Afrika Statistika

Afrika Statistika
Vol. 13 (2), 2018, pages 1683–1697.
DOI: http://dx.doi.org/10.16929/1683.128

ISSN 2316-090X

Empirical Estimation Based On Progressive First Failure Censored Generalized Pareto Data

GYAN PRAKASH*

Department of Community Medicine,
M. L. N. Medical College, Allahabad, U. P., India.

Received on February 01, 2018; Accepted on April 04, 2018

Abstract. The Progressive First-Failure (PFF) censoring scheme is considered in the present article for the Empirical Bayes estimation. The approximate confidence intervals and the Bayes estimation for the unknown parameters under the empirical Bayesian technique are obtained for the Generalized Pareto distribution. The improved approximate confidence intervals are discussed also. A simulation technique is applied here for illustrating the methods based on different censoring plans, those are the special cases of PFF censoring scheme.

Résumé. Le schéma de censure du premier échec progressif (PFF) est considéré dans le présent article pour l'estimation empirique de Bayes. Les intervalles de confiance approximatifs et l'estimation de Bayes pour les paramètres inconnus sous la méthode bayésienne empirique sont obtenus pour la distribution de Pareto généralisée. Les intervalles de confiance approximatifs améliorés sont également discutés. Une technique de simulation est appliquée ici pour illustrer les méthodes basées sur différents plans de censure, qui sont les cas particuliers du schéma de censure PFF.

Key words: Generalized Pareto Distribution, Empirical Bayes Estimation, Progressive First-Failure (PFF) Censoring, Approximate Confidence Intervals (ACI).
AMS 2010 Mathematics Subject Classification : 62A15, 62F15, 65C05.

---

*GYAN PRAKASH (ggyanji@yahoo.com)

1. Introduction

The probability density function and cumulative density function of the considered generalized Pareto distribution are given as

$$f(x;\sigma,\theta)=\frac{1}{\sigma\theta}\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}-1}\ ;\ 0<x<\sigma,\theta>0, \tag{1}$$

and

$$F(x;\sigma,\theta)=1-\left(1-\frac{x}{\sigma}\right)^{\frac{1}{\theta}}\ ;\ 0<x<\sigma,\theta>0. \tag{2}$$

Here, the parameters θ and σ are the shape parameter and scale parameter respectively. See Castillo & Hadi (1977) for more details about the fundamental properties of the underlying model. A little few literature are available for generalized Pareto distribution. Grimshaw (1993) was obtained Maximum likelihood estimator and studying their properties. Rezaei *et al.* (2010) discussed about the estimation of $P[Y<X]$. Prakash (2012) obtained the central coverage Bayes prediction bound length of the said distribution. Both known and unknown cases of scale parameter have considered in the estimation of bound length. In (2014), Azimi *et al.* (2014) obtained some Bayes estimator of life parameters for the generalized Pareto distribution under Progressive censoring.

In the present article, PFF censoring scheme is considered for the inference. The ACI and an improved ACI are obtained by using a log-transformation. The log-transformation improved the performance of the ACI. The Bayes estimation is obtained under two different asymmetric loss functions for both parameters of the generalized Pareto distribution. The Empirical Bayesian approach have used for improving the performances of the Bayes estimators. The performances of the present procedures based on five different censoring techniques, those are the special cases of the PFF censoring are illustrated by a simulation technique.

2. First-Failure Progressive Censoring Scheme

In life testing, a censoring is very common because of time limitation and other restrictions on data collection. The censoring occurs when exact lifetimes are available for a part of units under study. The most common censoring test criterion is Type-II censoring and is beneficial for saving time and money.

The generalization of Type-II censoring is better known as the Progressive censoring scheme and is useful in such cases when the live test units removed, other than the final termination point. Last one decade Progressive censoring scheme has received considerable interest among the researchers. A little few of them Balakrishnan & Sandhu (1995), Balakrishnan & Aggarwala (2000), Kundu (2008), Lee *et al.* (2009), Raqab *et al.* (2010), Fu *et al.* (2012), Al-Zahrani & Al-Sobhi (2013), Prakash (2015) and Prakash (2016).

Following Johnson (1964), a life test in which experimenter might decide to group the test units into several sets, each as an assembly of test units, and then run all the test units simultaneously until occurrence the First-Failure in each group. Jun *et al.* (2006) was extended this plan for a bearing manufacturer and time of first-failure were observed from each group.

