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SEMI-NORMAL STRUCTURE AND BEST PROXIMITY PAIR RESULTS IN CONVEX METRIC SPACES

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Dedicated to my late friend Hassan Shams (1981–2004), an outstanding math student

Communicated by R. E. Curto.

ABSTRACT. A new geometric notion on a nonempty and convex pair of subsets of a convex metric space X, called semi-normal structure, is introduced and used to investigate the existence of best proximity pairs for a new class of mappings, called strongly noncyclic relatively C-nonexpansive. We also study the structure of minimal sets of strongly noncyclic relatively C-nonexpansive mappings in the setting of convex metric spaces.

1. INTRODUCTION

Let (X, d) be a metric space. A self-mapping $T : X \to X$ is called *C-contraction* if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$. The class of C-contractions was introduced by Chatterjea in [8]. It was proved in [8] that if X is complete metric space, every C-contraction self-mapping defined on X has a unique fixed point ([8]). Note that the C-contraction self-mappings may not be continuous.

We say that a self-mapping $T : X \to X$ is *C-nonexpansive* if

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] \quad \forall x, y \in X.$$

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Now, let A and B be two nonempty subsets of a metric space (X,d). A mapping $T: A\cup B\to A\cup B$ is said to be a *noncyclic mapping* provided that $T(A)\subseteq A$ and $T(B)\subseteq B$. For this class of mappings, we consider the following minimization problem: Find

$$\min_{x\in A} d(x,Tx), \quad \min_{y\in B} d(y,Ty) \quad \text{and} \quad \min_{(x,y)\in A\times B} d(x,y).$$

A point $(p,q)\in A\times B$ is said to be a *best proximity pair* of the noncyclic mapping T provided that (p,q) is a solution of the above minimization problem, that is,

$$Tp=p, \quad Tq=q \quad \text{and} \quad d(p,q)=dist(A,B),$$

where, $dist(A,B):=\inf\{d(x,y):(x,y)\in A\times B\}$. We mention that in [1], the authors investigated sufficient conditions to ensure the existence of best proximity pairs for noncyclic mappings.

Let (A,B) be a nonempty pair of subsets of a metric space (X,d). A mapping $T: A\cup B\to A\cup B$ is said to be *noncyclic relatively nonexpansive* if T is noncyclic and $d(Tx,Ty)\le d(x,y)$ for all $(x,y)\in A\times B$.

Eldred, Kirk and Veeramani ([5]) established the existence of a best proximity pair for noncyclic relatively nonexpansive mappings by using a geometric notion of *proximal normal structure* in the setting of Banach spaces.

We shall say that a pair (A,B) in a metric space (X,d) satisfies a property if both A and B satisfy that property. For instance, (A,B) is closed if and only if both A and B are closed; $(A,B)\subseteq(C,D)\Leftrightarrow A\subseteq C,$ and $B\subseteq D$. We shall also adopt the following notations.

$$\delta_x(A):=\sup\{d(x,y)\colon y\in A\} \text{ for all } x\in X,$$

$$\delta(A,B):=\sup\{d(x,y)\colon x\in A,\, y\in B\},$$

$$diam(A):=\delta(A,A).$$

$$A_0:=\{x\in A: d(x,y)=dist(A,B) \;\text{ for some }\; y\in B\},$$

$$B_0:=\{y\in B: d(x,y)=dist(A,B) \;\text{ for some }\; x\in A\}.$$

We note that if A and B are nonempty, weakly compact and convex subsets of a Banach space X, then (A_0,B_0) must be a nonempty pair in X.

Definition 1.1. A Banach space X is said to be
(i) *uniformly convex* if there exists a strictly increasing function $\delta:(0,2]\to[0,1]$ such that the following implication holds for all $x,y,p\in X, R>0$ and $r\in[0,2R]$:

$$\begin{cases}\|x-p\|\le R,\\ \|y-p\|\le R,\\ \|x-y\|\ge r\end{cases} \Rightarrow \|\frac{x+y}{2}-p\|\le(1-\delta(\frac{r}{R}))R;$$

(ii) *strictly convex* if the following implication holds for all $x,y,p\in X$ and $R>0$:

$$\begin{cases}\|x-p\|\le R,\\ \|y-p\|\le R,\\ x\ne y\end{cases} \Rightarrow \|\frac{x+y}{2}-p\|< R.$$

In the current paper, we introduce a new class of mappings, called *strongly noncyclic relatively C-nonexpansive* and study sufficient conditions which ensure the existence of best proximity pairs for this class of mappings in the setting of *convex metric spaces*. Moreover, we survey the structure of minimal sets for this class of noncyclic mappings and show that results alike to the celebrated Goebel–Karlovitz Lemma ([6, 10, 11]) for nonexpansive self-mappings can be obtained for strongly noncyclic relatively C-nonexpansive mappings.

