=0}^n A_i)/\log n < \infty$, and
- A satisfies Gorenstein condition, that is,

$$\underline{\mathrm{Ext}}^i_A(k, A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ k(l) \text{ for some } l \in \mathbf{Z} & \text{if } i = d. \end{cases}$$

A commutative algebra A is AS-regular if and only if A is a polynomial algebra. A 2-dimensional AS-regular algebra generated in degree 1 is either of the form

$$k_J[x,y] \quad \text{or} \quad k_\lambda[x,y].$$

Classification of 3-dimensional AS-regular algebras generated in degree 1 was attacked by Artin and Schelter in their paper [1]. Later Artin, Tate and Van den Bergh [2] completed the classification of 3-dimensional AS-regular algebras generated in degree 1 by using geometric approach.

4. The Case $r = 2$

In this section, we see methods and results of the classification in the case $r = 2$.

Lemma 4.1. *If $f = \alpha x^2 + \beta xy + \gamma yx + \delta y^2$ is a quadratic relation, then f is reducible if and only if $\alpha\delta - \beta\gamma = 0$.*

Proof. The last two matrices in (3-1) are the canonical form of invertible matrices up to scalar multiplication. Hence we have

$$\begin{aligned} f \text{ is reducible} &\Leftrightarrow k\langle x, y\rangle/(f) \cong k\langle x, y\rangle/(x^2) \text{ or } k\langle x, y\rangle/(xy) \\ &\Leftrightarrow \text{there exists } P \in GL_2(k) \text{ such that} \\ &\qquad PM_fP^{-1} \approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\Leftrightarrow \det M_f \ (= \det PM_fP^{-1}) = \alpha\delta - \beta\gamma = 0. \end{aligned}$$

□

PROPOSITION 4.2. *Let $k\langle x, y\rangle/(f_1, f_2)$ be a quadratic algebra. Then there exist at least one and at most two linearly independent redusible relations in $(f_1, f_2)_2 = kf_1 + kf_2$.*

PROOF. For each quadratic relation $f = \alpha x^2 + \beta xy + \gamma yx + \delta y^2$, we define the point $p_f := (\alpha, \beta, \gamma, \delta) \in \mathbf{P}^3$. If we define

$$X := \{p_{sf_1+tf_2} \mid (s, t) \in \mathbf{P}^1\} \subset \mathbf{P}^3, \quad Y := \mathscr{V}(x_0x_3 - x_1x_2) \subset \mathbf{P}^3,$$

then dim $X = 1$, deg $X = 1$, dim $Y = 2$ and deg $Y = 2$, so the total number of intersection points of X and Y counted with their multiplicities is $2 (= \deg X \deg Y)$ by Bezout's Theorem [3, Theorem 18.3]. Since $p_f \in Y$ if and only if $\alpha\delta - \beta\gamma = 0$, the result now follows by Lemma 4.1. □

COROLLARY 4.3. *Every quadratic algebra $k\langle x, y\rangle/(f_1, f_2)$ can be made either of the form*

$$k\langle x, y\rangle/(x^2, \alpha x^2 + \beta xy + \gamma yx + \delta y^2) \quad \text{or} \quad k\langle x, y\rangle/(xy, \alpha x^2 + \beta xy + \gamma yx + \delta y^2).$$

PROOF. Proposition 4.2 says that there exists at least one redusible relation in $(f_1, f_2)_2$, so this corollary is proved. □

PROPOSITION 4.4. *A quadratic algebra of the form $k\langle x, y\rangle/(x^2, \alpha x^2 + \beta xy + \gamma yx + \delta y^2)$ is isomorphic to one of the following*:

$$k\langle x, y\rangle/(x^2, xy), \quad k\langle x, y\rangle/(x^2, yx), \quad k\langle x, y\rangle/(x^2, xy - \lambda yx),$$
$$k\langle x, y\rangle/(x^2, y^2), \quad k\langle x, y\rangle/(x^2, y^2 - xy)$$

where $\lambda \neq 0$.

