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NOTE ON HERMITIAN JACOBI FORMS

By

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Abstract. We compare the spaces of Hermitian Jacobi forms (HJF) of weight k and indices 1, 2 with classical Jacobi forms (JF) of weight k and indices 1, 2, 4. Upper bounds for the order of vanishing of HJF at the origin are obtained. We compute the rank of HJF as a module over elliptic modular forms and prove the algebraic independence of the generators in case of index 1. Some related questions are discussed.

1. Introduction

Hermitian Jacobi forms of integer weight and index are defined for the Hermitian Jacobi group over the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K. (See section 2.1.) They were defined and studied by K. Haverkamp in [7]. In [3] differential operators were constructed from the Taylor expansion of Hermitian Jacobi forms in analogy to that for classical Jacobi forms in [5] and a certain subspace of Hermitian Jacobi forms was realized as a subspace of a direct product of elliptic modular forms for the full modular group. The structural properties of index 1 froms were treated in [11].

In this paper we treat classical Jacobi forms as an intermediate space between Hermitian Jacobi forms and elliptic modular forms. We present some of the structural properties of index 2 forms using the restriction maps $\pi_\rho : J_{k,m}(\mathcal{O}_K) \to J_{k,N(\rho)m}$ defined by $\pi_\rho \phi(\tau, z_1, z_2) = \phi(\tau, \rho z, \bar{\rho} z)$ ($\rho \in \mathcal{O}_K$, see [7]). Since we do not have (at present) the order of vanishing of a Hermitian Jacobi form at the origin (which is known to be $2m$, m being the index; in the case of classical Jacobi forms), computations involving the Taylor expansions is not very fruitful for

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$m \geq 2$. Therefore many of the arguments in this paper rely on the Theta decomposition (see section 2.1.1) of such forms.

The main results are in sections 3, 4 and 5. The purpose of this note is to look at the structure of index 2 forms by comparing them with the classical Jacobi forms. In sections 3 and 4 we relate Hermitian Jacobi forms with classical Jacobi forms via several exact sequences. In section 4.5, we give upper bounds for the order of vanishing of Hermitian Jacobi forms at the origin. We compute the rank of index m forms of weight a multiple of 2 and 4 (denoted as $J_{n*,m}(\mathcal{O}_K)$, $n = 2, 4$) as a module over the algebra of elliptic modular forms (denoted as M_*) and prove the algebraic independence of the 3 generators (see [11] for their description) of $J_{4*,1}(\mathcal{O}_K)$ over M_*. Unlike the classical Jacobi forms, we find that the number of homogeneous products of degree m of the index 1 generators is less than the rank of $J_{n*,m}(\mathcal{O}_K)$ over M_* for $m \geq 2$ (see Proposition 5.1 and section 6, remark 3). In the final section, we discuss several related questions on Hermitian Jacobi forms.

2. Notations and Definitions

We let $e(z) := e^{2\pi i z}$ unless otherwise mentioned. In the rest of the paper we will use the standard notation $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

2.1. Hermitian Jacobi forms. Let $\mathcal{H}$ be the upper half plane. Let $K = \mathbf{Q}(i)$ and $\mathcal{O}_K = \mathbf{Z}[i]$ be it's ring of integers. Let $\Gamma_1(\mathcal{O}_K) = \{\varepsilon M \mid M \in SL(2, \mathbf{Z}), \varepsilon \in \mathcal{O}_K^\times\}$. The Hermitian Jacobi group over $\mathcal{O}_K$ is $\Gamma^J(\mathcal{O}_K) = \Gamma_1(\mathcal{O}_K) \ltimes \mathcal{O}_K^2$.

Definition 2.1. The space of Hermitian Jacobi forms for $\Gamma^J(\mathcal{O}_K)$ of weight k and index m, where k, m are positive integers, consists of holomorphic functions on $\mathcal{H} \times \mathbf{C}^2$ satisfying:

$$\phi(\tau, z_1, z_2) = \phi|_{k,m}\varepsilon M(\tau, z_1, z_2)$$

$$:= \varepsilon^{-k}(c\tau + d)^{-k} e^{-2\pi i m c z_1 z_2/(c\tau + d)} \phi\left(M\tau, \frac{\varepsilon z_1}{c\tau + d}, \frac{\bar{\varepsilon} z_2}{c\tau + d}\right)$$

$$\text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } SL(2, \mathbf{Z}),\ \varepsilon \in \mathcal{O}_K^\times, \tag{2.1}$$

$$\phi(\tau, z_1, z_2) = \phi|_m[\lambda, \mu]$$

$$:= e^{2\pi i m (N(\lambda)\tau + \bar{\lambda} z_1 + \lambda z_2)} \phi(\tau, z_1 + \lambda\tau + \mu, z_2 + \bar{\lambda}\tau + \bar{\mu})$$

$$\text{for all } \lambda, \mu \text{ in } \mathcal{O}_K, \tag{2.2}$$

and having a Fourier expansion of the form:

$$\phi(\tau, z_1, z_2) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathcal{O}_K^{\#} \\ nm \geq N(r)}} c_\phi(n, r) e^{2\pi i (n\tau + rz_1 + \bar{r}z_2)}, \tag{2.3}$$

where $\mathcal{O}_K^{\#} = \frac{i}{2}\mathcal{O}_K$ (the inverse different of $K|\mathbf{Q}$) and $N : K \to \mathbf{Q}$ is the norm map. The (finite dimensional) complex vector space of Hermitian Jacobi forms of weight k and index m is denoted by $J_{k,m}(\mathcal{O}_K)$. We say that ϕ is a Jacobi *cusp* form if it is true that $c_\phi(n, r) = 0$ for $nm = N(r)$. The space of Jacobi cusp forms of weight k and index m is denoted $J_{k,m}^{cusp}(\mathcal{O}_K)$. The theory of Hermitian Jacobi forms, especially the theory of Hecke operators for them, have been studied by K. Haverkamp in [6], [7].

2.1.1. *Theta Decomposition.* Hermitian Jacobi forms admit a Theta decomposition analogous to that of classical Jacobi forms. Let $\phi \in J_{k,m}(\mathcal{O}_K)$ have the Fourier expansion as in equation (2.3). It is known that the Fourier coefficients $c_\phi(n, r)$ $(n \in \mathbf{Z}, r \in \mathcal{O}_K^{\#})$ of ϕ depend only on $r \ (\mathrm{mod}\ m\mathcal{O}_K)$ and $D = nm - N(r)$. Thus, we can rewrite the Fourier development of ϕ to get the `theta decomposition`:

$$\phi(\tau, z_1, z_2) = \sum_{s \in \mathcal{O}_K^{\#}/m\mathcal{O}_K} h_s(\tau) \cdot \theta_{m,s}^H(\tau, z_1, z_2), \quad \text{where}$$

$$h_s(\tau) := \sum_{\substack{L=0 \\ N(s)+L/4 \in m\mathbf{Z}}}^{\infty} c_s(L) e^{2\pi i L\tau/4m} \quad \text{and}$$

$$\theta_{m,s}^H(\tau, z_1, z_2) := \sum_{r \equiv s \pmod{mO_K}} e\left(\frac{N(r)}{m}\tau + rz_1 + \bar{r}z_2\right). \tag{2.4}$$

We take as a set of representatives of $\mathcal{O}_K^{\#}$ in $\mathcal{O}_K^{\#}/m\mathcal{O}_K$ as the set

$$\mathscr{S}_m := \left\{\frac{a}{2} + i\frac{b}{2} \middle| a, b \in \mathbf{Z}/2m\mathbf{Z}\right\}.$$

We call $\theta_{m,s}^H$ the Hermitian Theta function of index m, type s and h_s the Theta components of ϕ. We will also denote the Theta components of $\phi \in J_{k,m}(\mathcal{O}_K)$ by $h_{a,b}$ $\left(\frac{a+ib}{2} \in \mathscr{S}_m\right)$ and the Hermitian Theta functions of weight 1 and index m by $\theta_{m;a,b}^H$ (or by $\theta_{m;s}^H$, $s = \frac{a+ib}{2} \in \mathscr{S}_m$) in this paper, but we drop the index unless there is a danger of confusion. Also we denote by $\theta_{m,\mu}(\tau, z)$ ($\mu \pmod{2m}$) the classical Jacobi-Theta functions defined as

$$\theta_{m,\mu}(\tau, z) := \sum_{r \in \mathbf{Z}, r \equiv \mu \pmod{2m}} q^{r^2/4m} \zeta^r; \quad q = e^{2\pi i \tau}, \zeta = e^{2\pi i z}.$$