2. Preliminaries

In [17], Takahashi introduced the notion of convexity in metric spaces as follows.

Definition 2.1. Let (X, d) be a metric space and $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on X provided that for each $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \mathcal{W}(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$ is called a convex metric space, which is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces. But a Frechet space is not necessary a convex metric space. The other examples of convex metric spaces which are not imbedded in any Banach space can be founded in [17].

To describe our results, we need some definitions and preliminary facts from the reference [17].

Definition 2.2. A subset K of a convex metric space $(X, d, \mathcal{W})$ is said to be a *convex set* provided that $\mathcal{W}(x, y; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Proposition 2.3. *Let $(X, d, \mathcal{W})$ be a convex metric space and let $\mathcal{B}(x; r)$ denote the closed ball centered at $x \in X$ with radius $r \geq 0$. Then $\mathcal{B}(x; r)$ is a convex subset of X.*

Proposition 2.4. *Let $\{K_\alpha\}_{\alpha\in A}$ be a family of convex subsets of X, then $\bigcap_{\alpha\in A} K_\alpha$ is also a convex subset of X.*

Definition 2.5. A convex metric space $(X, d, \mathcal{W})$ is said to have *property (C)* if every bounded decreasing net of nonempty, closed and convex subsets of X has a nonempty intersection.

For example every weakly compact convex subset of a Banach space has property (C). The next example ensures that condition (C) is natural as well in the metrical setting.

Example 2.6. ([15]) Let $\mathcal{H}$ be a Hilbert space and let X be a nonempty closed subset of $\{x \in \mathcal{H} : \|x\| = 1\}$ such that if $x, y \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, then $\frac{\alpha x+\beta y}{\|\alpha x+\beta y\|} \in X$ and $diam(X) \leq \frac{\sqrt{2}}{2}$, where $diam(X) := \sup\{d(x, y) : x, y \in X\}$. Let $d(x, y) := \cos^{-1}(< x, y >)$ for all $x, y \in X$, where $<,>$ is the inner product of $\mathcal{H}$. If we define the convex structure $\mathcal{W} : X \times X \times I \to X$ with $\mathcal{W}(x, y, \lambda) := \frac{\lambda x+(1-\lambda)y}{\|\lambda x+(1-\lambda)y\|}$, then (X, d) is a complete convex metric space which has the property (C) (for more information see Example 2 of [15]).

Definition 2.7. ([9]) A convex metric space $(X, d, \mathcal{W})$ is said to have *property (D)* provided that for each x_1, x_2, y_1, y_2 in X we have

$$d(\mathcal{W}(x_1, x_2, \frac{1}{2}), \mathcal{W}(y_1, y_2, \frac{1}{2})) < \frac{1}{2}[d(x_1, y_1) + d(x_2, y_2)].$$

It is clear that every strictly convex Banach space is a convex metric space which satisfies the property (D).

Let A be a nonempty subset of a convex metric space $(X, d, \mathcal{W})$. The *closed and convex hull* of a set A will be denoted by $\overline{cov}(A)$ and defined as below.

$$\overline{cov}(A) := \bigcap\{C : C \text{ is a closed and convex subset of } X \text{ such that } C \supseteq A\}.$$

The next lemmas will be used in our results.

Lemma 2.8. *([2]) Let (K_1, K_2) be a nonempty pair of a convex metric space $(X, d, \mathcal{W})$. Then $\delta(K_1, K_2) = \delta(\overline{cov}(K_1), \overline{cov}(K_2))$.*

Lemma 2.9. *([12]) Let A be a nonempty subset of a convex metric space $(X, d, \mathcal{W})$. Then*

$$\delta_x(A) = \delta_x(\overline{cov}(A)), \quad \forall x \in X.$$

Definition 2.10. Let A be a nonempty subset in a metric space (X, d). A point p in A is said to be a *diametral point* if $\delta_p(A) = diam(A)$.