PROOF. Since x^2 is contained in the relations, we first make the second relation f so that $M_f = \begin{pmatrix} \beta & 0 \\ \delta & -\gamma \end{pmatrix}$. If we define $P := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \in GL_2(k)$, then $P\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}P^{-1} \approx \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so conjugating by P fixes the relation x^2, hence we can replace f by f'

so that $M_{f'} = PM_fP^{-1} = \begin{pmatrix} \beta+p\delta & -p(\beta+\gamma+p\delta) \\ \delta & -(\gamma+p\delta) \end{pmatrix}$ by Lemma 3.1. Moreover, since x^2 is contained in the relations, we again replace the relation f' so that $M_{f'} = \begin{pmatrix} \beta+p\delta & 0 \\ \delta & -(\gamma+p\delta) \end{pmatrix}$.

(1) The case $\delta = 0$: If $\gamma = 0$, then $M_{f'} \approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so $f' = xy$. If $\beta = 0$, then $M_{f'} \approx \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, so $f' = yx$. Otherwise, $M_{f'} \approx \begin{pmatrix} 1 & 0 \\ 0 & \gamma\beta^{-1} \end{pmatrix}$, so $f' = xy + \gamma\beta^{-1}yx$.

(2) The case $\delta \neq 0$: We take $p = -\gamma\delta^{-1}$. If $\beta = \gamma$, then $M_{f'} \approx \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, so $f' = y^2$. If $\beta \neq \gamma$, then $M_{f'} \approx \begin{pmatrix} \beta-\gamma & 0 \\ \delta & 0 \end{pmatrix}$, so $f' = y^2 - xy$ by multiplying $-\delta(\beta-\gamma)^{-1}$ to x. □

Proposition 4.5. *A quadratic algebra of the form $k\langle x, y\rangle/(xy, \alpha x^2 + \beta xy + \gamma yx + \delta y^2)$ is isomorphic to one of the following*:

$$k\langle x, y\rangle/(xy, y^2), \qquad k\langle x, y\rangle/(xy, yx), \qquad k\langle x, y\rangle/(xy, x^2),$$

$$k\langle x, y\rangle/(xy, x^2 - y^2), \quad k\langle x, y\rangle/(xy, x^2 - yx), \quad k\langle x, y\rangle/(xy, x^2 - yx + \mu y^2)$$

where $\mu \neq 0$.

Proof. We may assume $\alpha \neq 0$, $\gamma \neq 0$ or $\delta \neq 0$. The result follows by appropriately multiplying to x and/or y by scalars, and the fact that $k\langle x, y\rangle/(y(y+x), xy) \cong k\langle x, y\rangle/(xy, (x-y)x)$. □

Now $k\langle x, y\rangle/(xy, x^2)$ appears both in Proposition 4.4 and Proposition 4.5. Moreover $k\langle x, y\rangle/(x^2, yx)$ in Proposition 4.4 is isomorphic to $k\langle x, y\rangle/(xy, y^2)$ in Proposition 4.5. There might be other isomorphisms between algebras in Proposition 4.4 and Proposition 4.5. Therefore we check whether two algebras in Proposition 4.4 and Proposition 4.5 are not isomorphic by using Theorem 2.2 and Hilbert series.

Except the following two cases

$$A := k\langle x, y\rangle/(x^2, y^2 - xy) \overset{?}{\leftrightarrow} k\langle x, y\rangle/(xy, x^2 - yx + y^2) =: A' \quad (\mu = 1), \tag{4-1}$$

$$B := k\langle x, y\rangle/(x^2, xy - \lambda yx) \overset{?}{\leftrightarrow} k\langle x, y\rangle/(x^2, xy - \lambda' yx) =: B' \tag{4-2}$$

where $\lambda \neq \lambda'$, we can check that all algebras in Table 1 are non-isomorphic to one another by Theorem 2.2 and Theorem 2.9. In the case (4-1), $\Gamma_A \xrightarrow[\sim]{\sigma\times\sigma} \Gamma_{A'}$ is given by $\sigma = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, and ${}^t\sigma$ induces an isomorphism $A' \cong A$. In the case (4-2), whatever we take $\sigma \in \mathrm{Aut}_k \mathbf{P}^1$ such that $\Gamma_B \xrightarrow[\sim]{\sigma\times\sigma} \Gamma_{B'}$, ${}^t\sigma$ does not induce an isomorphism $B' \to B$, hence $B' \not\cong B$. Hence we complete the classification up to isomorphism in the case $r = 2$ as follows.