2.1.2. *Transformation Formulas for the Theta components.* The Theta components of $\phi \in J_{k,m}(\mathcal{O}_K)$ (see [6, p. 46, 47]) have the following transformation properties under $SL(2, \mathbf{Z})$ and $\mathcal{O}_K^\times$ (:= The group of units in $\mathcal{O}_K$):

$$h_s|_{k-1} T = e^{-2\pi i N(s)/m} h_s, \tag{2.5}$$

$$h_s|_{k-1} S = \frac{i}{4m} \sum_{s' \in \mathcal{O}_K^\#/m\mathcal{O}_K} e^{-4\pi i \operatorname{Re}(\bar{s}s')/m} h_{s'}, \tag{2.6}$$

$$h_s|_{k-1} \varepsilon I = \varepsilon h_{\varepsilon s}, \quad \varepsilon \in \mathcal{O}_K^\times. \tag{2.7}$$

In the above transformation properties, the slash operations are the usual ones in view of the fact that $h_s \in M_{k-1}(\Gamma(4m))$ for all $s \in \mathcal{O}_K^\#/m\mathcal{O}_K$. (See [6].)

2.1.3. *Other related spaces of Modular forms.* We denote the space of Jacobi forms of weight k and index m for the Jacobi group $SL(2, \mathbf{Z}) \ltimes \mathbf{Z}^2$ by $J_{k,m}$ (see [5] for their definition and properties), elliptic modular (resp. cusp) forms of weight k for a congruence subgroup Γ of $SL(2, \mathbf{Z})$ by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$). When $\Gamma = SL(2, \mathbf{Z})$, we denote the corresponding spaces by M_k (resp. S_k).

Further, we define the following space of modular forms with a multiplier system. Let ω be the linear character of $SL(2, \mathbf{Z})$ defined by $\omega(T) = i$, $\omega(S) = i$. Then,

Definition 2.2 ([1], [10]).

$$M_{k-1}(SL(2, \mathbf{Z}), \bar{\omega}) := \{\text{The space of holomorphic functions } f : \mathcal{H} \to \mathbf{C} \text{ bounded at infinity and satisfying } f|_{k-1} S = \bar{\omega}(S) f,\ f|_{k-1} T = \bar{\omega}(T) f\}.$$

2.1.4. *Restriction maps and Differential operators.* The main restriction maps that would be used in this paper are

$$\pi_\rho : J_{k,m}(\mathcal{O}_K) \to J_{k,N(\rho)m}, \quad \phi(\tau, z_1, z_2) \mapsto \phi(\tau, \rho z, \bar{\rho} z) \quad (\rho \in \mathcal{O}_K).$$

$$D_0 : J_{k,m} \to M_k, \qquad \phi(\tau, z) \mapsto \phi(\tau, 0). \tag{2.8}$$

Occasionally we will refer to the Taylor development of a Hermitian Jacobi form ϕ with Taylor coefficients $\chi_{\alpha,\beta}$, around the origin $(0,0)$:

$$\phi(\tau, z_1, z_2) = \sum_{\alpha,\beta\geq 0} \chi_{\alpha,\beta}(\tau) z_1^{\alpha} z_2^{\beta}. \tag{2.9}$$

We will use the following well-known differential operator on Jacobi forms, constructed from it's Taylor expansion around the origin $z = 0$ (see [5] for more details):

$$D_2 : J_{k,m} \to S_{k+2}, \quad \phi(\tau, z) \mapsto \left(\frac{k}{2\pi i m} \frac{\partial^2 \phi}{\partial z^2} - 2 \frac{\partial \phi}{\partial \tau} \right)_{z=0}. \tag{2.10}$$

3. Comparision of $J_{k,1}$ and $J_{k,1}(\mathcal{O}_K)$

3.1. As introduced in section 2.1.1, we take $\mathcal{S}_1 = \{0, \frac{i}{2}, \frac{1}{2}, \frac{1+i}{2}\}$ as the set of representatives of $\mathcal{O}_K^{\#}$ in $\mathcal{O}_K^{\#}/\mathcal{O}_K$. ($\cong \frac{\mathbf{Z}}{2\mathbf{Z}} \times \frac{\mathbf{Z}}{2\mathbf{Z}}$.) For convenience, in this section we drop the index (which is 1) of the Hermitian Jacobi-Theta functions and denote them by $\theta_{i,j}^H$, and the corresponding Theta components by $h_{i,j}$ where $\{i, j\} \in \{0, 1\}$. We denote the Jacobi Theta functions of index 1 by $\theta_{1,0}(\tau, z)$, $\theta_{1,1}(\tau, z)$. Further we let

$$\vartheta_0(\tau) := \sum_{r\in\mathbf{Z}} e(r^2\tau), \quad \vartheta_1(\tau) := \sum_{r\equiv 1 \pmod 2} e\left(\frac{r^2}{4}\tau\right) \quad (\tau \in \mathcal{H}). \tag{3.1}$$

3.2. The case $k \equiv 2 \pmod 4$.

Theorem 3.1. (1) *Let $k \equiv 2 \pmod 4$. Then there is an exact sequence of vector spaces*

$$0 \to J_{k,1}(\mathcal{O}_K) \xrightarrow{\pi_1} J_{k,1} \xrightarrow{D_0} M_k \to 0. \tag{3.2}$$

(2) *Let $k \equiv 2 \pmod 4$. Then π_{1+i} is the zero map.*

Proof. Let $\phi \in J_{k,1}(\mathcal{O}_K)$. In this case $h_{0,0} = h_{1,1} = 0$ and $h_{0,1} = -h_{1,0}$ and we get that

$$\pi_1 \phi = h_{0,1}(\vartheta_1 \theta_{1,0} - \vartheta_0 \theta_{1,1}).$$

Since $\vartheta_1 \theta_0 - \vartheta_0 \theta_1 \neq 0$ (see [1]), we clearly have that π_1 is injective and $\mathrm{Im}(\pi_1) \subseteq \ker D_0$.

Let $\phi \in \ker D_0$. From [1, Theorem 1] we see that $\phi(\tau, z) = \varphi(\tau)(\vartheta_1\theta_{1,0} - \vartheta_0\theta_{1,1})$, where $\varphi \in M_{k-1}(SL(2, \mathbf{Z}), \varpi)$.

So, it is enough to prove $h_{0,1} \in M_{k-1}(SL(2, \mathbf{Z}), \varpi)$, which easily follows from equations (2.5) and (2.6). Since D_0 is surjective, we get (1).

(2) This follows easily from [4, p. 5] or from Lemma 4.9, so we omit the proof. □

From the above Theorem and the results of [1] we get an isomorphism of $J_{k,1}(\mathcal{O}_K)$ to S_{k+2}, which was also obtained by R. Sasaki in [11].

Corollary 3.2. *Let* $k \equiv 2 \pmod 4$. *Then* $J_{k,1}(\mathcal{O}_K) \cong M_{k-1}(SL(2, \mathbf{Z}), \varpi)$.

Proof. This follows from the proof of Theorem 3.1, (1). □

Corollary 3.3. *Let* $k \equiv 2 \pmod 4$. *Then the composite*

$$J_{k,1}(\mathcal{O}_K) \overset{\pi_1}{\hookrightarrow} J_{k,1} \overset{D_2}{\to} S_{k+2} \tag{3.3}$$

gives an isomorphism of $J_{k,1}(\mathcal{O}_K)$ *to* S_{k+2}.