The notion of *normal structure* was introduced by Brodskil and Milman in [7] and then it was generalized by Takahashi in convex metric spaces as follows.

Definition 2.11. ([17]) A convex metric space $(X, d, \mathcal{W})$ is said to have normal structure if for each bounded, closed and convex subset K of X which contains at least two points, there exists an element $p \in K$ which is a nondiametral point.

By using this geometric notion the following fixed point theorem was established in [17].

Theorem 2.12. *Suppose that $(X, d, \mathcal{W})$ is a convex metric space such that X has the property (C). Let K be a nonempty bounded, closed and convex subset of X with normal structure. If $T : K \to K$ is a nonexpansive mapping, that is,*

$$d(Tx, Ty) \leq d(x, y), \quad \forall\ x, y \in K,$$

then T has a fixed point in K.

We mention that the previous theorem is an extension of *Kirk's fixed point theorem* ([13]) in the setting of convex metric spaces.

Some of interesting results regarding the existence and convergence of fixed points for various classes of nonexpansive mappings can be found in [3, 4, 14, 16].

Definition 2.13. ([9]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d). We say that the pair (A, B) is a *proximal compactness pair* provided that every net $(\{x_\alpha\}, \{y_\alpha\})$ of $A \times B$ satisfying the condition that $d(x_\alpha, y_\alpha) \to dist(A, B)$, has a convergent subnet in $A \times B$.

It is clear that if (A, B) is a compact pair in a metric space (X, d) then (A, B) is proximal compactness.

3. Best proximity pair results

We begin our main result of this section with the following geometric notion.

Definition 3.1. A convex pair (K_1, K_2) in a convex metric space $(X, d, \mathcal{W})$ is said to have semi-normal structure if for any bounded, closed and convex pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $\delta(H_1, H_2) > dist(H_1, H_2)$ and $dist(H_1, H_2) = dist(K_1, K_2)$, there exits $(x_1, x_2) \in H_1 \times H_2$ such that

$$d(x_1, x_2) = dist(K_1, K_2) \quad \& \quad \max\{\delta_{x_1}(H_2), \delta_{x_2}(H_1)\} < \delta(H_1, H_2).$$

Note that if in above definition $K_1 = K_2$, then (K_1, K_2) has semi-normal structure if and only if the set K_1 has normal structure in the sense of Brodskil and Milman ([7]).

Definition 3.2. Let (A, B) be a nonempty pair of subsets of a convex metric space $(X, d, \mathcal{W})$. A mapping $T : A \cup B \to A \cup B$ is said to be strongly noncyclic relatively C-nonexpansive provided that T is noncyclic and

$$d(Tx, Ty) = d(x, y), \ \forall (x, y) \in A \times B \quad \text{with} \quad d(x, y) = dist(A, B),$$

and

$$d(Tx, Ty) \le \min\{d(x, Ty), d(y, Tx)\}, \forall (x, y) \in A \times B \text{ with } d(x, y) > dist(A, B).$$

Definition 3.3. Let (A, B) be a nonempty pair of subsets of a convex metric space $(X, d, \mathcal{W})$. A mapping $T : A \cup B \to A \cup B$ is said to be noncyclic relatively C-nonexpansive provided that T is noncyclic and

$$d(Tx, Ty) = d(x, y), \ \forall (x, y) \in A \times B \quad \text{with} \quad d(x, y) = dist(A, B),$$

and

$$d(Tx, Ty) \le \frac{1}{2}[d(x, Ty) + d(y, Tx)], \forall (x, y) \in A \times B \text{ with } d(x, y) > dist(A, B).$$

It is clear that the class of noncyclic relatively C-nonexpansive mappings contains the class of strongly noncyclic relatively C-nonexpansive mappings as a subclass.

The next theorem guarantees the existence of best proximity pairs for strongly noncyclic relatively C-nonexpansive mappings in convex metric spaces.