In an experiment required to remove some sets of test units, before observing the *first-failures* in the sets, the test plan is called PFF censoring scheme (Wu & Ku? (2009). Following Wu & Ku? (2009) the progressive first-failure censoring scheme is described as

Let us assume from $(n \times m)$ live test units, there are n independent groups with m items within each group are putting on a life test. When the *first-failures* X_1^R is occurred, the R_1 units and the group in which the *first-failures* is observed are randomly removed from the test. Similarly, when the second failure X_2^R is observed, the R_2 units and the group in second failure was observed, are removed from the test. The test will run until the $k^{th}; (\leq n)$ failure X_k^R is observed. Suppose, $X_1^R < X_2^R < ... < X_k^R$ are the progressively first-failure censored order statistics of size k with pre assumed progressive censoring scheme $R = (R_1, R_2, ..., R_k)$ follows the relation

$$\sum_{j=1}^{k} R_j = n - k.$$

Let us assume $(n \times m)$ items in the life test are from the generalized Pareto distribution given in Eq. (1). Then the joint probability density function under PFF censored order statistic is defined as

$$f(\theta, \sigma | \underline{x}) = C_p m^k \prod_{i=1}^{k} f\left(x_i^R; \sigma, \theta\right) \left(1 - F\left(x_i^R; \sigma, \theta\right)\right)^{m(R_i+1)-1}; \tag{3}$$

where C_p is a progressive normalizing constant. (See Prakash (2016) for more details).

The First-Failure-censoring scheme has advantages in term of reducing test time, in which more items are used but only k of $n \times m$ items are failures. Some special cases were included in the PFF censored scheme

1. The joint probability density function under PFF censored order statistic given in Eq. (3), converted to the joint probability density function under First-Failure censoring scheme when $R_1 = R_2 = ... = R_k = 0$.
2. For $m = 1$, the Eq. (3) is represented the joint probability density function under Progressive Type-II censored order statistic.
3. When $m = 1, R_1 = R_2 = ... = R_{k-1} = 0$ and $R_k = n - k$, the joint probability density function under PFF censored order statistic is simply convert into the

joint probability density function under the Type-II censored order statistic and, is for complete sample case when $m = 1$ and $R_1 = R_2 = ... = R_k = 0$.

Remark: All these cases are considered in present article for the numerical illustration.

Using Eq. (1) and Eq. (2) in Eq. (3), the joint probability density function under PFF censoring scheme is obtained as:

$$f(\theta, \sigma|\underline{x}) \propto \prod_{i=1}^{k} \left\{ \frac{1}{\sigma\theta} \left(1 - \frac{x_i}{\sigma}\right)^{\frac{1}{\theta}-1} \right\} \left\{ \left(1 - \frac{x_i}{\sigma}\right)^{\frac{1}{\theta}} \right\}^{m(R_i+1)-1}$$

$$\Rightarrow f(\theta, \sigma|\underline{x}) \propto \sigma^{-k}\, \theta^{-k}\, e^{-T_0(\underline{x},\sigma)}\, exp\left(\frac{T_1(\underline{x}, \sigma)}{\theta}\right); \quad (4)$$

where $T_0(\underline{x}, \sigma) = \sum_{i=1}^{k} log\left(\frac{\sigma - x_i}{\sigma}\right)$ and $T_1(\underline{x}, \sigma) = m \sum_{i=1}^{k} (R_i + 1)\, log\left(\frac{\sigma - x_i}{\sigma}\right)$.

3. Parameter Estimation

3.1. Maximum Likelihood Estimation

The logarithm of the joint probability distribution given in Eq. (4) is

$$L\,(say) = log\, f(\theta, \sigma|\underline{x}) = -k\, log\, \sigma - k\, log\, \theta - T_0(\underline{x}, \sigma) + \frac{T_1(\underline{x}, \sigma)}{\theta}. \quad (5)$$

Differentiating Eq. (5) with respect to the parameters θ and σ respectively, and equating it to zero. If ML estimation are denoted by $\hat{\theta}_{ML}$ and $\hat{\sigma}_{ML}$ of the parameters θ and σ respectively, then, one can be obtained as the solution of these equations

$$\hat{\theta}_{ML} = -\frac{T_1(\underline{x}, \sigma)}{k} \quad (6)$$

and

$$\sum_{i=1}^{k} \left(\frac{x_i}{\hat{\sigma}_{ML} - x_i}\right) \left\{ \frac{m}{\hat{\theta}_{ML}} (1 + R_i) - 1 \right\} - k = 0. \quad (7)$$

The Eq. (6) & Eq. (7) cannot be solved analytically; a numerical method (Newton-Raphson method) must be employed to solve these two equations for numerical finding of ML estimate $\hat{\sigma}_{ML}$ and $\hat{\theta}_{ML}$.

3.2. Approximate Confidence Interval

The asymptotic variances and co-variances of the ML estimates of the parameters are obtained by elements of the inverse of the Fisher information matrix. However, the exact mathematical expressions are very difficult to obtain. Hence, the observed asymptotic variance-covariance matrix for the ML Estimation is obtained as

$$\begin{bmatrix} -\frac{\partial^2}{\partial\theta^2}L & -\frac{\partial^2}{\partial\theta\partial\sigma}L \\ -\frac{\partial^2}{\partial\sigma\partial\theta}L & -\frac{\partial^2}{\partial\sigma^2}L \end{bmatrix}^{-1}_{\left(\hat{\theta}_{ML},\hat{\sigma}_{ML}\right)} = \begin{bmatrix} Var\left(\hat{\theta}_{ML}\right) & Cov\left(\hat{\theta}_{ML},\hat{\sigma}_{ML}\right) \\ Cov\left(\hat{\sigma}_{ML},\hat{\theta}_{ML}\right) & Var\left(\hat{\sigma}_{ML}\right) \end{bmatrix} \quad (8)$$