Proof. The result follows from [1, Theorem 2], which in the case $N = 1$ says that $D_2 : J_{k,1} \to S_{k+2}$ gives an isomorphism of $\ker\{D_0 : J_{k,1} \to M_k\}$ to the space $S^{\circ}_{k+2} := \left\{ f \in S_{k+2} \mid \varphi := \frac{f}{\xi} \in M_{k-1}(SL(2, \mathbf{Z}), \varpi) \right\}$ (where ω is defined in section 2.1.3), and $\xi = \vartheta_1\vartheta_0' - \vartheta_0\vartheta_1'$ (where ϑ_i, $i = 0, 1$ are defined in equation (3.1)). But $S^{\circ}_{k+2} = S_{k+2}$ when $N = 1$, since by [1, Proposition 2], $\xi \in S_3(SL(2, \mathbf{Z}), \omega)$. From equation (3.2) we have $\ker\{D_0 : J_{k,1} \to M_k\} = \mathrm{Im}(\pi_1)$, on which D_2 induces an isomorphism by the above. Therefore the Corollary follows. □

We define $J_{k,1}(\mathcal{O}_K, N)$ to be the space of Hermitian Jacobi forms of weight k and index 1 for the congruence subgroup $\Gamma_0(N)$ in the usual way. It is immediate that the same proof as in Theorem 3.1 applies to this case when $k \equiv 2 \pmod 4$ (see also [1] where the case of classical Jacobi forms is done) and we have an exact sequence of vector spaces

$$0 \to J_{k,1}(\mathcal{O}_K, N) \overset{\pi_1}{\to} J_{k,1}(N) \overset{D_0}{\to} M_k(N) \tag{3.4}$$

Corollary 3.4. *Let* $N > 1$. *Then* $J_{2,1}(\mathcal{O}_K, N) = 0$.

Proof. A result of T. Arakawa, S. Böcherer [2] says that D_0 in (3.4) is injective when $k = 2$ and $N > 1$. Therefore the Corollary follows from equation (3.4). □

3.3. The case $k \equiv 0 \pmod 4$

Theorem 3.5. (1) *Let* $k \equiv 0 \pmod 4$. *Then there is an exact sequence of vector spaces*

$$0 \longrightarrow S_{k+4} \xrightarrow{\xi^{-1}|_{S_{k+4}}} J_{k,1}(\mathcal{O}_K) \xrightarrow{\pi_1} J_{k,1} \longrightarrow 0, \tag{3.5}$$

where $\xi : J_{k,1}(\mathcal{O}_K) \to M_k \oplus S_{k+2} \oplus S_{k+4}$ *is the isomorphism given in* [11].
(2) *Let* $k \equiv 0 \pmod 4$. *Then* π_{1+i} *induces an isomorphism of* $J_{k,1}(\mathcal{O}_K)$ *to* $J_{k,2}$.

Proof. (1) follows directly from Lemma 3.7 given below.

(2) When $k \equiv 0 \pmod 4$, in the Theta decomposition of $\phi \in J_{k,1}(\mathcal{O}_K)$, we have $h_{0,1} = h_{1,0}$. Let $\phi \in \ker \pi_{1+i}$. From the Theta decomposition of $\pi_{1+i}\phi$ (see [4, p. 5]) we easily deduce that in this case $h_{0,1} = h_{1,0} = 0$, $(a_0^2 - a_2^2)h_{0,0} = (a_0^2 - a_2^2)h_{1,1} = 0$. But $a_0^2 \not\equiv a_2^2$ since Wr_2 does not vanish on $\mathcal{H}$ (see the proof of *Step 1* of Theorem 4.11). Hence, the kernel is trivial. Moreover, from Corollary 3.6, considering the dimensions, we conclude that π_{1+i} is an isomorphism. □

Corollary 3.6. *Let* $k \equiv 0 \pmod 4$. *Then*

$$J_{k,2} \xrightarrow{D_0+D_2+D_4} M_k \oplus S_{k+2} \oplus S_{k+4} \xrightarrow{\xi^{-1}} J_{k,1}(\mathcal{O}_K)$$

is an isomorphism, where ξ *is as in Theorem* 3.5.

Proof. In fact, each arrow is an isomorphism. The first map is injective by [5] and dimension count shows that it is an isomorphism. □

Remark 3.1. In the above Theorem, it is clear that if $f \in S_{k+4}$,

$$\xi^{-1}f = \{\phi \in J_{k,1}(\mathcal{O}_K) \,|\, \chi_{2,2} - 12\chi_{0,4} = f\}$$

Lemma 3.7. *The following diagram*

$$\begin{array}{ccc} J_{k,1}(\mathcal{O}_K) & \xrightarrow{\pi_1} & J_{k,1} \\ \cong \big\downarrow \xi & & \cong \big\downarrow D_0+(\pi i/k)D_2 \\ M_k \oplus S_{k+2} \oplus S_{k+4} & \xrightarrow{pr.} & M_k \oplus S_{k+2} \end{array}$$

is commutative.

Proof. The proof is immediate from definitions. First, $(pr. \circ \xi)\phi = \chi_{0,0} - \frac{2\pi i}{k}\chi'_{0,0} + \chi_{1,1}$. On the other hand, from the Taylor expansion of ϕ around the origin with Taylor coefficients $\chi_{\alpha,\beta}$ we get,

$$\left(D_0 + \frac{\pi i}{k} D_2\right) \circ \pi_1 \phi = \chi_{0,0} + (\chi_{0,2} + \chi_{2,0} + \chi_{1,1}) - \frac{2\pi i}{k}\chi'_{0,0}$$

$$= \chi_{0,0} - \frac{2\pi i}{k}\chi'_{0,0} + \chi_{1,1},$$

since $\chi_{0,2} = \chi_{2,0} = 0 = \chi_{1,0} = \chi_{0,1}$ when $k \equiv 0 \pmod 4$. In fact, $\chi_{\alpha,\beta} = 0$ unless $\alpha - \beta \equiv k \pmod 4$ follows from first transformation rule (2.1) for Hermitian Jacobi forms. □

Remark 3.2. We remark here that from the Fourier expansion (2.3) of a Hermitian Jacobi form ϕ of index 1, we get $\phi(\tau, z_1, z_2) = \phi(\tau, z_2, z_1)$ if $k \equiv 0 \pmod 4$ and hence in it's Taylor expansion we have $\chi_{\alpha,\beta} = \chi_{\beta,\alpha}$ $\forall \alpha, \beta \geq 0$. Therefore the isomorphism ξ in Theorem 3.5 is also given by $\phi \mapsto \chi_{0,0} + \xi_{1,1} + \xi_{2,2} - 12(\chi_{0,4})$. (For the definition of $\xi_{1,1}$ and $\xi_{2,2}$, see [3] or [11].) Hence the four Taylor coefficients $\chi_{0,0}$, $\chi_{0,4}$, $\chi_{1,1}$, $\chi_{2,2}$ determine ϕ, as expected in analogy with classical Jacobi forms.

4. Hermitian Jacobi forms of Index 2

In this section we consider Hermitian Jacobi forms of index 2 by relating them to classical Jacobi forms and elliptic modular forms via several restriction maps. Let $\mathcal{D} := 2i\mathcal{O}_K$, the Different of K. We use a representation of the group defined for a positive integer m:

$$G_m := \{\mu \in \mathcal{O}_K/m\mathcal{D} \mid N(\mu) \equiv 1 \pmod{4m}\}.$$

For a positive integer m, we consider the representation of G_m defined in [6]:

$$\rho_m : G_m \to Aut(J_{k,m}(\mathcal{O}_K)), \quad \mu \mapsto W_\mu,$$

where W_μ is defined by $W_\mu(h^t \cdot \Theta_m^H) = h^{(\mu)} \cdot \Theta_m^H$, $h^{(\mu)} := (h_{\mu s})_{s \in \mathcal{O}_K^\# / m\mathcal{O}_K}$;

$$\Theta_m^H(\tau, z_1, z_2) := (\theta_{m,s}^H(\tau, z_1, z_2))_{s \in \mathcal{O}_K^\# / m\mathcal{O}_K} \in \mathbf{C}^{4m^2}, \quad h = (h_s)_{s \in \mathcal{O}_K^\# / m\mathcal{O}_K}. \tag{4.1}$$