Theorem 3.4. *Let (A, B) be a nonempty, bounded, closed and convex pair in a convex metric space $(X, d, \mathcal{W})$ such that X has the properties (C) and (D), A_0 is nonempty and (A, B) is a proximal compactness pair. Suppose that $T : A \cup B \to A \cup B$ is a strongly noncyclic relatively C-nonexpansive mapping. If (A, B) has semi-normal structure, then T has a best proximity pair.*

Proof. Suppose that $\mathfrak{F}$ denote the collection of all nonempty, closed and convex pairs (E, F) which are subsets of (A, B) and such that T is noncyclic on $E \cup F$ and $d(x, y) = dist(A, B)$ for some $(x, y) \in E \times F$. Since A_0 is nonempty, $(A, B) \in \mathfrak{F}$. Also, $\mathfrak{F}$ is partially ordered by revers inclusion, that is, $(E_1, F_1) \preceq (E_2, F_2) \Leftrightarrow (E_2, F_2) \subseteq (E_1, F_1)$. Let $\{(E_\alpha, F_\alpha)\}_\alpha$ be a descending chain in $\mathfrak{F}$. Put $E := \bigcap E_\alpha$ and $F := \bigcap F_\alpha$. By the fact that X has the property (C), we deduce that (E, F) is a nonempty pair and we know that (E, F) is closed. It is easy to see that T

is noncyclic on $E \cup F$. Moreover, by Proposition 2.4, (E, F) is a convex pair. Now, let $(x_\alpha, y_\alpha) \in E_\alpha \times F_\alpha$ be such that $d(x_\alpha, y_\alpha) = dist(A, B)$. Since (A, B) is proximal compactness, (x_α, y_α) has a convergent subsequence say $(x_{\alpha_i}, y_{\alpha_i})$ such that $x_{\alpha_i} \to x \in A$ and $y_{\alpha_i} \to y \in B$. Hence,

$$d(x, y) = \lim_i d(x_{\alpha_i}, y_{\alpha_i}) = dist(A, B).$$

Therefore, there exists an element $(x, y) \in E \times F$ such that $d(x, y) = dist(A, B)$. So, every increasing chain in $\mathfrak{F}$ is bounded above with respect to revers inclusion relation. Then by using Zorn's Lemma we obtain a minimal element for $\mathfrak{F}$, say (K_1, K_2). Suppose that

$$\delta(K_1, K_2) = dist(K_1, K_2)(= dist(A, B)).$$

Then for each $(x, y) \in (K_1, K_2)$ we have $d(x, y) = dist(K_1, K_2)$. We now claim that both K_1 and K_2 are singleton. Assume that $x_1, x_2 \in K_1$ such that $x_1 \neq x_2$. Then for all $y_1, y_2 \in K_2$ we have

$$d(x_1, y_1) = d(x_2, y_2) = dist(A, B).$$

Since (K_1, K_2) is a convex pair, $\mathcal{W}(x_1, x_2, \frac{1}{2}) \in K_1$ and $\mathcal{W}(y_1, y_2, \frac{1}{2}) \in K_2$. By the fact that X has the property (D) we conclude that

$$\begin{aligned} dist(A, B) &\leq d(\mathcal{W}(x_1, x_2, \frac{1}{2}), \mathcal{W}(y_1, y_2, \frac{1}{2})) \\ &< \frac{1}{2}[d(x_1, y_1) + d(x_2, y_2)] = dist(A, B), \end{aligned}$$

which is a contradiction. Hence, K_1 is singleton. Similarly, we can see that K_2 is a singleton set. This implies that the noncyclic mapping T has a best proximity pair in this case and we are finished. Now, suppose that

$$\delta(K_1, K_2) > dist(K_1, K_2)(= dist(A, B)).$$

Let $(p, q) \in K_1 \times K_2$ be such that $d(p, q) = dist(A, B)$. Consider the nonempty, closed and convex pair $(\overline{cov}(T(K_1)), \overline{cov}(T(K_2))) \subseteq (K_1, K_2)$. We have $(Tp, Tq) \in (\overline{cov}(T(K_1)), \overline{cov}(T(K_2)))$, and so,

$$dist(A, B) \leq dist(\overline{cov}(T(K_1)), \overline{cov}(T(K_2))) \leq d(Tp, Tq) = d(p, q) = dist(A, B).$$