Table 1. Point schemes and Hilbert series

A	Γ_A	$H_A(t)$
$k\langle x, y\rangle/(x^2, xy)$	$(0,1) \times \mathbf{P}^1$	$(1+t)/(1-t)$
$k\langle x, y\rangle/(x^2, yx)$	$\mathbf{P}^1 \times (0,1)$	$(1+t)/(1-t)$
$k\langle x, y\rangle/(x^2, xy - \lambda yx)$	$(0,1) \times (0,1)$	$(1+t)/(1-t)$
$k\langle x, y\rangle/(x^2, y^2)$	$(0,1) \times (1,0) \cup (1,0) \times (0,1)$	$(1+t)/(1-t)$
$k\langle x, y\rangle/(x^2, y^2 - xy)$	$(0,1) \times (1,0) \cup (1,1) \times (0,1)$	$1+2t+2t^2+t^3$
$k\langle x, y\rangle/(xy, yx)$	$(0,1) \times (0,1) \cup (1,0) \times (1,0)$	$(1+t)/(1-t)$
$k\langle x, y\rangle/(xy, x^2 - y^2)$	$(0,1) \times (1,0)$	$1+2t+2t^2$
$k\langle x, y\rangle/(xy, x^2 - yx)$	$(0,1) \times (0,1) \cup (1,1) \times (1,0)$	$(1+t-t^3)/(1-t)$
$k\langle x, y\rangle/(xy, x^2 - yx + y^2)$	$(0,1) \times (1,1) \cup (1,1) \times (1,0)$	$1+2t+2t^2+t^3$
$k\langle x, y\rangle/(xy, x^2 - yx + \mu y^2)$	$(0,1) \times (\mu,1) \cup (1,1) \times (1,0)$	$1+2t+2t^2$

where $\lambda \neq 0$, $\mu \neq 0, 1$.

THEOREM 4.6. *Every quadratic algebra of the form* $k\langle x, y\rangle/(f_1, f_2)$ *is isomorphic to exactly one of the following*:

$$k\langle x, y\rangle/(x^2, xy),$$
$$k\langle x, y\rangle/(x^2, yx),$$
$$k\langle x, y\rangle/(x^2, y^2),$$
$$k\langle x, y\rangle/(x^2, y^2 - xy),$$
$$k\langle x, y\rangle/(x^2, xy - \lambda yx) =: S_\lambda, \quad (\lambda \neq 0)$$
$$k\langle x, y\rangle/(xy, yx),$$
$$k\langle x, y\rangle/(xy, x^2 - yx),$$
$$k\langle x, y\rangle/(xy, x^2 - y^2),$$
$$k\langle x, y\rangle/(xy, x^2 - yx + \mu y^2) =: T_\mu \quad (\mu \neq 0, 1)$$

where

$$S_\lambda \cong S_{\lambda'} \Leftrightarrow \lambda' = \lambda, \quad T_\mu \cong T_{\mu'} \Leftrightarrow \mu' = \mu.$$

Next we classify quadratic algebras in Theorem 4.6 up to graded Morita equivalence. We check whether two algebras are not graded Morita equivalent by using Theorem 2.7 and Theorem 2.9.

Except the following three cases

$$A := k\langle x, y\rangle/(x^2, xy - \lambda yx) \overset{?}{\leftrightarrow} k\langle x, y\rangle/(x^2, xy - \lambda' yx) =: A', \tag{4-3}$$

$$B := k\langle x, y\rangle/(x^2, y^2) \overset{?}{\leftrightarrow} k\langle x, y\rangle/(xy, yx) =: B', \tag{4-4}$$

$$C := k\langle x, y\rangle/(xy, x^2 - yx + \mu y^2) \overset{?}{\leftrightarrow} k\langle x, y\rangle/(xy, x^2 - yx + \mu' y^2) =: C' \tag{4-5}$$

where $\lambda \neq \lambda'$, $\mu \neq \mu'$, we can check that all algebras in Theorem 4.6 are non-graded Morita equivalent to one another by Theorem 2.7 and Theorem 2.9. In the case (4-3), if we define $\varphi \in \mathrm{Aut}_k A$ by $\varphi(x) = x$, $\varphi(y) = \lambda'\lambda^{-1}y$, then $\theta = \{\varphi^i\}$ is a twisting system such that $A' \cong A^\theta$, so $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$. In the case (4-4), if we define $\psi \in \mathrm{Aut}_k B$ by $\psi(x) = y$, $\psi(y) = x$, then $\eta = \{\psi^i\}$ is a twisting system such that $B' \cong B^\eta$, so $\mathrm{GrMod}\, B \cong \mathrm{GrMod}\, B'$.