Accordingly we have a decomposition of $J_{k,m}(\mathcal{O}_K)$:

$$J_{k,m}(\mathcal{O}_K) = \bigoplus_{\eta \in G_m^*} J_{k,m}^\eta(\mathcal{O}_K), \tag{4.2}$$

where G_m^* is the group of characters of G_m and

$$J_{k,m}^{\eta}(\mathcal{O}_K) := \{\phi \in J_{k,m}(\mathcal{O}_K) \mid W_\mu \phi = \eta(\mu)\phi \ \ \forall \mu \in G_m\}. \tag{4.3}$$

Now we note that $G_2 = \mathcal{O}_K^\times \cong \mathbf{Z}/4\mathbf{Z}$ via $i \mapsto 1,\ -1 \mapsto 2,\ -i \mapsto 3,\ 1 \mapsto 0$. Also,

$$G_2^* = \{\eta_\alpha := (x \mapsto e^{2\pi i \alpha x/4}); x, \alpha \in \mathbf{Z}/4\mathbf{Z}\}.$$

The following Lemmas give the Theta decomposition of the images of Hermitian Jacobi forms of index 2 under the restriction maps. We define for convenience of notation $a_\mu := \theta_{2,\mu}(\tau, 0)$ $(\mu \in \mathbf{Z}/4\mathbf{Z})$ and $b_\mu := \theta_{4,\mu}(\tau, 0)$ $(\mu \in \mathbf{Z}/8\mathbf{Z})$.

Let $\phi \in J_{k,2}(\mathcal{O}_K)$ have the Theta decomposition as in equation (2.4) with Theta components h_s as explained in section 2.1.1.

Lemma 4.1. *Let* $\pi_1\phi = \sum_{\mu \in \mathbf{Z}/4\mathbf{Z}} H_\mu(\tau) \cdot \theta_{2,\mu}(\tau, z)$ *be it's Theta decomposition, where* $H_\mu = (-1)^k H_{-\mu}$ $(\mu \in \mathbf{Z}/4\mathbf{Z})$. *Then,*

$$H_0 = h_{0,0}a_0 + h_{0,1}a_1 + h_{0,2}a_2 + h_{0,3}a_3, \tag{4.4}$$

$$H_1 = h_{1,0}a_0 + h_{1,1}a_1 + h_{1,2}a_2 + h_{1,3}a_3, \tag{4.5}$$

$$H_2 = h_{2,0}a_0 + h_{2,1}a_1 + h_{2,2}a_2 + h_{2,3}a_3. \tag{4.6}$$

Proof. Let $s \in \mathcal{S}_2$. The effect of π_1 on $\theta_{2;s}^H$ is given below.

$$\begin{aligned}
\pi_1\theta_{2;s}^H &= \sum_{\substack{r \equiv s \pmod{2\mathcal{O}_K} \\ r \in \mathcal{O}_K^\#}} e\left(\frac{N(r)}{2}\tau + 2\,\mathrm{Re}(r) \cdot z\right) \\
&= \sum_{\substack{\mathrm{Re}(2r) \equiv \mathrm{Re}(2s) \pmod{4\mathbf{Z}} \\ \mathrm{Re}(2r) \in \mathbf{Z}}} e\left(\frac{(\mathrm{Re}(2r))^2}{8}\tau + \mathrm{Re}(2r) \cdot z\right) \\
&\quad \times \sum_{\substack{\mathrm{Im}(2r) \equiv \mathrm{Im}(2s) \pmod{4\mathbf{Z}} \\ \mathrm{Im}(2r) \in \mathbf{Z}}} e\left(\frac{(\mathrm{Im}(2r))^2}{8}\tau\right) \\
&= \theta_{2,\mathrm{Re}(2s)}(\tau, z) \cdot a_{\mathrm{Im}(2s)}.
\end{aligned}$$

This shows that $\pi_1\phi = \sum_{\mu \in \mathbf{Z}/4\mathbf{Z}} \left(\sum_{\substack{s \in \mathcal{S}_2 \\ \mathrm{Re}(2s) = \mu}} h_s \cdot a_{\mathrm{Im}(2s)} \right) \theta_{2,\mu}(\tau, z)$, which proves the Lemma. □

Lemma 4.2. *Let* $\pi_{1+i}\phi = \sum_{\mu \in \mathbf{Z}/8\mathbf{Z}} \overline{H}_\mu(\tau) \cdot \theta_{4,\mu}(\tau, z)$ *be it's Theta decomposition, where* $\overline{H}_\mu = (-1)^k \overline{H}_{-\mu}$ $(\mu \in \mathbf{Z}/8\mathbf{Z})$. *Then,*

$$\overline{H}_0 = h_{0,0}b_0 + h_{1,1}b_2 + h_{2,2}b_4 + h_{3,3}b_6, \tag{4.7}$$

$$\overline{H}_1 = h_{1,0}b_1 + h_{2,1}b_3 + h_{3,2}b_5 + h_{0,3}b_7, \tag{4.8}$$

$$\overline{H}_2 = h_{2,0}b_2 + h_{3,1}b_4 + h_{0,2}b_6 + h_{1,3}b_0, \tag{4.9}$$

$$\overline{H}_3 = h_{3,0}b_3 + h_{0,1}b_5 + h_{1,2}b_7 + h_{2,3}b_1, \tag{4.10}$$

$$\overline{H}_4 = h_{0,0}b_4 + h_{1,1}b_6 + h_{2,2}b_0 + h_{3,3}b_2. \tag{4.11}$$

Proof. We note that $2(1+i)\mathcal{O}_K = 4\mathcal{O}_K \cup (2(1+i) + 4\mathcal{O}_K)$ (disjoint union) as abelian groups. Let $s = \frac{\mu}{2} + i\frac{\lambda}{2} \in \mathcal{S}_2$. We have

$$\begin{aligned} U_{1+i}\theta^H_{2,s}(\tau, z_1, z_2) &= \sum_{r \equiv s \pmod{2\mathcal{O}_K}} e\left(\frac{N(r)}{2}\tau + (1+i)rz_1 + (1-i)\bar{r}z_2\right) \\ &= \sum_{r' \equiv (1+i)s \pmod{2(1+i)\mathcal{O}_K}} e\left(\frac{N(r')}{4}\tau + r'z_1 + \bar{r}'z_2\right) \\ &= \sum_{r' \equiv (\mu-\lambda)/2 + i((\mu+\lambda)/2) \pmod{4\mathcal{O}_K}} e\left(\frac{N(r')}{4}\tau + r'z_1 + \bar{r}'z_2\right) \\ &\quad + \sum_{r' \equiv (\mu-\lambda+4)/2 + i((\mu+\lambda+4)/2) \pmod{4\mathcal{O}_K}} e\left(\frac{N(r')}{4}\tau + r'z_1 + \bar{r}'z_2\right), \end{aligned}$$

from which the Lemma follows easily. □

Remark 4.1. From the transformation $h_s|_{k-1}\varepsilon I = \varepsilon h_{\varepsilon s}$ $(\varepsilon \in \mathcal{O}_K^\times)$, we conclude that

$$h_{a,b} = i^k h_{-b,a}, \quad h_{a,b} = (-1)^k h_{-a,-b}. \left(\frac{a+ib}{2} \in \mathcal{O}_K^\#/m\mathcal{O}_K.\right) \tag{4.12}$$

From the direct-sum decomposition (4.2) or from above equation (4.12) we see that $J_{k,2}(\mathcal{O}_K) = J^{\eta_\alpha}_{k,2}(\mathcal{O}_K)$ for $k + \alpha \equiv 0 \pmod 4$.