Also,

$$T(\overline{cov}(T(K_1))) \subseteq T(K_1) \subseteq \overline{cov}(T(K_1)),$$

and similarly, $T(\overline{cov}(T(K_2))) \subseteq \overline{cov}(T(K_2))$. Therefore, the mapping

$$T : \overline{cov}(T(K_1)) \cup \overline{cov}(T(K_2)) \to \overline{cov}(T(K_1)) \cup \overline{cov}(T(K_2))$$

is noncyclic. So, $(\overline{cov}(T(K_1)), \overline{cov}(T(K_2))) \in \mathfrak{F}$. It now follows from the fact that (K_1, K_2) is the minimal element of $\mathfrak{F}$,

$$K_1 = \overline{cov}(T(K_1)) \quad \& \quad K_2 = \overline{cov}(T(K_2)).$$

Put

$$r_1 := \delta_p(K_2), \quad r_2 := \delta_q(K_1) \quad \text{and} \quad r := \max\{r_1, r_2\}.$$

Since (A, B) has the semi-normal structure, we have

$$r < \delta(K_1, K_2).$$

Suppose that

$$C_r(K_2) := K_1 \bigcap (\cap_{x\in K_2} \mathcal{B}(x;r)) \quad \& \quad C_r(K_1) := K_2 \bigcap (\cap_{x\in K_1} \mathcal{B}(x;r)).$$

Propositions 2.3 and 2.4 conclude that $(C_r(K_2), C_r(K_1)) \subseteq (K_1, K_2)$ is a nonempty, closed and convex pair. Besides, it is easy to see that for $(x, y) \in K_1 \times K_2$,

$$(x, y) \in C_r(K_2) \times C_r(K_1) \Leftrightarrow K_2 \subseteq \mathcal{B}(x;r), \quad K_1 \subseteq \mathcal{B}(y;r).$$

Also, $(p, q) \in C_r(K_2) \times C_r(K_1)$ which deduces that

$$dist(C_r(K_2), C_r(K_1)) \leq d(p, q) = dist(K_1, K_2) \leq dist(C_r(K_2), C_r(K_1)).$$

We now assert that T is noncyclic on $C_r(K_2) \cup C_r(K_1)$. Let $u \in C_r(K_2)$. Since T is strongly noncyclic relatively C-nonexpansive, for each $v \in K_2$ we have

$$d(Tu, Tv) \leq \min\{d(u, Tv), d(v, Tu)\} \leq r,$$

which concludes that $Tv \in B(Tu;r)$. Therefore, $T(K_2) \subseteq \mathcal{B}(Tu;r)$ and so,

$$K_2 = \overline{cov}(T(K_2)) \subseteq \mathcal{B}(Tu;r),$$

that is, $Tu \in C_r(K_2)$. Thus, $T(C_r(K_2)) \subseteq C_r(K_2)$. Similarly, we can see that $T(C_r(K_1)) \subseteq C_r(K_1)$. Hence, T is noncyclic on $C_r(K_2) \cup C_r(K_1)$. The minimality of (K_1, K_2) implies that

$$C_r(K_2) = K_1 \quad \& \quad C_r(K_1) = K_2.$$

So, $K_1 \subseteq \bigcap_{v\in K_2} \mathcal{B}(v;r)$. Then for each $u \in K_1$ we have $\delta_u(K_2) \leq r$. Thus,

$$\delta(K_1, K_2) = \sup_{u\in K_1} \delta_u(K_2) \leq r,$$

which is a contradiction since $r < \delta(K_1, K_2)$.

□

The next corollary obtains from Theorem 3.4, immediately.

Corollary 3.5. *Let (A, B) be a nonempty, weakly compact and convex pair in a strictly convex Banach space X such that (A, B) has semi-normal structure. If $T : A \cup B \to A \cup B$ is a strongly noncyclic relatively C-nonexpansive mapping, then T has a best proximity pair.*

The next theorem guarantees the existence of best proximity pairs in uniformly convex Banach spaces.