The second order derivatives of the log-likelihood equation are

$$\frac{\partial^2}{\partial\theta^2}L = \frac{k}{\theta^2} + 2\,\frac{T_1\left(\underline{x},\sigma\right)}{\theta^3},$$

$$\frac{\partial^2}{\partial\sigma^2}L = \sigma^{-2}\left(k + \sum_{i=1}^{k}\frac{x_i\left(2\sigma - x_i\right)}{\left(\sigma - x_i\right)^2}\left\{1 - \frac{m}{\theta}\left(1 + R_i\right)\right\}\right)$$

and

$$\frac{\partial^2}{\partial\theta\partial\sigma}L = \frac{\partial^2}{\partial\sigma\partial\theta}L = -\sigma^{-1}\sum_{i=1}^{k}\left(\frac{x_i}{\sigma - x_i}\right)\left\{\frac{m}{\theta^2}\left(1 + R_i\right)\right\}.$$

All the expressions of second derivative involve the unknown parameters. Hence, the Fisher information matrix can be obtained by replacing its ML estimators. The asymptotic normality of the ML estimation is used for the computation of ACI for the unknown parameters θ and σ. Hence, $(1-\epsilon)\,100\%$ confidence intervals for the parameters θ and σ are given respectively as

$$\hat{\theta}_{ML} \mp Z_{\epsilon/2}\sqrt{var\left(\hat{\theta}_{ML}\right)} \quad (9)$$

and

$$\hat{\sigma}_{ML} \mp Z_{\epsilon/2}\sqrt{var\left(\hat{\sigma}_{ML}\right)}. \quad (10)$$

Here, $Z_{\epsilon/2}$ is the percentile of the standard normal distribution with right-tail probability $\epsilon/2$. The applicability of normal approximation of ML estimation is in small sample size. Following Meeker & Escobar (1998), a log-transformation can be considered for improvements in the performance of the normal approximation. Thus, $(1-\epsilon)\,100\%$ improved approximate confidence intervals for the parameters θ and σ are obtained as

$$\left\{\hat{\theta}_{ML}\,exp\left(-\frac{Z_{\epsilon/2}\sqrt{Var\left(\hat{\theta}_{ML}\right)}}{\hat{\theta}_{ML}}\right), \hat{\theta}_{ML}\,exp\left(\frac{Z_{\epsilon/2}\sqrt{Var\left(\hat{\theta}_{ML}\right)}}{\hat{\theta}_{ML}}\right)\right\} \quad (11)$$

and

$$\left\{\hat{\sigma}_{ML}\,exp\left(-\frac{Z_{\epsilon/2}\sqrt{Var\left(\hat{\sigma}_{ML}\right)}}{\hat{\sigma}_{ML}}\right), \hat{\sigma}_{ML}\,exp\left(\frac{Z_{\epsilon/2}\sqrt{Var\left(\hat{\sigma}_{ML}\right)}}{\hat{\sigma}_{ML}}\right)\right\}. \quad (12)$$

3.3. Bayes Estimation When Both Parameter Is Unknown

In the present section, both parameters of the underlying model given in Eq. (1) are considered as a random variable. Prakash (2012) considered the following joint prior distribution for the parameters θ and σ as

$$\pi_{(\theta,\sigma)} = \left(\frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{-\alpha-1}\, exp\left(-\frac{\beta}{\theta}\right)\right)\left(\frac{1}{\phi}\right)\;;\;\theta>0,\alpha>0,\beta>0,\phi>0,0\le\sigma\le\phi. \tag{13}$$

Using Eq. (4) and Eq. (13), the joint posterior density and marginal posterior density corresponding the parameters of the generalized Pareto distribution are obtained as

$$\pi^*_{(\theta,\sigma)} = \xi\theta^{-k-\alpha-1}\, exp\left(-\frac{\beta-T_1\left(\underline{x},\sigma\right)}{\theta}\right)\sigma^{-k}\, e^{-T_0(\underline{x},\sigma)} \tag{14}$$

$$\pi^*_{\theta} = \xi\theta^{-k-\alpha-1}\, exp\left(-\frac{\beta}{\theta}\right)\int_{\sigma} exp\left(-T_0\left(\underline{x},\sigma\right)+\frac{T_1\left(\underline{x},\sigma\right)}{\theta}\right)\sigma^{-k}\, d\sigma \tag{15}$$

and

$$\pi^*_{\sigma} = \xi\,\Gamma\left(k+\alpha\right)\frac{\sigma^{-k}\, e^{-T_0(\underline{x},\sigma)}}{\left(\beta-T_1\left(\underline{x},\sigma\right)\right)^{k+\alpha}}; \tag{16}$$

where $\xi = \left\{\Gamma(k+\alpha)\int_{\sigma}\sigma^{-k}e^{-T_0(\underline{x},\sigma)}\left(\beta-T_1\left(\underline{x},\sigma\right)\right)^{-k-\alpha}\, d\sigma\right\}^{-1}$