4.1. $\eta = \eta_1$. In this case $k \equiv 3 \pmod 4$, and it is easy to see from equation (4.12) that $h_{0,0} = h_{2,2} = h_{0,2} = h_{2,0} = 0$, and after a calculation,

$$h_{0,3} = -h_{0,1}, \quad h_{1,0} = -ih_{0,1}, \quad h_{1,3} = -ih_{1,1}, \quad h_{2,1} = ih_{1,2}, \quad h_{2,3} = -ih_{1,2}, \tag{4.13}$$

$$h_{3,0} = ih_{0,1}, \quad h_{3,1} = ih_{1,1}, \quad h_{3,2} = -h_{1,2}, \quad h_{3,3} = -h_{1,1}. \tag{4.14}$$

Using Lemma 4.2 we have from equation (4.13) and (4.14) that

$$\pi_{1+i}\phi(\tau,z) = \sum_{\mu \pmod 8} \overline{H}_\mu \theta_{4,\mu}(\tau,z) \quad \text{where } \overline{H}_0 = \overline{H}_4 = 0, \tag{4.15}$$

$$\overline{H}_1 = -(1+i)h_{0,1}b_1 - (1-i)h_{1,2}b_3, \quad \overline{H}_2 = ih_{1,1}(b_4 - b_0), \tag{4.16}$$

$$\overline{H}_3 = (1+i)h_{0,1}b_3 + (1-i)h_{1,2}b_1; \tag{4.17}$$

from which we conclude that π_{1+i} is injective. But for $k > 4$, from [6, Satz 2.5] we get $\dim J_{k,2}^{Eis}(\mathcal{O}_K) = 0$. Also for $k > 4$, using the Trace formula (see [7, Theorem 3], or [6, Korollar 2.5, p. 92]) we get $\dim J_{k,2}(\mathcal{O}_K) = \frac{k-3}{4} = \dim J_{k,2}$, where the last equality follows from [5, Cor. Theorem 9.2]). When $k = 3$, $J_{3,4} = 0$, hence so is $J_{3,2}(\mathcal{O}_K)$. Therefore,

Proposition 4.3. *Let* $k \equiv 3 \pmod 4$. *Then* π_{1+i} *induces an isomorphism of* $J_{k,2}(\mathcal{O}_K)$ *to* $J_{k,4}$.

4.2. $\eta = \eta_2$**.** In this case $k \equiv 2 \pmod 4$ we have $h_{0,0} = h_{2,2} = 0$ and using equation (4.12),

$$h_{0,3} = h_{0,1}, \quad h_{1,0} = -h_{0,1}, \quad h_{1,3} = -h_{1,1}, \quad h_{2,1} = -h_{1,2}, \quad h_{2,3} = -h_{1,2}, \tag{4.18}$$

$$h_{3,0} = -h_{0,1}, \quad h_{3,1} = -h_{1,1}, \quad h_{3,2} = h_{1,2}, \quad h_{3,3} = h_{1,1}. \tag{4.19}$$

The transformation formulae of $h_{0,1}$, $h_{0,2}$, $h_{1,1}$, $h_{1,2}$ under S, T are given in [4, p. 11].

Lemma 4.4. *Let* $k \equiv 2 \pmod 4$. *Then in the Theta decomposition of* $\phi \in J_{k,2}(\mathcal{O}_K)$, $h_{1,1} \in M_{k-1}(SL(2,\mathbf{Z}),\varpi)$, *where* ω *is the linear character of* $SL(2,\mathbf{Z})$ *defined by* $\omega(T) = \omega(S) = i$.

Proof. From [4, p. 11] we get $h_{1,1} \in M_{k-1}(SL(2,\mathbf{Z}),\varpi)$; since $h_{1,1}$ is already a modular form for $\Gamma(8)$, it is holomorphic at infinity. □

Lemma 4.5. $k \equiv 2 \pmod 4$. *Then* $J_{k,2}^{Spez}(\mathcal{O}_K) = \{0\}$.

Proof. This follows easily from the fact that [6, Proposition 5.6] the Eichler-Zagier homomorphism $\iota : J_{k,2}^{Spez}(\mathcal{O}_K) \to M_{k-1}(\Gamma_0(8), (\frac{-4}{\cdot}))$ (see [6, Proposition 5.6] for details) defined by $\iota(\phi)(\tau) = \sum_{s \in \mathcal{O}_K^\#/2\mathcal{O}_K} h_s(8\tau)$, is injective. In this case we clearly have $\iota(\phi)(\tau) = 0$ from equations (4.18) and (4.19). □

The following Lemma will be used in the proof of the next Theorem.

Lemma 4.6. *Let* $p, q \in \mathbf{Z}/4\mathbf{Z}$ *so that* $\frac{p}{2} + i\frac{q}{2} \in \mathscr{S}_2$. *Then*

$$\frac{\partial^6}{\partial z_1^6}(\theta^H_{p,q}(\tau, z_1, z_2) - \theta^H_{q,p}(\tau, z_1, z_2))_{z_1=z_2=0}$$

$$= 2(16\pi i)^3((a_p'''a_q - a_q'''a_p) + 15(a_q''a_p' - a_p''a_q'))$$

Proof. We omit the computation, which can be found in [4]. □

Theorem 4.7. *Let* $k \equiv 2 \pmod 4$. *We have the following exact sequence of vector spaces*

$$0 \to S_{k+2} \times S_{k+6} \xrightarrow{\sigma} J_{k,2}(\mathcal{O}_K) \xrightarrow{\pi_1} J_{k,2} \xrightarrow{D_0} M_k \to 0 \tag{4.20}$$

—The map σ is defined as follows. We will prove that,

$$\ker \pi_1 \overset{I_1}{\cong} M_{k-1}(SL(2,\mathbf{Z}), \bar{\varpi}) \times S_{k+6};$$

$$\phi \mapsto (h_{1,1}, D_0(6)(\phi - h_{1,1}(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3}))) \tag{4.21}$$

where $D_0(6)\phi = \chi_{6,0}$, the coefficient of z_1^6 in the Taylor expansion of ϕ around $z_1 = z_2 = 0$ (see [3] for the definition of Differential operators D_ν, $\nu \in \mathbf{Z}_{\geq 0}$).

We define the isomorphism I_2 of S_{k+2} to $M_{k-1}(SL(2,\mathbf{Z}), \bar{\varpi})$ given by $I_2 f = f/\xi$, ξ as in the proof of Corollary 3.3, see [1].

Then σ is merely the composition of the following:

$$S_{k+2} \times S_{k+6} \xrightarrow{I_2 \times id.} M_{k-1}(SL(2,\mathbf{Z}), \bar{\varpi}) \times S_{k+6} \xrightarrow{I_1^{-1}} \ker \pi_1 \hookrightarrow J_{k,2}(\mathcal{O}_K). \tag{4.22}$$

Proof. We divide the proof into 3 steps.

Step 1. $\ker \pi_1 = \mathrm{Im}(\sigma)$. From the description of the map σ in equation (4.22), it is enough to prove the assertion about $\ker \pi_1$ in equation (4.21). Let ϕ be in $\ker \pi_1$. From Lemma 4.1 (the notation are as stated at the beginning of this section), and using equations (4.18), (4.19) we get

$$\frac{h_{0,1}}{a_2} = \frac{-h_{0,2}}{2a_1} = \frac{h_{1,2}}{a_0} := \psi. \tag{4.23}$$

ψ is well defined since it is well known that a_μ ($\mu \in \mathbf{Z}/4\mathbf{Z}$) never vanish on $\mathscr{H}$. Therefore

$$\phi = \psi(a_2(\theta^H_{0,1} + \theta^H_{0,3} - \theta^H_{1,0} - \theta^H_{3,0}) - 2a_1(\theta^H_{0,2} - \theta^H_{2,0})$$

$$+ a_0(\theta^H_{1,2} - \theta^H_{2,1} - \theta^H_{2,3} + \theta^H_{3,2})) + h_{1,1}(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3}). \tag{4.24}$$

From [4, p. 11] we get the following transformation formulas for ψ:

$$\psi\left(-\frac{1}{\tau}\right) = \frac{1-i}{\sqrt{2}}\tau^{k-3/2}\psi(\tau), \quad \psi(\tau+1) = -\frac{1-i}{\sqrt{2}}\psi(\tau). \tag{4.25}$$

Further, from [4, p. 11] (since the transformations of $(h_{0,1}, h_{0,2}, h_{1,2})$ and $h_{1,1}$ under S, T are independent of each other) we conclude that $h_{1,1}(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3}) \in J_{k,2}(\mathcal{O}_K)$ and hence so is $\phi - h_{1,1}(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3})$.