Theorem 3.6. *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a strongly noncyclic relatively C-nonexpansive mapping. Then T has a best proximity pair.*

Proof. Suppose that T has not a best proximity pair. We get a contradiction by showing that (A, B) has semi-normal structure. Let (K_1, K_2) be a closed and convex subset of (A, B) such that $\delta(K_1, K_2) > dist(K_1, K_2)$ and $dist(K_1, K_2) = dist(A, B)$. Let $(p, q) \in K_1 \times K_2$ be such that $\|p - q\| = dist(K_1, K_2)(= dist(A, B))$. Since T is strongly noncyclic relatively C-nonexpansive, we have

$$\|Tp - Tq\| = \|p - q\| = dist(A, B).$$

Now, we must have $p \neq Tp$ or $q \neq Tq$ by the fact that T has not best proximity pair. It now follows from the strictly convexity of X that

$$\|\frac{p+Tp}{2} - \frac{q+Tq}{2}\| = dist(A,B).$$

Put $R := \delta(K_1, K_2)$ and $r := \min\{\|p - Tp\|, \|q - Tq\|\}$. Also, set $p^\star := \frac{p+Tp}{2}$ and $q^\star := \frac{Tq+q}{2}$. Now, for all $y \in K_2$ we have

$$\begin{cases} \|p - y\| \leq R, \\ \|Tp - y\| \leq R, \\ \|p - Tp\| \geq r. \end{cases}$$

Since X is a uniformly convex Banach space, we conclude that

$$\|p^\star - y\| \leq (1 - \delta(\frac{r}{R}))R, \quad \forall y \in K_2,$$

Hence, $\delta_{p^\star}(K_2) < R$. Similarly, we can see that $\delta_{q^\star}(K_1) < R$. Therefore,

$$\|p^\star - q^\star\| = dist(A,B) \quad \& \quad \max\{\delta_{p^\star}(K_2), \delta_{q^\star}(K_1)\} < \delta(K_1, K_2).$$

That is, (A, B) has semi-normal structure.

□

Here, we prove a *new fixed point theorem* by using the geometric notion of normal structure in convex metric spaces.

Theorem 3.7. *Suppose that $(X, d, \mathcal{W})$ is a convex metric space such that X has the property (C). Let A be a nonempty, bounded, closed and convex subset of X with normal structure. If $T : A \to A$ is a strongly C-nonexpansive, that is,*

$$d(Tx, Ty) \leq \min\{d(x, Ty), d(y, Tx)\} \quad \forall x, y \in A,$$

then T has a fixed point in A.

Proof. Invoking property (C) and Zorn's Lemma, we obtain a subset K of A which is minimal with respect to being nonempty, closed, convex and T-invariant. So, we must have $\overline{cov}(T(K)) = K$. Suppose that $diam(K) > 0$. By the fact that X has normal structure there exist $p \in K$ and $r > 0$ such that $\delta_p(K) < r < diam(K)$. Put

$$C := \{x \in K : K \subseteq \mathcal{B}(x; r)\}.$$

Note that $p \in C$ and then C is nonempty. Moreover, it is easy to see that

$$C = K \bigcap (\cap_{x \in K} \mathcal{B}(x; r)),$$

that is, C is a closed and convex subset of a convex metric space X. Now, let $x \in C$. Then for each $y \in K$ we have

$$d(Tx, Ty) \leq \min\{d(x, Ty), d(Tx, y)\} \leq r.$$

This implies that $Ty \in \mathcal{B}(Tx; r)$ for each $y \in K$ and so,

$$K = \overline{cov}(T(K)) \subseteq \mathcal{B}(Tx; r).$$

Hence, $Tx \in C$, that is, $T : C \to C$. Minimality of K deduces that $C = K$. Thus, $diam(K) \leq r$ which is a contradiction. Then $diam(K) = 0$ and hence, K consists of a single point which must be a fixed point of T.

□

Remark 3.8. We note that a convex metric space $(X, d, \mathcal{W})$ need not to have the condition (D), in Theorem 3.7.

Let us illustrate Theorem 3.7 with the following example.

Example 3.9. Let $X := [-1, 1]$ and define a metric d on X by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \max\{|x|, |y|\}, & \text{if } x \neq y. \end{cases}$$

Define $\mathcal{W} : X \times X \times I \to X$ with

$$\mathcal{W}(x, y, \lambda) = \lambda \min\{|x|, |y|\},$$

for each $x, y \in X$ and $\lambda \in I$. We show that $\mathcal{W}$ is a convex structure on X. Let $x, y \in X$ and $\lambda \in I$. We may assume that $|x| \leq |y|$. Then for each $u \in X$ we have

$$\begin{aligned} d(u, \mathcal{W}(x, y, \lambda)) &= \max\{|u|, \lambda \min\{|x|, |y|\}\} \\ &= \max\{|u|, \lambda|x|\} \leq \max\{|u|, |x|\} \\ &= \lambda \max\{|u|, |x|\} + (1 - \lambda) \max\{|u|, |x|\} \\ &\leq \lambda \max\{|u|, |x|\} + (1 - \lambda) \max\{|u|, |y|\} \\ &= \lambda d(u, x) + (1 - \lambda) d(u, y). \end{aligned}$$