The Bayes estimators of the unknown parameters are obtained in this section under asymmetric loss function. Several authors have recognized that, the use of squared error loss function in Bayesian analysis is inappropriate in case when the overestimation is more serious than the underestimation and vice versa. An asymmetric loss function which is the result of a minor modification in squared error loss, named as invariant squared error loss function (ISELF) and is defined as

$$L\left(\hat{\theta},\theta\right) = \left(\theta^{-1}\partial\right)^2\;;\;\partial=\hat{\theta}-\theta. \tag{17}$$

The Bayes estimator $\hat{\theta}_I$ corresponding to the parameter θ under ISELF is obtained as

$$\hat{\theta}_I = \frac{\int_{\theta}\theta^{-1}\cdot\,\pi^*_{\theta}\, d\theta}{\int_{\theta}\theta^{-2}\cdot\,\pi^*_{\theta}\, d\theta}$$

$$= \frac{\int_{\theta}\theta^{-k-\alpha-2}exp\left(-\frac{\beta}{\theta}\right)\int_{\sigma} exp\left(-T_0\left(\underline{x},\sigma\right)+\frac{T_1(\underline{x},\sigma)}{\theta}\right)\sigma^{-k}\, d\sigma\, d\theta}{\int_{\theta}\theta^{-k-\alpha-3}exp\left(-\frac{\beta}{\theta}\right)\int_{\sigma} exp\left(-T_0\left(\underline{x},\sigma\right)+\frac{T_1(\underline{x},\sigma)}{\theta}\right)\sigma^{-k}\, d\sigma\, d\theta}$$

$$\Rightarrow \hat{\theta}_I = \frac{\int_\sigma exp\left(-T_0\left(\underline{x},\sigma\right)\right)\left(\beta - T_1\left(\underline{x},\sigma\right)\right)^{-k-\alpha-1}\sigma^{-k}\,d\sigma}{\Gamma(k+\alpha+1)\int_\sigma exp\left(-T_0\left(\underline{x},\sigma\right)\right)\left(\beta - T_1\left(\underline{x},\sigma\right)\right)^{-k-\alpha-2}\sigma^{-k}\,d\sigma}. \tag{18}$$

Similarly, the Bayes estimator $\hat{\sigma}_I$ for parameter σ under ISELF is obtained as

$$\hat{\sigma}_I = \frac{\int_\sigma exp\left(-T_0\left(\underline{x},\sigma\right)\right)\left(\beta - T_1\left(\underline{x},\sigma\right)\right)^{-k-\alpha}\sigma^{-k-1}\,d\sigma}{\int_\sigma exp\left(-T_0\left(\underline{x},\sigma\right)\right)\left(\beta - T_1\left(\underline{x},\sigma\right)\right)^{-k-\alpha}\sigma^{-k-2}\,d\sigma}. \tag{19}$$

There are few situations where overestimation and underestimation can lead to different values. For example, when we estimate the average reliable working life of the components of a spaceship or an aircraft, overestimation is usually more serious than underestimation. In such situation, the LINEX loss function (Varian (1975)) may provide useful results. Following Singh *et al.* (2007), the modified version of the LINEX loss function (LLF) is defined for any estimate $\hat{\theta}$ corresponding to the parameter θ as

$$L\left(\partial\right) = e^{a\partial} - a\partial - 1\;;\;\partial = \frac{\hat{\theta} - \theta}{\theta}.$$

See Singh *et al.* (2007) for more details. The Bayes estimation for the parameter θ under LLF is obtained by simplifying following equality

$$\int_\theta \theta^{-k-\alpha-2} exp\left(\frac{T_1\left(\underline{x},\sigma\right) + a\,\hat{\theta}_L - \beta}{\theta}\right)\int_\sigma exp\left(-T_0\left(\underline{x},\sigma\right)\right)\sigma^{-k}\,d\sigma\,d\theta =$$

$$e^a\int_\theta \theta^{-k-\alpha-2} exp\left(\frac{T_1\left(\underline{x},\sigma\right) - \beta}{\theta}\right)\int_\sigma exp\left(-T_0\left(\underline{x},\sigma\right)\right)\sigma^{-k}\,d\sigma\,d\theta.$$

Thus, the Bayes estimator $\hat{\theta}_L$ (say) is the solution of the following equality

$$\int_\sigma \frac{\sigma^{-k}\,exp\left(-T_0\left(\underline{x},\sigma\right)\right)}{\left(\beta - a\hat{\theta}_L - T_1\left(\underline{x},\sigma\right)\right)^{k+\alpha+1}}d\sigma \;=\; e^a\int_\sigma \frac{\sigma^{-k}\,exp\left(-T_0\left(\underline{x},\sigma\right)\right)}{\left(\beta - T_1\left(\underline{x},\sigma\right)\right)^{k+\alpha+1}}d\sigma. \tag{20}$$

Similarly, the Bayes estimator $\hat{\sigma}_L$ (say) for the parameter σ is the solution of the following equality

$$\int_\sigma \frac{\sigma^{-k-1}\,exp\left(-T_0\left(\underline{x},\sigma\right) + a\frac{\hat{\sigma}_L}{\sigma}\right)}{\left(\beta - T_1\left(\underline{x},\sigma\right)\right)^{k+\alpha}}d\sigma \;=\; e^a\int_\sigma \frac{\sigma^{-k-1}\,exp\left(-T_0\left(\underline{x},\sigma\right)\right)}{\left(\beta - T_1\left(\underline{x},\sigma\right)\right)^{k+\alpha}}d\sigma. \tag{21}$$

The Eq. (18) – Eq. (21) cannot be solved analytically; a numerical method employed to solve these equations numerical.