We define $\ker \pi_1^\circ := \{\phi \in \ker \pi_1 \mid h_{1,1} = 0\}$. By the same reasoning as in the above paragraph,

$$\ker \pi_1 \cong M_{k-1}(SL(2,\mathbf{Z}), \bar{\varpi}) \times \ker \pi_1^\circ \quad \text{via}$$

$$\phi \mapsto (h_{1,1}, \phi - h_{1,1}(\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3})),$$

using Lemma 4.4 and that $\theta^H_{1,1} - \theta^H_{1,3} - \theta^H_{3,1} + \theta^H_{3,3} \not\equiv 0$.

We now prove that $D_0(6) : \ker \pi_1^\circ \to S_{k+6}$ is an isomorphism.

Let $\phi \in \ker \pi_1^\circ$. From equation (4.24) and Lemma 4.6 we get

$$\begin{aligned} D_0(6)\phi &= 2c\psi a_2((a_0'''a_1 - a_1'''a_0) + 15(a_1''a_0' - a_0''a_1')) - 2c\psi a_1((a_0'''a_2 - a_2'''a_0) \\ &\quad + 15(a_2''a_0' - a_0''a_2')) + 2c\psi a_0((a_1'''a_2 - a_2'''a_1) + 15(a_2''a_1' - a_1''a_2')) \\ &= 15c\psi \cdot Wr_2(\tau) = 15c'\psi\eta^{15}(\tau) \end{aligned}$$

where $c = 2(16\pi i)^3$, $c' = c\left(\frac{\pi i}{4}\right)^3 2!4!$ and $Wr_m(\tau) = 2^{m-1}\det(\theta^{(\nu)}_{m,\mu}(\tau,0)_{0\le\nu,\mu\le m})$ is the Jacobi-Theta Wronskian of order m. The equality $Wr_2(\tau) = \left(\frac{\pi i}{4}\right)^3 2!4!\eta^{15}(\tau)$ ($\eta(\tau)$ being the Dedekind's η-function) follows from [8].

$D_0(6)|_{\ker \pi_1^\circ}$ is clearly injective. It's surjectivity is equally clear from the transformation properties of η. From the definition of the map σ, this finishes *Step* 1.

Step 2. The case $k = 2$. We claim $J_{2,2}(\mathcal{O}_K) = 0$. Indeed, using *Step* 1 for the description of $\ker \pi_1$ we find $\dim J_{2,2}(\mathcal{O}_K) = \dim \ker \pi_1 + \dim \mathrm{Im}(\pi_1) = \dim S_4 + \dim S_8 = 0$, since $J_{2,2} = 0$. The Theorem is trivially true in this case.

Step 3. $\ker D_0 = \mathrm{Im}(\pi_1)$ for $k > 4$. We have $D_0(\mathcal{O}_K)\phi = \chi_{0,0} = 0$ (see Remark 3.2). So $\mathrm{Im}(\pi_1) \subseteq \ker D_0$. But a direct check (considering $k \equiv 2, 6, 10 \pmod{12}$) shows

$$\begin{aligned} \dim \mathrm{Im}(\pi_1) &= \dim J_{k,2}(\mathcal{O}_K) - \dim \ker \pi_1 = \left[\frac{k-1}{3}\right] - \dim S_{k+2} - \dim S_{k+6} \\ &= \dim J_{k,2} - \dim M_k = \dim \ker D_0 \end{aligned}$$

since D_0 is surjective. This completes the proof of the Theorem. □

4.3. $\eta = \eta_3$. In this case $k \equiv 1 \pmod 4$. It is easy to see using equation (4.12) that $h_{0,0} = h_{2,2} = h_{0,2} = h_{2,0} = 0$, and after a calculation,

$$h_{0,3} = -h_{0,1}, \quad h_{1,0} = ih_{0,1}, \quad h_{1,3} = ih_{1,1}, \quad h_{2,1} = -ih_{1,2}, \quad h_{2,3} = ih_{1,2}, \tag{4.26}$$

$$h_{3,0} = -ih_{0,1}, \quad h_{3,1} = -ih_{1,1}, \quad h_{3,2} = -h_{1,2}, \quad h_{3,3} = -h_{1,1} \tag{4.27}$$

Exactly the same argument as in the case $\eta = \eta_1$ applies here. For $k > 4$, we use that $\dim J_{k,2}(\mathcal{O}_K) = \frac{k-5}{4}$ $(= \dim J_{k,4})$, whereas $J_{1,2}(\mathcal{O}_K) \overset{\pi_{1+i}}{\hookrightarrow} J_{1,4} = 0$. Hence,

Proposition 4.8. *Let* $k \equiv 1 \pmod 4$. *Then* π_{1+i} *induces an isomorphism of* $J_{k,2}(\mathcal{O}_K)$ *to* $J_{k,4}$.

4.4. $\eta = \eta_0$. In this case $k \equiv 0 \pmod 4$ and,

$$h_{0,1} = h_{0,3} = h_{1,0} = h_{3,0}, \quad h_{0,2} = h_{2,0}, \tag{4.28}$$

$$h_{1,2} = h_{2,1} = h_{2,3} = h_{3,2}, \quad h_{1,1} = h_{1,3} = h_{3,1} = h_{3,3}. \tag{4.29}$$

We prove two Lemmas which will be used in the proof of the next Theorem. First we define the map:

Definition 4.1 ([6]).

$$U_\rho : J_{k,m}(\mathcal{O}_K) \to J_{k,N(\rho)m}(\mathcal{O}_K); \quad \phi(\tau, z_1, z_2) \mapsto \phi(\tau, \rho z_1, \bar{\rho} z_2). \tag{4.30}$$

Lemma 4.9. *Let* $\phi \in J_{k,1}(\mathcal{O}_K)$ *be with the Theta decomposition* $\phi = h_0\theta^H_{1;0} + h_{1/2}\theta^H_{1;1/2} + h_{i/2}\theta^H_{1;1/2} + h_{(1+i)/2}\theta^H_{1;1/2}$. *Then the Theta decomposition of* $U_{1+i}\phi$ *is given by*:

$$\begin{aligned} U_{1+i}\phi = h_0\theta^H_{0,0} + h_0\theta^H_{2,2} + h_{1/2}\theta^H_{1,1} + h_{1/2}\theta^H_{3,3} + h_{i/2}\theta^H_{3,1} \\ + h_{i/2}\theta^H_{1,3} + h_{(1+i)/2}\theta^H_{0,2} + h_{(1+i)/2}\theta^H_{2,0}. \end{aligned} \tag{4.31}$$

Proof. First we note $(1+i)\mathcal{O}_K = 2\mathcal{O}_K \cup ((1+i) + 2\mathcal{O}_K)$ (disjoint union) as abelian groups. Let $s = \frac{x}{2} + i\frac{y}{2} \in \mathcal{S}_2$. We have

$$U_{1+i}\theta^H_{1,s}(\tau, z_1, z_2) = \sum_{r \equiv s \pmod{\mathcal{O}_K}} e(N(r)\tau + (1+i)rz_1 + (1-i)\bar{r}z_2) \tag{4.32}$$

$$= \sum_{r' \equiv (x-y)/2 + i((x+y)/2) \pmod{(1+i)\mathcal{O}_K}} e\left(\frac{N(r')}{2}\tau + r'z_1 + \bar{r}'z_2\right). \tag{4.33}$$

Using the above formula and that $(1+i)\mathcal{O}_K = 2\mathcal{O}_K \cup ((1+i)+2\mathcal{O}_K)$, we see that

$$U_{1+i}\theta^H_{1;0} = \theta^H_{0,0} + \theta^H_{2,2}; \quad U_{1+i}\theta^H_{1;1/2} = \theta^H_{1,1} + \theta^H_{3,3}; \tag{4.34}$$

$$U_{1+i}\theta^H_{1;i/2} = \theta^H_{3,1} + \theta^H_{1,3}; \quad U_{1+i}\theta^H_{1;(1+i)/2} = \theta^H_{0,2} + \theta^H_{2,0}. \tag{4.35}$$

The Lemma now follows at once. □

Lemma 4.10. *Let* $\phi_{4,1}$ *be the basis element of* $J_{4,1}(\mathcal{O}_K)$ *given in* [11]. *Then,* $\{U_{1+i}\phi_{4,1}, \phi_{4,1}|V_2\}$ *is a basis of* $J_{4,2}(\mathcal{O}_K)$.