This implies that $(X, d, \mathcal{W})$ is a convex metric space. Now, let E be a nonempty convex subset of X. Then $\mathcal{W}(x, y, \lambda) \in E$ for each $x, y \in E$ and $\lambda \in I$. If $\lambda = 0$, then we conclude that $0 \in E$. Therefore, the convex metric space $(X, d, \mathcal{W})$ must be have the property (C). Suppose that $A := [0, 1]$. Thus, A is a bounded closed and convex subset of X. Note that every convergent sequence in this metric space converges to $0 \in A$. Let $T : A \to A$ be a mapping defined as

$$Tx = \begin{cases} 0, & \text{if } x = 1, \\ x, & \text{if } x \neq 1. \end{cases}$$

We claim that T is strongly C-nonexpansive. For this purpose it is sufficient to consider $x = 1$ and $y \neq 1$. Then $d(Tx, Ty) = y$, $d(x, Ty) = 1$ and $d(y, Tx) = y$. We now have

$$d(Tx, Ty) \leq \min\{d(x, Ty), d(y, Tx)\}.$$

It now follows from Theorem 3.7 that T has a fixed point. It is interesting to note that the existence of fixed point for the mapping T cannot be obtained from Theorem 2.12 due to Takahashi because of the mapping T is not continuous.

We now raise the next problem.

Question 3.1. It is interesting to ask whether Theorem 3.4 holds whenever T is noncyclic relatively C-nonexpansive.

4. Additional results

In this section, we study the structure of minimal sets of strongly noncyclic relatively C-nonexpansive in the setting of convex metric spaces. We begin our main results of this section with the following existence theorem.

Theorem 4.1. *Let (A, B) be a nonempty, bounded, closed and convex pair in a convex metric space $(X, d, \mathcal{W})$ such that X has the properties (C) and (D). Assume that $T : A \cup B \to A \cup B$ is a noncyclic mapping such that*

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)] + (1 - 2\alpha)dist(A, B),$$

for some $\alpha \in (0, \frac{1}{2})$ and for all $(x, y) \in A \times B$. Then T has a best proximity pair.

Proof. By using Zorn's Lemma and by the fact that X has the property (C), we obtain a nonempty, closed and convex pair (K_1, K_2) in X which is minimal with respect to being invariant under noncyclic mapping T. So, we must have $K_1 = \overline{cov}(T(K_1))$ and $K_2 = \overline{cov}(T(K_2))$. Take $a \in K_1$. Then $K_2 \subseteq \mathcal{B}(a; \delta_a(K_2))$. Now, if $y \in K_2$ then

$$\begin{aligned} d(Ta, Ty) &\leq \alpha[d(a, Ty) + d(Ty, a)] + (1 - 2\alpha)dist(A, B) \\ &\leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B). \end{aligned}$$

Put $\rho := 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B)$. Then $Ty \in \mathcal{B}(Ta; \rho)$ for each $y \in K_2$ and so,

$$K_2 = \overline{cov}(T(K_2)) \subseteq \mathcal{B}(Ta; \rho),$$

which deduces that

$$\delta_{Ta}(K_2) \leq \rho, \quad \forall a \in K_1.$$

Similar argument concludes that for each $b \in K_2$ we have $\delta_{Tb}(K_1) \leq \rho$. Set

$$E_1 := \{x \in K_1 : \delta_x(K_2) \leq \rho\}, \quad E_2 := \{y \in K_2 : \delta_y(K_1) \leq \rho\}.$$