3.4. Bayes Estimation When Shape Parameter Is Known

When the shape parameter σ is known, the inverted Gamma distribution is considered here as the conjugate prior for the parameter θ, having the probability density function

$$\pi_\theta \propto \theta^{-\alpha-1}\, exp\left(-\frac{\beta}{\theta}\right)\; ;\, \theta > 0, \alpha > 0, \beta > 0. \tag{22}$$

The posterior density corresponding to the prior π_θ for the parameter θ is

$$\pi_\theta^{**} = \frac{\left(\beta - T_1\left(\underline{x}, \sigma\right)\right)^{k+\alpha}}{\Gamma(k+\alpha)}\, \theta^{-(k+\alpha+1)}\, exp\left(-\frac{\beta - T_1\left(\underline{x}, \sigma\right)}{\theta}\right). \tag{23}$$

The Bayes estimator for the scale parameter θ corresponding to ISELF and LLF corresponding to posterior π_θ^{**} are obtained as

$$\tilde{\theta}_I = \frac{\beta - T_1\left(\underline{x}, \sigma\right)}{k+\alpha+1} \tag{24}$$

and

$$\tilde{\theta}_L = \left(\frac{\beta - T_1\left(\underline{x}, \sigma\right)}{a}\right)\left\{1 - exp\left(-\frac{a}{k+\alpha+1}\right)\right\}. \tag{25}$$

4. Empirical Bayes Estimation

The ML estimate method is one of the best method for estimating the hyper-parameter. Based on empirical Bayesian approach the unknown hyper-parameter β (when α is known) is estimated. Hence, under the empirical Bayesian approach, we begin with the Bayesian model:

Since,

$$x_{(i)}|\theta \sim f\left(x; \sigma, \theta\right),\; i = 1, 2, ..., n$$

and

$$\theta|\alpha, \beta \sim \pi_\theta.$$

As all the units have identical generalized Pareto distribution, the marginal density of x, say $f(x)$, can be obtained as

$$f(x) = \int_\theta f\left(\theta, \sigma|\underline{x}\right) \cdot \pi_\theta\, d\theta$$

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, \sigma^{-k} e^{-T_0(\underline{x},\sigma)} \int_\theta exp\left(-\frac{\beta - T_1\left(\underline{x}, \sigma\right)}{\theta}\right) \theta^{-\alpha-k-1}\, d\theta$$

$$\Rightarrow f(x) = \frac{\beta^\alpha \Gamma(\alpha+k)}{\Gamma(\alpha)}\, \frac{\sigma^{-k} e^{-T_0(\underline{x},\sigma)}}{\left(\beta - T_1\left(\underline{x}, \sigma\right)\right)^{\alpha+k}}. \tag{26}$$

The ML estimate of the hyper parameter β based on $f(x)$ is

$$\hat{\beta}_{ML} = -\frac{\alpha}{k} T_1\left(\underline{x}, \sigma\right). \tag{27}$$

The Empirical Bayes estimators corresponding to the parameters are obtained by replacing the hyper-parameter β with its ML estimate $\hat{\beta}_{ML}$ given in Eq (27). Thus, the Empirical Bayes estimators for the parameter θ when σ unknown, are given as

$$\hat{\theta}_{EI} = \left(-\frac{\alpha+k}{k}\right) \frac{\int_\sigma \sigma^{-k}\, exp\left(-T_0\left(\underline{x}, \sigma\right)\right)\left(T_1\left(\underline{x}, \sigma\right)\right)^{-k-\alpha-1}\, d\sigma}{\Gamma(k+\alpha+1)\int_\sigma \sigma^{-k}\, exp\left(-T_0\left(\underline{x}, \sigma\right)\right)\left(T_1\left(\underline{x}, \sigma\right)\right)^{-k-\alpha-2}\, d\sigma} \tag{28}$$

and

$$\int_\sigma \frac{\sigma^{-k}\, exp\left(-T_0\left(\underline{x}, \sigma\right)\right)}{\left(ak\hat{\theta}_{EL} + T_1\left(\underline{x}, \sigma\right)(\alpha+k)\right)^{k+\alpha+1}}\, d\sigma = e^a \int_\sigma \frac{\sigma^{-k}\, exp\left(-T_0\left(\underline{x}, \sigma\right)\right)}{\left(T_1\left(\underline{x}, \sigma\right)(\alpha+k)\right)^{k+\alpha+1}}\, d\sigma. \tag{29}$$