Proof. The Taylor expansion of $\phi_{4,1}$ around $z_1 = z_2 = 0$ can be computed to be

$$\phi_{4,1}(\tau, z_1, z_2) = 2E_4 + \pi i E_4' z_1 z_2 + \cdots;$$

from which the proof follows easily by writing down the corresponding Taylor expansions of $U_{1+i}\phi_{4,1}$ and $\phi_{4,1}|V_2$. □

Theorem 4.11. *Let* $k \equiv 0 \pmod 4$. *We have the following exact sequence of vector spaces*

$$0 \longrightarrow J_{k,2}(\mathcal{O}_K) \xrightarrow{\pi_1 \times \pi_{1+i}} J_{k,2} \times J_{k,4} \xrightarrow{\Lambda(2)-\Lambda(4)} M_k \times S_{k+2} \longrightarrow 0 \tag{4.36}$$

where $\Lambda(m) := D_0 + D_2 : J_{k,m} \to M_k \times S_{k+2}$.

Proof. We divide the proof into two steps.

Step 1. *Injectivity of* $\pi_1 \times \pi_{1+i}$. Let $\phi \in \ker \pi_1 \times \pi_{1+i}$. From Lemma 4.1 and 4.2 (the notation is as stated at the beginning of this section), we get $(b_1^2 - b_3^2)h_{0,1} = (b_1^2 - b_3^2)h_{1,2} = 0$.

Now, $\theta_{m,\mu}(\tau,0) \neq \theta_{m,\nu}(\tau,0)$ for $\mu \neq \nu$ $(0 \leq \mu, \nu \leq m)$, $\tau \in \mathcal{H}$. Otherwise the Wronskian Wr_m of $\theta_{m,\mu}$ $(0 \leq \mu \leq m)$, would be identically zero on $\mathcal{H}$ contradicting the fact that it is a non-zero multiple of Dedekind's η-function (see [8]).

Therefore $b_1^2(\tau) \neq b_3^2(\tau)$ $(\tau \in \mathcal{H})$ which implies $h_{0,1} = h_{1,2} = 0$. It follows that $h_{1,1} = h_{0,2} = 0$, and $(b_0^2 - b_4^2)h_{0,0} = (b_0^2 - b_4^2)h_{2,2} = 0$. By the above, we get $h_{0,0} = h_{2,2} = 0$. Hence $\phi = 0$.

Step 2. $\mathrm{Im}(\pi_1 \times \pi_{1+i}) \subseteq \ker(\Lambda(2) - \Lambda(4))$. We use the Taylor expansions of the Jacobi forms involved. Let $\phi(\tau, z_1, z_2) = \sum_{\alpha,\beta \geq 0} \chi_{\alpha,\beta}(\tau) z_1^\alpha z_2^\beta \in J_{k,2}(\mathcal{O}_K)$ be the

Taylor expansion of ϕ around $z_1 = z_2 = 0$. Then the Taylor development of $\pi_1\phi$ and $\pi_{1+i}\phi$ are

$$\pi_1\phi = \chi_{0,0} + \chi_{1,1}z^2 + (\chi_{0,4} + \chi_{2,2} + \chi_{4,0})z^4 + \cdots, \tag{4.37}$$

$$\pi_{1+i}\phi = \chi_{0,0} + 2\chi_{1,1}z^2 - 4(\chi_{0,4} - \chi_{2,2} + \chi_{4,0})z^4 + \cdots, \tag{4.38}$$

from which it easily follows that $\Lambda(2)\pi_1\phi = \Lambda(4)\pi_{1+i}\phi$.

$\mathrm{Im}(\pi_1 \times \pi_{1+i}) = \ker(\Lambda(2) - \Lambda(4))$. We show that they have the same dimension (for $k \geq 4$). $\dim \mathrm{Im}(\pi_1 \times \pi_{1+i}) = \dim J_{k,2}(\mathcal{O}_K) = \frac{k}{2}$ (see Lemma 4.10 for $k = 4$) whereas, $\dim \ker(\Lambda(2) - \Lambda(4)) = \dim J_{k,2} + \dim J_{k,4} - \dim M_k - \dim S_{k+2} = \frac{k}{2}$ (follows easily).

Finally, $\Lambda(2) - \Lambda(4)$ is clearly surjective since $\Lambda(2)$ is surjective. (Recall that from [5] we know $D_0 + D_2 + D_4 : J_{k,2} \to M_k \oplus S_{k+2} \oplus S_{k+4}$ is an isomorphism.) □

4.5. Order of vanishing at origin. For $\phi \in J_{k,m}(\mathcal{O}_K)$ let $\phi(\tau, z_1, z_2) = \sum_{\alpha,\beta \geq 0} \chi_{\alpha,\beta}(\tau) z_1^\alpha z_2^\beta$ be the Taylor expansion around $z_1 = z_2 = 0$. Define a non-negative integer $\varrho_{k,m}\phi$ by

$$\varrho_{k,m}\phi = \min\{\alpha + \beta \,|\, \chi_{\alpha,\beta}(\tau) \not\equiv 0\} \quad \text{if} \quad \phi \not\equiv 0, \tag{4.39}$$

$$= \infty \qquad\qquad \text{otherwise.} \tag{4.40}$$

i.e., $\varrho_{k,m}\phi$ can be interpeted as the order of vanishing of ϕ at the origin. From the relation with Jacobi forms, we can give upper bounds on $\varrho_{k,m}\phi$ for any $\phi \in J_{k,m}(\mathcal{O}_K)$ $(m = 1, 2)$.

PROPOSITION 4.12. (i) *Let* $\phi \in J_{k,1}(\mathcal{O}_K)$ *be non zero. Then*

$$0 \leq \varrho_{k,1}\phi \leq 2 \ \textit{if} \ k \equiv 2 \pmod 4; \quad 0 \leq \varrho_{k,1}\phi \leq 4 \ \textit{if} \ k \equiv 0 \pmod 4. \tag{4.41}$$

(ii) *Let* $\phi \in J_{k,2}(\mathcal{O}_K)$. *Then*

$$0 \leq \varrho_{k,2}\phi \leq 5 \ \textit{if} \ k \equiv 1, 3 \pmod 4; \quad 0 \leq \varrho_{k,2}\phi \leq 8 \ \textit{if} \ k \equiv 0, 2 \pmod 4. \tag{4.42}$$

PROOF. All of these assertions except the case $k \equiv 2 \pmod 4$, $m = 2$ follow easily from Propositions 4.8, 4.3 and Theorem 4.11 and the corresponding result for Jacobi forms (see [5, p. 37]). In the case $k \equiv 2 \pmod 4$, $m = 2$ we have $J_{k,2}(\mathcal{O}_K) \xhookrightarrow{\pi_1 \times \pi_{1+i}} J_{k,2} \times J_{k,4}$ (as in the case $k \equiv 0 \pmod 4$). This follows from Lemmas 4.1 and 4.2 and equations (4.18) and (4.19). For convenience, we give the proof. Let $\phi \in J_{k,2}(\mathcal{O}_K)$, with Theta components $h_{a,b}$ $(a, b \in \mathcal{S}_2)$. From $\pi_{1+i}\phi = 0$, we get $h_{1,1} = 0$ and from $\pi_1\phi = 0$ that (using $h_{0,0} = h_{2,2} = 0$)

$$\begin{pmatrix} 2a_1 & a_2 & 0 \\ a_0 & 0 & a_2 \\ 0 & a_0 & 2a_1 \end{pmatrix} \begin{pmatrix} h_{0,1} \\ h_{0,2} \\ h_{1,2} \end{pmatrix} = 0$$

Since $\det\begin{pmatrix} 2a_1 & a_2 & 0 \\ a_0 & 0 & a_2 \\ 0 & a_0 & 2a_1 \end{pmatrix} = -4a_0a_1a_2$, we get the injectivity and hence the Proposition. □

5. Rank of $J_{n*,m}(\mathcal{O}_K)$ over M_* and Algebraic Independence of $\phi_{4,1}$, $\phi_{8,1}$, $\phi_{12,1}$

We refer the reader to [11] for the definition of the index 1 forms $\phi_{4,1}$, $\phi_{8,1}$, $\phi_{12,1}$ which form a basis for $J_{4*,1}(\mathcal{O}_K) := \bigoplus_{k \geq 0} J_{4k,1}(\mathcal{O}_K)$ as a module over M_*.