We now consider the case (4-5). Suppose that there is a twisting system $\tau = \{\tau_i\}$ such that $C' \cong C^\tau$. We can present C and C' as

$$C = k\langle x, y\rangle/(xy, (x - y)(x - \mu y)), \quad C' = k\langle x, y\rangle/(xy, (x - y)(x - \mu' y)).$$

Note that C and C' have exactly two linealy independent reducible relations as presented above by Proposition 4.2. Thus, since τ is a twisting system, τ_i satisfies

$$\begin{cases} \tau_i(x)\tau_{i+1}(y) = xy \\ \tau_i(x - y)\tau_{i+1}(x - \mu' y) = (x - y)(x - \mu y) \end{cases} \quad \text{or} \quad \begin{cases} \tau_i(x)\tau_{i+1}(y) = (x - y)(x - \mu y) \\ \tau_i(x - y)\tau_{i+1}(x - \mu' y) = xy \end{cases}$$

for any i. Then we can check that $\mu + \mu' = 1$ is necessary condition for τ to be a twisting system by calculations. Moreover if $\mu + \mu' = 1$, then we construct a twisting system τ by

$$\begin{cases} \tau_{2i}(x) = \mu' x \\ \tau_{2i}(y) = x - \mu y \\ \tau_{2i}(x^2) = \mu' x^2 \\ \tau_{2i}(yx) = x^2 - \mu yx \end{cases}, \quad \begin{cases} \tau_{2i+1}(x) = -x + y \\ \tau_{2i+1}(y) = y \\ \tau_{2i+1}(x^2) = \mu' x^2 - \mu' yx \\ \tau_{2i+1}(yx) = x^2 - \mu' yx \end{cases}$$

for all $i \geq 0$. (Since $H_C(t) = H_{C'}(t) = 1 + 2t + 2t^2$, it is enough to check

$$\tau_j((ax + by)\tau_1(cx + dy)) = \tau_j(ax + by)\tau_{j+1}(cx + dy)$$

for all j.)

Hence we complete classification up to graded Morita equivalence in the case $r = 2$ as follows.

THEOREM 4.7. *Every quadratic algebra of the form* $k\langle x, y\rangle/(f_1, f_2)$ *is graded Morita equivalent to exactly one of the following*:

$$k\langle x, y\rangle/(x^2, xy),$$

$$k\langle x, y\rangle/(x^2, yx),$$

$$k\langle x, y\rangle/(x^2, y^2 - xy),$$

$$k\langle x, y\rangle/(x^2, xy - yx),$$

$$k\langle x, y\rangle/(xy, yx),$$

$$k\langle x, y\rangle/(xy, x^2 - yx),$$

$$k\langle x, y\rangle/(xy, x^2 - y^2),$$

$$k\langle x, y\rangle/(xy, x^2 - yx + \mu y^2) =: T_\mu \quad (\mu \neq 0, 1)$$

where

$$\mathrm{GrMod}\ T_\mu \cong \mathrm{GrMod}\ T_{\mu'} \Leftrightarrow \mu' = \mu \ \textit{or}\ \mu + \mu' = 1.$$

5. The Case $r = 3$

DEFINITION 5.1. Let $A = k\langle X\rangle/(R)$ be a quadratic algebra. Let V be a k-vector space having a basis $X = \{x_1, \dots, x_n\}$, and V^* a k-dual space of V. If the elements $\xi_i \in V^*$ are defined by $\xi_i(x_i) = \delta_{ij}$, then $X^* = \{\xi_1, \dots, \xi_n\}$ is a basis of V^* where δ_{ij} is the Kronecker delta. The quadratic dual of A is defined by

$$A^! = k\langle X^*\rangle/(R^\perp), \qquad R^\perp = \{\lambda \in k\langle X^*\rangle_2 \mid \lambda(f) = 0 \ \ \forall f \in R\}$$

where if $f = \sum_{i,j} \alpha_{ij} x_i x_j$, $\lambda = \sum_{i,j} \alpha_{ij}^* \xi_i \xi_j$, then $\lambda(f) = \sum_{i,j} \alpha_{ij} \alpha_{ij}^*$.