Similarly, the empirical Bayes estimator for shape parameter σ when both parameter unknown, are given as

$$\hat{\sigma}_{EI} = \frac{\int_\sigma exp\left(-T_0\left(\underline{x}, \sigma\right)\right)\left(T_1\left(\underline{x}, \sigma\right)\right)^{-k-\alpha}\ \sigma^{-k-1}\, d\sigma}{\int_\sigma exp\left(-T_0\left(\underline{x}, \sigma\right)\right)\left(T_1\left(\underline{x}, \sigma\right)\right)^{-k-\alpha}\ \sigma^{-k-2}\, d\sigma} \tag{30}$$

and

$$\int_\sigma \frac{\sigma^{-k-1}\, exp\left(-T_0\left(\underline{x}, \sigma\right) + a\frac{\hat{\sigma}_{EL}}{\sigma}\right)}{\left(T_1\left(\underline{x}, \sigma\right)\right)^{\alpha+k}}\, d\sigma = e^a \int_\sigma \frac{\sigma^{-k-1}\, exp\left(-T_0\left(\underline{x}, \sigma\right)\right)}{\left(T_1\left(\underline{x}, \sigma\right)\right)^{\alpha+k}}\, d\sigma. \tag{31}$$

When shape parameter is assume to be known, the empirical Bayes estimator of the parameter θ are given as

$$\tilde{\theta}_{EI} = \left(-\frac{\alpha+k}{k}\right)\left(\frac{T_1\left(\underline{x}, \sigma\right)}{k+\alpha+1}\right)$$

and

$$\tilde{\theta}_{EL} = \left(\frac{(\alpha+k)T_1\left(\underline{x}, \sigma\right)}{ak}\right)\left\{exp\left(-\frac{a}{k+\alpha+1}\right) - 1\right\}.$$

5. Numerical Analysis Based on Simulation

In the present section, a simulation study has been performed for the analysis of the proposed methods. The Monte Carlo simulation technique was used for generating 10,000 PFF censored samples for each simulation (Based on algorithms described in Balakrishnan & Sandhu (1995)).

Table 1. Special Cases of PFF Censoring Scheme

Case	m	k	$R_i; 1, 2, ...k,$	Different Censoring Plans
1	5	05	1 2 0 2 1	First-Failure Progressive Type-II Censoring
2	5	05	0 0 0 0 0	Progressive Type-II Censoring
3	1	05	1 2 0 2 1	First-Failure Censoring
4	1	05	0 0 0 0 25	Type-II Censoring
5	1	05	0 0 0 0 0	Complete Sample
1	5	10	1 0 0 5 0 0 1 4 2 1	First-Failure Progressive Type-II Censoring
2	5	10	0 0 0 0 0 0 0 0 0 0	Progressive Type-II Censoring
3	1	10	1 0 0 5 0 0 1 4 2 1	First-Failure Censoring
4	1	10	0 0 0 0 0 0 0 0 0 20	Type-II Censoring
5	1	10	0 0 0 0 0 0 0 0 0 0	Complete Sample

The samples were simulated for different values of $n = 30, \sigma = 2, m = 5$, hyper-parametric values $(\alpha, \beta) = (2.10, 0.34), (3.60, 3.28), (5.00, 6.90)$ with different k. In this section, we also encounter different special cases of PFF censoring in empirical Bayesian analysis. The different special cases of PFF censoring scheme along with different values of k are given in Table (1).

The ML estimates $\hat{\theta}_{ML}$ and $\hat{\sigma}_{ML}$ of the parameters θ and σ respectively are computed from the solution of Eq. (6) and Eq. (7), by using Newton-Raphson iteration method and presented in Table (2). It is observed from the table is that; the value of ML estimate is increasing as the prior parameter α or the censoring size increases and decreasing when prior parameter β increases.

Table 2. Maximum Likelihood Estimate

$n = 30$	$\beta \rightarrow$	0.34	3.28	6.90	0.34	3.28	6.90
$m = 5$	$\alpha \downarrow$	$k = 05$			$k = 10$		
$\hat{\theta}_{ML}$	2.10	1.2607	0.9546	0.7893	1.2947	0.9893	0.8968
	3.60	1.3446	0.9712	0.8219	1.3626	1.2359	1.0334
	5.00	1.3801	1.0284	0.8567	1.4001	1.3171	1.0917
$\hat{\sigma}_{ML}$	2.10	1.5617	1.1856	1.0903	1.6117	1.2803	1.1978
	3.60	1.6656	1.1942	1.1429	1.6856	1.5482	1.2665
	5.00	1.7241	1.2714	1.2507	1.8701	1.6114	1.3357

An improved ACI based on log-transformation have obtained and presented in Table (3) for selected set of parameters at significance levels $\epsilon = 90\%, 95\%, 99\%$. It has seen that the ACI for both parameters increases when censoring size increases or the significance levels increase. An opposite trend have seen when set of prior parameters increase.

