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ON SOME MATRIX DIOPHANTINE EQUATIONS[1]

By

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Abstract. Let $A \in M_n(\boldsymbol{C})$, $n \geq 2$ be the matrix which has at least one real eigenvalue $\alpha \in (0, 1)$. If the matrix equation

$$A^x + A^y + A^z = A^w \tag{1}$$

is satisfied in positive integers x, y, z, w, then $max\{x - w, y - w, z - w\} \geq 1$. If suppose that the matrix A has at least one real eigenvalue $\alpha > \sqrt{2}$ and the equation (1) is satisfied in positive integers x, y, z and w, then $max\{x - w, y - w, z - w\} = -1$. Moveover, we investigate the solvability of the matrix equations (1) and

$$A^x + A^y = A^z \tag{2}$$

for the non-negative real $n \times n$ matrices, where $|\det A| > 1$, in positive integers x, y, z, w for (1) and x, y, z for (2). Using the well-known theorem of Perron-Frobenius we obtain some informations concerning solvability these equations.

1. Introduction

We give necessary conditions for solvability of the equation (1) in some positive integers x, y, z, w under some restrictions for $A \in M_n(\boldsymbol{C})$, $n \geq 2$ concerning eigenvalues of the matrix A.

A. Grytczuk proved in [7] that if the matrix $A \in M_n(\boldsymbol{C})$, $n \geq 2$ has at least one real eigenvalue $\alpha > \sqrt{2}$ and the equation (2) is satisfied in positive integers x, y, z, then $max\{x - z, y - z\} = -1$. In [6] A. Grytczuk and J. Grytczuk found necessary and sufficient conditions for $A \in M_n(\boldsymbol{Z})$, $n \geq 2$ to satisfy the equation

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(2) for some positive integers x, y, z. Earlier M. Le and C. Li [12] remarked that if A is an integral 2×2 matrix, then the equation (2) has a solution for $x = mr$, $y = ms$, $z = mt$, where $m > 2$, r, s, t are positive integers, if and only if A is a nilpotent matrix or $\det A = Tr\, A = 1$. Another proof of this result is given by A. Grytczuk in [8].

We note that for $X = A^r$, $Y = A^s$, $Z = A^t$ we obtain from (2) the Fermat's equation

$$X^m + Y^m = Z^m. \tag{3}$$

In 1966 R. Z. Domiaty [4] discovered that the equation (3) has infinitely many solutions in $M_2(\boldsymbol{Z})$ for $m = 4$. Some results relating to the equation of Fermat in the set of matrices have been described by P. Ribenboim in monograph [15]. In 1995 A. Wiles [19], R. Taylor and A. Wiles [17] proved that (3) has no solutions in nonzero integers X, Y, Z if $m > 2$. An important problem is to give a necessary and sufficient condition for solvability the equation (3) in the set of matrices. The solvability of (3) in $GL_2(\boldsymbol{Z})$ was first investigated by L. N. Vaserestein [18]. A. Khazanov in [11] gave necessary and sufficient conditions for solvability (3) for X, Y, Z belonging to $SL_2(\boldsymbol{Z})$, $SL_3(\boldsymbol{Z})$, $GL_3(\boldsymbol{Z})$. A. Gryczuk [9] proved some necessary condition to satisfy (3) in integral 2×2 matrices X, Y, Z, and in [5] he gave an extension of this result. Studies connected with Khazanov's results effected H. Qin [14]. In [2] Z. Cao and A. Grytczuk investigated the Fermat's equation (3) for $X, Y, Z \in G(k, \pm 1)$, where

$$G(k, \pm 1) = \left\{ \begin{pmatrix} r & s \\ ks & r \end{pmatrix} : r, s \in \boldsymbol{Z}, \det\begin{pmatrix} r & s \\ ks & r \end{pmatrix} = \pm 1 \right\},$$

k is a fixed square-free positive integer and for $X, Y, Z \in G(k, a)$, where

$$G(k, a) = \left\{ \begin{pmatrix} r & s \\ ks & r \end{pmatrix} : r, s \in \boldsymbol{Z}, \det\begin{pmatrix} r & s \\ ks & r \end{pmatrix} = a \right\},$$

k is a fixed positive integer and a is a fixed integer. Moveover, they proved [2] that an equation $X^m + Y^m + Z^m = W^m$, where $X, Y, Z, W \in G(k, a)$, $k > 1$ is a fixed square-free integer, does not hold, except when $X + Y = 0$ or $Y + Z = 0$ or $Z + X = 0$ and $(m, 2) = 1$. In [3] Z. Cao and A. Grytczuk gave a necessary and sufficient condition for solvability (3) for $X, Y, Z \in \overline{SL_2(\boldsymbol{Z})}$. Z. Patay and A. Szakacs [13] studied the equation of Fermat (3) in $SL_3(\boldsymbol{Z})$ and in irreducible elements of the rings $M_2(\boldsymbol{Z})$ and $M_3(\boldsymbol{Z})$.

2. Basic Lemmas

Lemma 1 (Schur [1]). *Let A be an $n \times n$ complex matrix. Then there is a unitary matrix P such that*

$$P^*AP = \begin{pmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ & \lambda_2 & & \vdots \\ & & \ddots & \\ 0 & & \lambda_n & \end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A.

Lemma 2. *Let A be an $n \times n$ complex matrix with $|\det A| > 1$ and let $\alpha(A) = \max_{1 \le j \le n} |\alpha_j|$, where α_j are the eigenvalues of A for $j = 1, 2, \dots, n$. Then*

$$\alpha(A) > 1 + \frac{\log|\det A|}{n}.$$

The proof of this Lemma is given by A. Grytczuk and M. Szałkowski in [10].