EXAMPLE 5.2. Example of quadratic duals.
- $A = k\langle x, y\rangle \leftrightarrow A^! = k\langle \xi, \eta\rangle/(\xi^2, \xi\eta, \eta\xi, \eta^2)$.
- $A = k\langle x, y\rangle/(xy - \lambda yx) \leftrightarrow A^! = k\langle \xi, \eta\rangle/(x^2, \lambda\xi\eta + \eta\xi, \eta^2)$.
- $A = k\langle x, y\rangle/(x^2, y^2) \leftrightarrow A^! = k\langle \xi, \eta\rangle/(\xi\eta, \eta\xi)$.

The following result is well-known (cf. [4]). We will include the proof for the reader's convenience.

THEOREM 5.3. *Let* $A = k\langle X\rangle/(R)$, $B = k\langle X\rangle/(S)$ *be quadratic algebras, and* $A^!$, $B^!$ *the quadratic duals. Then a graded* k*-algebra homomorphism* $\sigma : A \to B$

induces a graded k-algebra homomorphism ${}^t\sigma : B^! \to A^!$. *In particular if* σ *is an isomorphism, then so is* ${}^t\sigma$. *Thus*

$$A \cong B \Leftrightarrow A^! \cong B^!.$$

Proof. Set $A^! = k\langle X^*\rangle/(R^\perp)$, $B^! = k\langle X^*\rangle/(S^\perp)$. Let $\sigma|_{A_1}$ be the restriction map of σ to A_1, and $\tilde{\sigma} : k\langle X\rangle \to k\langle X\rangle$ the natural extended map of $\sigma|_{A_1}$. Moreover let ${}^t\tilde{\sigma} : k\langle X^*\rangle \to k\langle X^*\rangle$ be the natural extended map of ${}^t(\sigma|_{A_1}) : B_1^! \to A_1^!$. We will see that $B^! \to A^!$ is induced by ${}^t\tilde{\sigma}$.

$$\begin{tikzcd} k\langle X^*\rangle \arrow[r, "{}^t\tilde{\sigma}"] \arrow[d, two heads, "\pi'"'] & k\langle X^*\rangle \arrow[d, two heads, "\pi"] \\ B^! \arrow[r, dotted] & A^!. \end{tikzcd}$$

For any $f \in R$, $\tilde{\sigma}(f)$ is in S. For any $\lambda \in S^\perp$, it follows from ${}^t\tilde{\sigma}(\lambda)(f) = \lambda(\tilde{\sigma}(f)) = 0$ that ${}^t\tilde{\sigma}(\lambda) \in R^\perp$. Thus we can define

$${}^t\sigma : B^! \to A^!, \quad \pi'(g) \mapsto \pi\,{}^t\tilde{\sigma}(g) \quad (g \in k\langle X^*\rangle)$$

where $\pi : k\langle X^*\rangle \to A^!$ and $\pi' : k\langle X^*\rangle \to B^!$ are the natural maps. If we think of ${}^t\sigma$ as $\sigma^!$, then we can easily check that $(-)^!$ is a contravariant functor from the category of quadratic algebras in n variables to itself, so $A^! \cong B^!$ is induced by $A \cong B$. It follows from the fact $A \cong (A^!)^!$ that $A \cong B$ if and only if $A^! \cong B^!$. $\square$

Corollary 5.4. *Every quadratic algebra of the form* $k\langle x, y\rangle/(f_1, f_2, f_3)$ *is isomorphic to exactly one of the following*:

$$k\langle x, y\rangle/(xy, yx, y^2) \cong (k\langle x, y\rangle/(x^2))^!,$$

$$k\langle x, y\rangle/(x^2, yx, y^2) \cong (k\langle x, y\rangle/(xy))^!,$$

$$k\langle x, y\rangle/(x^2 + xy, xy + yx, y^2) \cong k_J[x, y]^!,$$

$$k\langle x, y\rangle/(x^2, \lambda xy + yx, y^2) \cong k_\lambda[x, y]^! \quad (\lambda \neq 0)$$

where

$$k_\lambda[x, y]^! \cong k_{\lambda'}[x, y]^! \Leftrightarrow \lambda' = \lambda^{\pm 1}.$$

Proposition 5.5. *Let* $A = k\langle X\rangle/(R)$ *be a quadratic algebra, and* $\varphi \in \mathrm{Aut}_k A$. *Set* $\theta = \{\theta_i = \varphi^i\}$, ${}^t\theta^{-1} = \{(({}^t\varphi)^{-1})^i = ({}^t\varphi)^{-i}\}$. *Then*

$$(A^\theta)^! = (A^!)^{{}^t\theta^{-1}}.$$

where ${}^t\varphi$ is the graded k-algebra automorphism of $A^!$ given by applying Theorem 5.3 to φ.