Table 3. Approximate Confidence Length

$n = 30$	$\epsilon \rightarrow$	90%	95%	99%	90%	95%	99%
$m = 5$	(α, β)	$k = 05$			$k = 10$		
	(2.10, 0.34)	1.0786	1.1016	1.2218	1.0846	1.1046	1.2598
θ	(3.60, 3.28)	1.0315	1.0691	1.0717	1.0655	1.0706	1.1191
	(5.00, 6.90)	1.0107	1.0305	1.0299	1.0384	1.0435	1.0392
	(2.10, 0.34)	1.1956	1.2356	1.3508	1.2096	1.2396	1.3638
σ	(3.60, 3.28)	1.1865	1.1996	1.2601	1.1925	1.2001	1.2707
	(5.00, 6.90)	1.1594	1.1645	1.1202	1.1607	1.1715	1.1409

The Bayes risks of the empirical Bayes estimators corresponding to the parameters θ and σ based on different censoring plans of PFF censoring scheme, as discussed in Table (1), have been presented in the Table (4-6).

It is observed that, the empirical Bayes risk increases when parameter α or censored sample size k increases for all the considered values. Similar properties also have seen when, the shape parameter of LLF $'a'$ increase for LLF risk criterion. It is observable that, the Bayes risk increase when censoring pattern changed. The smaller risks magnitude has noted for PFF, then progressive censoring and large risks in magnitude for a complete sample case. Further, there is no any clear trend has observed between the Type-II and First-failure censoring. However, one may prefer the PFF over the other censoring scheme for the selected parametric set of values.

Table 4. Empirical Bayes Risk Under ISELF When σ Unknown

$k \downarrow$	$\alpha \downarrow$	Different Censoring Plans				
		1	2	3	4	5
		$\hat{\theta}_{EI}$				
	2.10	0.8619	0.8905	0.9018	1.0116	1.0712
05	3.60	0.9306	0.9530	1.0479	1.0227	1.1487
	5.00	0.9536	0.9876	1.1231	1.0667	1.1604
	2.10	0.8705	0.9125	0.9539	1.0242	1.0756
10	3.60	0.9491	0.9758	1.0670	1.0371	1.1542
	5.00	0.9684	0.9919	1.1312	1.0787	1.1631
		$\hat{\sigma}_{EI}$				
	2.10	0.9486	0.9801	1.0462	0.9925	1.1689
05	3.60	0.9596	1.0118	1.1103	1.0139	1.1842
	5.00	0.9907	1.0309	1.1783	1.2236	1.2248
	2.10	0.9598	1.0043	1.0108	1.0605	1.1838
10	3.60	0.9745	1.0139	1.0287	1.1252	1.1903
	5.00	0.9958	1.0617	1.071	1.1735	1.2801

Table 5. Empirical Bayes Risk Under LLF When σ Unknown

$k\downarrow$	$a\downarrow$	$\alpha\downarrow$	Different Censoring Plans				
			1	2	3	4	5
$\hat{\theta}_{EL}$							
05	0.50	2.10	0.8319	0.8536	0.8846	0.9137	1.0189
		3.60	0.8525	0.9143	0.9616	0.9711	1.0541
		5.00	0.9218	0.9608	1.0148	0.9978	1.1022
	1.00	2.10	0.8501	0.8783	0.8895	0.9597	1.0565
		3.60	0.9178	0.9400	0.9678	0.9824	1.1123
		5.00	0.9405	0.9741	1.0717	1.0602	1.1214
10	0.50	2.10	0.8543	0.8785	0.9014	0.9363	1.0432
		3.60	0.9105	0.9364	0.9933	0.9805	1.0594
		5.00	0.9492	0.9712	1.0182	1.0571	1.1218
	1.00	2.10	0.8586	0.9007	0.9499	0.9766	1.0709
		3.60	0.9361	0.9725	0.9967	1.0202	1.1384
		5.00	0.9552	0.9783	1.0965	1.0786	1.1472
$\hat{\sigma}_{EL}$							
05	0.50	2.10	0.8606	0.8892	0.9004	0.9306	1.0296
		3.60	0.8706	0.9179	0.9198	0.9707	1.0704
		5.00	0.9043	0.9352	0.9492	0.9810	1.0884
	1.00	2.10	0.8995	0.9219	1.0011	0.9889	1.0504
		3.60	0.9018	0.9487	1.0148	1.0108	1.0844
		5.00	0.9346	0.9566	1.0573	1.0346	1.1011
10	0.50	2.10	0.8708	0.9111	0.9170	0.9678	1.0740
		3.60	0.8840	0.9198	0.9333	0.9846	1.0799
		5.00	0.9094	0.9632	1.0535	1.0124	1.1013
	1.00	2.10	0.9101	0.9417	1.0055	1.0073	1.0811
		3.60	0.9337	0.9607	1.0203	1.0589	1.0961
		5.00	0.9637	0.9855	1.0619	1.0901	1.1203