3. Results

Theorem 1. *Let $A \in M_n(\boldsymbol{C})$, $n \ge 2$. Suppose that the matrix A has at least one real eigenvalue $\alpha \in (0, 1)$. If the equation*

$$A^x + A^y + A^z = A^w \tag{1}$$

has a solution in positive integers x, y, z and w, then max$\{x - w, y - w, z - w\} \ge 1$.

Proof. By Lemma 1 there exist a unitary matrix P such that

$$A = P^*TP, \tag{4}$$

where T is the upper triangular matrix with the eigenvalues of the matrix A on main diagonal.

Let $\alpha = \alpha_s$, where $1 \le s \le n$, be real eigenvalue of A such that $\alpha_s \in (0, 1)$.

From (4) by induction follows that $A^k = P^*T^kP$, where T^k is the upper triangular matrix having on main diagonal eigenvalues α_j^k, $j = 1, 2, \dots, n$. Hence for arbitrary integers x, y, z, w we have

$$A^x = P^* T^x P, \quad A^y = P^* T^y P, \quad A^z = P^* T^z P, \quad A^w = P^* T^w P \tag{5}$$

Suppose that the equation (1) has a solution in some positive integers.

From (5) and (1) we obtain

$$T^x + T^y + T^z = T^w.$$

Comparing the elements on the main diagonals we get

$$\alpha_j^x + \alpha_j^y + \alpha_j^z = \alpha_j^w, \quad j = 1, 2, \ldots, n. \tag{6}$$

From (6) we have

$$\alpha_j^{x-w} + \alpha_j^{y-w} + \alpha_j^{z-w} = 1. \tag{7}$$

Since $\alpha_s \in (0, 1)$ and we assume that (7) holds, then exponents $x - w$, $y - w$, $z - w$ must be positive. If $x - w < 1$, then we would have to consider two cases: $x - w = 0$ or $x - w < 0$. If $x - w = 0$, then by (7) follows that $\alpha_s^{y-w} + \alpha_s^{z-w} = 0$, which is impossible.

Let $x - w = -t$, where t is a positive integer. Then $\alpha_s^{x-w} \geq 1$ and the equation (7) does not hold.

Therefore $max\{x - w, y - w, z - w\} \geq 1$ and the proof of Theorem 1 is complete. □

THEOREM 2. *Let $A \in M_n(\boldsymbol{C})$, $n \geq 2$. Suppose that the matrix A has at least one real eigenvalue $\alpha > \sqrt{2}$. If the equation* (1) *is satisfied in positive integers x, y, z, w, then $max\{x - w, y - w, z - w\} = -1$.*

PROOF. Supposing that $\alpha = \alpha_s > \sqrt{2}$, where $1 \leq s \leq n$ is a real eigenvalue of the matrix A, and using Schur's Lemma, we obtain the equation (7). Since $\alpha_s > \sqrt{2} > 1$, then exponents $x - w$, $y - w$, $z - w$ must be negative. Moveover, we have $\alpha_s^{-2} < \frac{1}{2}$. Therefore two exponents can be less than or equal to -2, and one exponent must be equal -1. Thus $x - w = -1$ or $y - w = -1$ or $z - w = -1$, which complete the proof. □

THEOREM 3. *Let $A \in M_n(\boldsymbol{R})$ be a non-negative matrix and $|\det A| > 1$. If the equation* (2) *has a solution in positive integers x, y, z, then*

$$\left(1 + \frac{\log|\det A|}{n}\right)^{x-z} + \left(1 + \frac{\log|\det A|}{n}\right)^{y-z} < 1.$$

PROOF. Using Lemma 1, we obtain the following equation

$$\alpha_j^x + \alpha_j^y = \alpha_j^z \tag{8}$$

where α_j are the eigenvalues of the matrix A, $j = 1, 2, \dots, n$.

Let $\alpha(A) = \max_{1 \le j \le n} |\alpha_j|$. By the theorem of Perron-Frobenius $\alpha(A)$ is the characteristic root of A. From (8) we have

$$(\alpha(A))^{x-z} + (\alpha(A))^{y-z} = 1.$$

Hence, by Lemma 2 it follows that

$$(\alpha(A))^{x-z} + (\alpha(A))^{y-z} > \left(1 + \frac{\log|\det A|}{n}\right)^{x-z} + \left(1 + \frac{\log|\det A|}{n}\right)^{y-z}.$$

Therefore

$$\left(1 + \frac{\log|\det A|}{n}\right)^{x-z} + \left(1 + \frac{\log|\det A|}{n}\right)^{y-z} < 1.$$

The proof of the Theorem 3 is complete. □

In similar way we can prove the following Theorem:

THEOREM 4. *Let $A \in M_n(\boldsymbol{R})$ be a non-negative matrix and* $|\det A| > 1$. *If the equation* (1) *is satisfield for positive integers x, y, z, w, then*

$$\left(1 + \frac{\log|\det A|}{n}\right)^{x-w} + \left(1 + \frac{\log|\det A|}{n}\right)^{y-w} + \left(1 + \frac{\log|\det A|}{n}\right)^{z-w} < 1.$$

Immediately, from Theorem 3 we get the following Corollary:

COROLLARY 1. *Let $A \in M_n(\boldsymbol{R})$ be the non-negative matrix with* $|\det A| > 1$. *If the equation* (2) *has a solution in positive integers x, y, z, then* $max\{x, y\} < z$.

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