PROOF. Since $\varphi \in \mathrm{Aut}_k A$, θ is a twisting system on A. By Theorem 5.3, $({}^t\varphi)^{-1}$ is in $\mathrm{Aut}_k A^!$, so ${}^t\theta^{-1}$ is a twisting system on $A^!$.

We set $A^!$, A^θ, $(A^\theta)^!$, $(A^!)^{{}^t\theta^{-1}}$ as follows:

$$\begin{tikzcd}
A^! = k\langle X^*\rangle/(R^\perp) \arrow[d, "\text{twist by } {}^t\theta^{-1}"] & A = k\langle X\rangle/(R) \arrow[l, "\text{quadratic dual}"'] \arrow[r, "\text{twist by } \theta"] & A^\theta = k\langle X\rangle/(R') \arrow[d, "\text{quadratic dual}"'] \\
(A^!)^{{}^t\theta^{-1}} = k\langle X^*\rangle/(R^{\perp\prime\prime}) \arrow[rr, equal, "?"] & & (A^\theta)^! = k\langle X^*\rangle/(R'^\perp).
\end{tikzcd}$$

Let $\sigma := \varphi|_{A_1}$. If $\lambda'' = \sum_{p,q}\alpha^*_{pq}\xi_p\xi_q \in R^{\perp\prime\prime}$ and $f' = \sum_{s,t}\alpha_{st}x_sx_t \in R'$, then $\sum_{p,q}\alpha^*_{pq}\,{}^t\theta_i^{-1}(\xi_p)\,{}^t\theta_{i+1}^{-1}(\xi_q) \in R^\perp$ and $\sum_{s,t}\alpha_{st}\theta_i(x_s)\theta_{i+1}(x_t) \in R$ for any i, so

$$\begin{aligned}
\lambda''(f') &= \sum_{p,q}\alpha^*_{pq}\xi_p\xi_q\left(\sum_{s,t}\alpha_{st}x_sx_t\right)\\
&= \sum_{p,q}\alpha^*_{pq}({}^t\sigma)^{-i}(\xi_p)({}^t\sigma)^{-(i+1)}(\xi_q)\left(\sum_{s,t}\alpha_{st}\sigma^i(x_s)\sigma^{i+1}(x_t)\right)\\
&= \sum_{p,q}\alpha^*_{pq}\,{}^t\theta_i^{-1}(\xi_p)\,{}^t\theta_{i+1}^{-1}(\xi_q)\left(\sum_{s,t}\alpha_{st}\theta_i(x_s)\theta_{i+1}(x_t)\right) = 0.
\end{aligned}$$

Hence $\lambda'' \in R'^\perp$. On the other hand, let $\mu = \sum_{p,q}\alpha^*_{pq}\xi_p\xi_q \in R'^\perp$. If $f = \sum_{s,t}\alpha_{st}x_sx_t \in R$, then for any i,

$$\begin{aligned}
&\sum_{p,q}\alpha^*_{pq}\,{}^t\theta_i^{-1}(\xi_p)\,{}^t\theta_{i+1}^{-1}(\xi_q)(f)\\
&\quad= \sum_{p,q}\alpha^*_{pq}({}^t\sigma)^i({}^t\theta_i^{-1})(\xi_p)({}^t\sigma)^{i+1}({}^t\theta_{i+1}^{-1})(\xi_q)\left(\sum_{s,t}\alpha_{st}\sigma^{-i}(x_s)\sigma^{-(i+1)}(x_t)\right)\\
&\quad= \sum_{p,q}\alpha^*_{pq}\xi_p\xi_q\left(\sum_{s,t}\alpha_{st}\sigma^{-i}(x_s)\sigma^{-(i+1)}(x_t)\right)\\
&\quad= \mu\left(\sum_{s,t}\alpha_{st}\theta_i^{-1}(x_s)\theta_{i+1}^{-1}(x_t)\right) = 0 \quad \left(\because \sum_{s,t}\alpha_{st}\theta_i^{-1}(x_s)\theta_{i+1}^{-1}(x_t) \in R'\right),
\end{aligned}$$