Remark 5.1. For $m \geq 1$, it is natural to see that $J_{n*,m}(\mathcal{O}_K)$ $(n = 2, 4)$ can be regarded as modules over $M_*^{(4)} := \mathbf{C}[E_4, E_6^2]$ and hence over M_* via the algebra isomorphism

$$M_* = \mathbf{C}[E_4, E_6] \xrightarrow{E_4 \mapsto E_4,\, E_6 \mapsto E_6^2} \mathbf{C}[E_4, E_6^2] = M_*^{(4)}. \tag{5.1}$$

We note here that, $E_6 \cdot J_{n*,m}(\mathcal{O}_K) \notin J_{n*,m}(\mathcal{O}_K)$. From the argument in [5, p. 97], we easily see that $J_{*,*}(\mathcal{O}_K)$ is free over M_*, and $J_{n*,m}(\mathcal{O}_K)$ is of finite rank $r_n(m)$ over M_*.

Proposition 5.1. (i) $r_4(m) = m^2 + 2$, (ii) $r_2(m) = 2(m^2 + 1)$.

Proof. The proof is immediate from the dimension formula of $J_{k,m}(\mathcal{O}_K)$ in [7, Theorem 3]. We find that $\dim J_{k,m}(\mathcal{O}_K) = (m^2 + 2) \dim M_k + f(m) + O(1)$, (respectively, $= m^2 \dim M_k + g(m) + O(1)$) where $f(m)$, $g(m)$ are functions depending only on m when $k \equiv 0 \pmod 4$ (resp. $k \equiv 2 \pmod 4$). Letting $k \to \infty$, we get (i). Since there can be no linear relation between the generators of weights $0 \pmod 4$ and $2 \pmod 4$ by Remark 5.1, (ii) follows. □

Proposition 5.2. $\phi_{4,1}$, $\phi_{8,1}$, $\phi_{12,1}$ *are algebraically independent over* M_*.

Proof. It is enough to prove the algebraic independence of $\psi_{8,1}$, $\tilde{\psi}_{16,1}$, $\psi_{16,1}$, where $\psi_{8,1} = E_4\phi_{4,1} - \phi_{8,1}$, $\psi_{12,1} = E_4\phi_{8,1} - \phi_{12,1}$, $\tilde{\psi}_{16,1} = 5E_4\psi_{12,1} - 3\psi_{16,1}$; $\psi_{8,1}$, $\psi_{12,1}$, $\psi_{16,1}$ being the generators of $J^{cusp}_{4*,1}(\mathcal{O}_K)$ over M_* (see [11] for their definition).

Let $f(X, Y, Z) = \sum_{a+b+c=m} Q_{a,b,c} X^a Y^b Z^c$ be a homogeneous polynomial over M_* of least degree m such that $f(\psi_{8,1}, \tilde{\psi}_{16,1}, \psi_{16,1}) = 0$. Applying the map π_1 in the above relation we get

$$\sum_{b+c=m} Q_{0,b,c}(\pi_1 \tilde{\psi}_{16,1})^b (\pi_1 \psi_{16,1})^c = 0, \quad \text{since } \pi_1 \psi_{8,1} = 0.$$

From Lemma 5.3 $\pi_1 \tilde{\psi}_{16,1} \neq 0$, $D_0 \pi_1 \tilde{\psi}_{16,1} = 0$, $D_0 \pi_1 \psi_{16,1} \neq 0$. Hence, the argument in [5, p. 90] for classical Jacobi forms applies, showing $Q_{0,b,c} = 0$ for all b, c such that $b + c = m$. Hence we have

$$\sum_{a+b+c=m, a \geq 1} Q_{a,b,c} \psi_{8,1}^a \tilde{\psi}_{16,1}^b \psi_{16,1}^c = 0,$$

giving an equation of lower degree. Hence the Proposition is proved. □

Lemma 5.3. (i) $\psi_{16,1} = -2^8 \Delta \phi_{4,1} + 2E_4^2 \phi_{8,1} - E_4 \phi_{12,1}$.
(ii) $D_0 \psi_{16,1} = -5 \cdot 2^8 \Delta E_4$, $D_0 \psi_{12,1} = -3 \cdot 2^8 \Delta$.
(iii) $\pi_1 \tilde{\psi}_{16,1} = \left(\frac{2}{9} E_4^3 + 3 \cdot 2^9 \Delta\right) E_{4,1} + \frac{8}{9} E_4 E_6 E_{6,1}$; $E_{4,1}$, $E_{6,1}$ *being the normalised Jacobi Eisenstein series, which are a basis of* $J_{2*,1}$ *over* M_*. $\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ *is the Discriminant cusp form of weight* 12.

Proof. The calculations follow from the Theta decompositions of $\phi_{4i,1}$ $(i = 1, 2, 3)$ given in [11] and using the Theta relations (see [9]): Let $\theta_s^H(\tau) := \theta_{1;a,b}^H(\tau, 0, 0)$, where $\frac{a+ib}{2} = s \in \mathscr{S}_1$. Then we have

$$\theta_{0,0}^H(\tau) = \frac{1}{2}(x^2 + y^2), \quad \theta_{0,1}^H(\tau) = \theta_{1,0}^H(\tau) = \frac{1}{2} z^2, \quad \theta_{1,1}^H(\tau) = \frac{1}{2}(x^2 - y^2);$$

where

$$x = \sum_{n \in \mathbf{Z}} e\left(\frac{n^2 \tau}{2}\right), \quad y = \sum_{n \in \mathbf{Z}} (-1)^n e\left(\frac{n^2 \tau}{2}\right), \quad z = \sum_{t \in 1/2 + \mathbf{Z}} e\left(\frac{t^2 \tau}{2}\right)$$

are the "Theta constants". We omit the calculations. □

6. Further Questions and Remarks

(1) The restriction maps that we use in this paper do not commute with Hecke operators. Nevertheless it is expected that the following should be true. There should exist finitely many algebraic integers $\rho_j \in \mathcal{O}_K$, $1 \leq j \leq n$

(where n depends only on the index m) such that we have an embedding/isomorphism

$$\pi_{\rho_1} \times \cdots \times \pi_{\rho_n} : J_{k,m}(\mathcal{O}_K) \hookrightarrow J_{k,mN(\rho_1)} \times \cdots \times J_{k,mN(\rho_n)},$$

where $N : K \to \mathbf{Q}$ is the norm map. From the results of this paper (see Theorem 4.11 and the proof of Proposition 4.12) this is true for $m = 1, 2$ and these cases suggest that ρ_j and n above should be related to the decomposition of m in $\mathcal{O}_K$.

(2) We know that for $k \equiv 0 \pmod 4$, $\dim J_{k,2}(\mathcal{O}_K) = \frac{k}{2} = 2(\dim M_{k-4} + \dim M_{k-8} + \dim M_{k-12})$. This suggests what the minimal weights of the 6 generators of $J_{4*,2}(\mathcal{O}_K)$ over M_* should be, but the calculations seem to be much more than that in the case of classical Jacobi forms.

(3) It would be interesting to write down $m^2 + 2$ forms in $J_{4*,m}(\mathcal{O}_K)$ which are linearly independent over M_*. We have $\binom{m+2}{2}$ such forms from Proposition 5.2. Since $m^2 + 2 - \binom{m+2}{2} = \binom{m-1}{2}$, we ask the following question. Let $A := \phi_{4,1}$, $B := \phi_{8,1}$, $C := \phi_{12,1}$. Does there exist a form ϕ_3 in $J_{k,3}(\mathcal{O}_K)$ (for suitable k) such that the set

$$\{A^a B^b C^c\}_{a+b+c=m} \cup \{\phi_3 A^a B^b C^c\}_{a+b+c=m-3}$$

consists of $m^2 + 2$ linearly independent forms (over M_*) in $J_{4*,m}(\mathcal{O}_K)$?

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