Table 6. Empirical Bayes Risk When σ Known

$k \downarrow$	$a \downarrow$	$\alpha \downarrow$	Different Censoring Plans				
			1	2	3	4	5
$\tilde{\theta}_{EI}$							
05		2.10	0.8819	0.9051	0.9812	0.9117	1.0213
		3.60	0.8936	0.9233	1.0191	0.9453	1.0615
		5.00	0.9365	0.9768	1.0301	0.9967	1.1041
10		2.10	0.9081	0.9358	1.0123	0.9513	1.0519
		3.60	0.9386	0.9431	1.0271	0.9651	1.0817
		5.00	0.9659	0.9861	1.0531	1.0007	1.1145
$\tilde{\theta}_{EL}$							
05	0.50	2.10	0.8171	0.8387	0.9091	0.8448	0.9463
		3.60	0.8279	0.8555	0.9442	0.8759	0.9835
		5.00	0.8677	0.9050	0.9545	0.9235	1.0230
	1.00	2.10	0.8345	0.8568	0.9096	0.8731	0.9478
		3.60	0.8657	0.8942	0.9459	0.9253	1.0265
		5.00	0.8768	0.9204	0.9765	0.9645	1.0373
10	0.50	2.10	0.8414	0.8671	0.9379	0.8814	0.9746
		3.60	0.8696	0.8739	0.9517	0.8942	1.0023
		5.00	0.8950	0.9137	0.9758	0.9272	1.0327
	1.00	2.10	0.8796	0.8962	0.9694	0.9101	1.0007
		3.60	0.8908	0.9332	0.9736	0.9411	1.0159
		5.00	0.9125	0.9401	1.0105	0.9603	1.0573

References

Al-Zahrani, B. and Al-Sobhi, M. (2013). On parameters estimation of Lomax distribution under general Progressive censoring. *Journal of Quality and Reliability Engineering*, 2013, 1 - 7.

Azimi, R., Fasihi, B. and Sarikhanbaglu, F. A. (2014). Statistical inference for generalized Pareto distribution based on progressive Type-II censored data with random removals. *International Journal of Scientific World*, 2 (1), 1 - 9.

Balakrishnan, N. and Aggarwala, R. (2000). Progressive Censoring: Theory, Methods, and Applications. *Birkhauser*, Boston.

Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulation algorithm for generating Progressively Type-II censored samples. *American Statistics*, 49, 229 - 230.

Castillo, E. and Hadi, A. S. (1977). Fitting the generalized Pareto distribution to data. *Journal of American Statistical Association*, 92, 1609 - 1620.

Fu, J., Xu, A. and Tang, Y. (2012). Objective Bayesian analysis of Pareto distribution under Progressive Type-II censoring. *Statistics and Probability Letters*, 82, 1829 - 1836.

Grimshaw, S. D. (1993). Computing maximum likelihood estimates for the generalized Pareto distribution. *Technometrics*, 35, 185 - 191.

Johnson, L. G. (1964). Theory and technique of variation research. *Elsevier*, Amsterdam.

Jun, C. H., Balamurali, S., and Lee, S. H. (2006). Variables sampling plans for Weibull distributed lifetimes under sudden death testing. *IEEE Transactions on Reliability*, 55 (1), 53 - 58.

Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of Progressive Censoring. *Technometrics*, 50 (2), 144 - 154.

Lee, W. C., Wu, J. W. and Hong, C. W. (2009). Assessing the lifetime performance index of products from progressively Type-II right censored data using Burr-XII model. *Mathematics and Computers in Simulation*, 79 (7), 2167 - 2179.

Meeker, W. Q., and Escobar, L. A. (1998). Statistical methods for reliability data. *John Wiley and Sons*, New York.

Prakash, G. (2012). Central coverage Bayes prediction intervals for the generalized Pareto distribution. *Statistics Research Letters*, 1 (1), 1 - 5.

Prakash, G. (2015). A comparative study based on Bayes estimation under progressively censored Rayleigh data. *Journal of Statistics Applications and Probability*, 4 (1), 23 - 30.

Prakash, G. (2016). Bayes prediction bound lengths under different censoring criterion: A comparative study. *International Journal of Intelligent Technologies and Applied Statistics*, 9 (1), 53 - 66.

Raqab, M. Z., Asgharzadeh, A. R. and Valiollahi, R. (2010). Prediction for Pareto distribution based on Progressively Type-II censored samples. *Computational Statistics and Data Analysis*, 54 (7), 1732 - 1743.

Rezaei, R., Tahmasbi, R. and Mahmoodi, M. (2010). Estimation of $P[Y < X]$ for generalized Pareto distribution. *Journal of Statistical Planning and Inference*, 140, 480 - 494.

Singh, D. C., Prakash, G. and Singh, P. (2007). Shrinkage testimators for the shape parameter of Pareto distribution using LINEX loss function. *Communications in Statistics-Theory and Methods*, 36, 741 - 753.

Varian, H. R. (1975). A Bayesian approach to real estate assessment. In: Feinberg, S. E., Zellner, A., eds. *Studies in Bayesian Econometrics and Statistics* in Honor of L. J. Savage. Amsterdem: North Holland, 195 - 208.

Wu, S. J. and Ku?, C. (2009). On estimation based on progressive first-failure censored sampling. *Computational Statistics and Data Analysis*, 53 (10), 3659 - 3670.