so $\sum_{p,q}\alpha^*_{pq}\,{}^t\theta_i^{-1}(\xi_p)\,{}^t\theta_{i+1}^{-1}(\xi_q) \in R^\perp$ for any i. Hence $\mu = \sum_{p,q}\alpha^*_{pq}\xi_p\xi_q \in R^{\perp\prime\prime}$. $\square$

Corollary 5.6. *Let* A, B *be quadratic algebras, and* $\theta = \{\theta_i = \varphi^i \mid \varphi \in \mathrm{Aut}_k A\}$. *Then*

$$B \cong A^{\theta} \Rightarrow \mathrm{GrMod}\, A^! \cong \mathrm{GrMod}\, B^!.$$

Proof. This follows from Theorem 5.3 and Proposition 5.5. □

Let

$$A = k\langle x, y\rangle/(xy, yx, y^2), \qquad B = k\langle x, y\rangle/(x^2, yx, y^2),$$

$$C = k\langle x, y\rangle/(x^2 + xy, xy + yx, y^2), \quad D = k\langle x, y\rangle/(x^2, \lambda xy + yx, y^2).$$

By Corollary 5.6,

$$\mathrm{GrMod}\, C \cong \mathrm{GrMod}\, k[x, y]^!_J \cong \mathrm{GrMod}\, k[x, y]^! \cong \mathrm{GrMod}\, k[x, y]^!_\lambda \cong \mathrm{GrMod}\, D.$$

The point scheme of each algebra is

$$\Gamma_A = (1, 0) \times (1, 0), \quad \Gamma_B = (1, 0) \times (0, 1), \quad \Gamma_C = \varnothing, \quad \Gamma_D = \varnothing,$$

so $\mathrm{GrMod}\, A \not\cong \mathrm{GrMod}\, C$ and $\mathrm{GrMod}\, B \not\cong \mathrm{GrMod}\, C$ by Theorem 2.7. The Hilbert series of each algebra is

$$H_A(t) = 1 + 2t + t^2 + t^3 + t^4 + t^5 \cdots = (1 + t - t^2)/(1 - t),$$

$$H_B(t) = H_C(t) = H_D(t) = 1 + 2t + t^2 = (1 + t)^2,$$

so $\mathrm{GrMod}\, A \not\cong \mathrm{GrMod}\, B$ by Theorem 2.9. Hence we complete the following classification.

Corollary 5.7. *Every quadratic algebra of the form* $k\langle x, y\rangle/(f_1, f_2, f_3)$ *is graded Morita equivalent to exactly one of the following*:

$$k\langle x, y\rangle/(xy, yx, y^2) \cong (k\langle x, y\rangle/(x^2))^!,$$

$$k\langle x, y\rangle/(x^2, yx, y^2) \cong (k\langle x, y\rangle/(xy))^!,$$

$$k\langle x, y\rangle/(x^2, xy + yx, y^2) \cong k[x, y]^!.$$

6. Properties of the Classified Quadratic Algebras

At the end of this paper, we describe several algebraic properties of the classified quadratic algebras. More specifically, we check properties like domain,

Table 2. List of Properties

A	domain	left noetherian	right noetherian	gldim A	GKdim A	Koszul	Hilbert series $H_A(t)$
$k\langle x, y\rangle$	Yes	No	No	1	∞	Yes	$1/(1-2t)$
$k\langle x, y\rangle/(x^2)$	No	No	No	∞	∞	Yes	$(1+t)/(1-t-t^2)$
$k\langle x, y\rangle/(xy)$	No	No	No	2	2	Yes	$1/(1-t)^2$
$k\langle x, y\rangle/(-x^2+xy-yx) \cdots \clubsuit$	Yes	Yes	Yes	2	2	Yes	$1/(1-t)^2$
$k\langle x, y\rangle/(xy-\lambda yx) \cdots \clubsuit\clubsuit$	Yes	Yes	Yes	2	2	Yes	$1/(1-t)^2$
$k\langle x, y\rangle/(x^2, xy)$	No	Yes	No	∞	1	Yes	$(1+t)/(1-t)$
$k\langle x, y\rangle/(x^2, yx)$	No	No	Yes	∞	1	Yes	$(1+t)/(1-t)$
$k\langle x, y\rangle/(x^2, y^2-xy)$	No	Yes	Yes	∞	0	No	$1+2t+2t^2+t^3$
$k\langle x, y\rangle/(x^2, xy-\lambda yx) \cdots \diamondsuit\diamondsuit$	No	Yes	Yes	∞	1	Yes	$(1+t)/(1-t)$
$k\langle x, y\rangle/(x^2, y^2) \cdots \heartsuit$	No	Yes	Yes	∞	1	Yes	$(1+t)/(1-t)$
$k\langle x, y\rangle/(xy, yx) \cdots \heartsuit$	No	Yes	Yes	∞	1	Yes	$(1+t)/(1-t)$
$k\langle x, y\rangle/(xy, x^2-yx)$	No	Yes	Yes	∞	1	No	$(1+t-t^3)/(1-t)$
$k\langle x, y\rangle/(xy, x^2-y^2)$	No	Yes	Yes	∞	0	No	$1+2t+2t^2$
$k\langle x, y\rangle/(xy, x^2-yx+\mu y^2) \cdots \#\#$	No	Yes	Yes	∞	0	No	$1+2t+2t^2$
$k\langle x, y\rangle/(xy, yx, y^2)$	No	Yes	Yes	∞	1	Yes	$(1+t-t^2)/(1-t)$
$k\langle x, y\rangle/(x^2, yx, y^2)$	No	Yes	Yes	∞	0	Yes	$(1+t)^2$
$k\langle x, y\rangle/(x^2+xy, xy+yx, y^2) \cdots \spadesuit$	No	Yes	Yes	∞	0	Yes	$(1+t)^2$
$k\langle x, y\rangle/(x^2, \lambda xy+yx, y^2) \cdots \spadesuit\spadesuit$	No	Yes	Yes	∞	0	Yes	$(1+t)^2$
$k\langle x, y\rangle/(x^2, xy, yx, y^2)$	No	Yes	Yes	∞	0	Yes	$1+2t$

where $\lambda \neq 0$, $\mu \neq 0, 1$. There exist some graded Morita equivalence between the algebras with the same mark like $\clubsuit, \diamondsuit, \heartsuit, \spadesuit, \#$.

noetherian property, global dimension, GK-dimension, Koszul property and Hilbert series. It is easy to check left and right noetherian property, Hilbert series and GK-dimension for the algebras in Table 2, so we give a few facts for determining global dimension and Koszul property.

Theorem 6.1 [6, Chapter 10, Theorem 4.2]. *Let A be a connected graded left (or right) noetherian k-algebra. If $H_A(t)^{-1} \notin \mathbf{Z}[t]$, then* $\operatorname{gldim} A = \infty$.

Example 6.2. Every classified quadratic algebra A in the cases $r = 2, 3, 4$ has $\operatorname{gldim} A = \infty$. This follows immediately from Theorem 6.1 and the Hilbert series in Table 2.

Next we recall the notion of Koszul. A connected graded k-algebra A is called Koszul if the minimal free resolution of k_A is of the form

$$\cdots \to \bigoplus A(-3) \to \bigoplus A(-2) \to \bigoplus A(-1) \to \bigoplus A \to k \to 0.$$

If A is Koszul, then A is a quadratic algebra. Moreover Yoneda algebra $E_A(k) = \bigoplus_{i \in \mathbf{N}} \underline{\mathrm{Ext}}^i_A(k,k)$ of A is isomorphic to the quadratic dual $A^!$ as graded algebras. Further if A is Koszul, then $A^!$ is also Koszul and we have the following equation

$$H_A(t)H_{A^!}(-t) = 1.$$

We refer to [7] for other basic properties of Koszul algebras.

Example 6.3. If $B = k\langle x, y\rangle/(xy, x^2 - y^2)$, then $B^! \cong B$. It follows from the Hilbert series $H_B(t) = 1 + 2t + 2t^2$ that $H_B(t)H_{B^!}(-t) \neq 1$. Hence B is not Koszul. Similarly we can check Koszul property for the algebras in Table 2.

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