Note that $T(K_1) \subseteq E_1$ and $T(K_2) \subseteq E_2$. On the other hand, it is easy to verify that

$$E_1 = [\bigcap_{y \in K_2} \mathcal{B}(y; \rho)] \cap K_1, \quad E_2 = [\bigcap_{x \in K_1} \mathcal{B}(x; \rho)] \cap K_2,$$

that is, (E_1, E_2) is a closed and convex pair in X. Let $x \in E_1$. Since $\delta_{Tx}(K_2) \leq \rho$, we have $Tx \in E_1$, i.e., $T(E_1) \subseteq E_1$. Similarly, we have $T(E_2) \subseteq E_2$. Hence, T is noncyclic on $E_1 \cup E_2$. Now, by the minimality of (K_1, K_2) we must have $E_1 = K_1$ and $E_2 = K_2$. Thus,

$$\delta_x(K_2) \leq \rho = 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)dist(A, B), \quad \forall x \in K_1.$$

Therefore,

$$\delta(K_1, K_2) = dist(A, B).$$

By the fact that the convex metric space X has the property (D) we conclude that both K_1 and K_2 are singleton. The conclusion then trivially follows.

□

Remark 4.2. Theorem 4.1 holds once the minimal sets K_1 and K_2 have been fixed and the noncyclic mapping $T : A \cup B \to A \cup B$ satisfies the following condition.

$$d(Tx, Ty) \leq r\delta(K_1, K_2) + (1 - r)dist(A, B),$$

for some $r \in (0, 1)$ and for all $(x, y) \in K_1 \times K_2$.

The next lemma guarantees the existence of *diametral pairs* for strongly noncyclic relatively C-nonexpansive mappings.

Lemma 4.3. *Let (A, B) be a nonempty, bounded, closed and convex pair of a convex metric space $(X, d, \mathcal{W})$ such that X has the properties (C) and (D). Let $T\colon A \cup B \to A \cup B$ be a strongly noncyclic relatively C-nonexpansive mapping and let $(K_1, K_2) \subseteq (A, B)$ be a minimal closed and convex pair which is T-invariant and such that $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$. Then each $(x^\star, y^\star) \in K_1 \times K_2$ with $d(x^\star, y^\star) = \operatorname{dist}(A, B)$ is a diametral pair (with respect to (K_1, K_2)), that is,*

$$\delta_{x^\star}(K_2) = \delta_{y^\star}(K_1) = \delta(K_1, K_2).$$

Proof. We have $K_1 = \overline{cov}(T(K_1))$ and $K_2 = \overline{cov}(T(K_2))$. Assume that there exists a pair $(x^\star, y^\star) \in K_1 \times K_2$ with $d(x^\star, y^\star) = \operatorname{dist}(A, B)$ which is not diametral pair. Then

$$\min\{\delta_{x^\star}(K_2), \delta_{y^\star}(K_1)\} < \delta(K_1, K_2).$$

Suppose that $r_1 := \delta_{x^\star}(K_2) < \delta(K_1, K_2)$ and $r_2 = \delta_{y^\star}(K_1)$. Set

$$\mathcal{C}_{r_1}(K_2) := K_1 \bigcap (\cap_{x \in K_2} \mathcal{B}(Tx; r_1)) \quad \& \quad \mathcal{C}_{r_2}(K_1) := K_2 \bigcap (\cap_{x \in K_1} \mathcal{B}(Tx; r_2)).$$

It is easy to see that $(x, y) \in \mathcal{C}_{r_1}(K_2) \times \mathcal{C}_{r_2}(K_1)$ if and only if

$$K_2 = \overline{cov}(T(K_2)) \subseteq \mathcal{B}(x; r_1), \ K_1 = \overline{cov}(T(K_1)) \subseteq \mathcal{B}(y; r_2).$$

Also, $(x^\star, y^\star) \in \mathcal{C}_{r_1}(K_2) \times \mathcal{C}_{r_2}(K_1)$ and so,

$$dist(\mathcal{C}_{r_1}(K_2), \mathcal{C}_{r_2}(K_1)) = dist(A, B).$$

Moreover, T is noncyclic on $\mathcal{C}_{r_1}(K_2) \cup \mathcal{C}_{r_2}(K_1)$. Indeed, if $p \in \mathcal{C}_{r_1}(K_2)$ then $p \in K_1$ and for each $y \in K_2$ we have $d(p, Ty) \leq r_1$. Since T is strongly noncyclic relatively C-nonexpansive,

$$d(Tp, Ty) \leq \min\{d(p, Ty), d(y, Tp)\} \leq r_1, \quad \forall y \in K_2.$$

Thus,

$$K_2 = \overline{cov}(T(K_2)) \subseteq \mathcal{B}(Tp; r_1),$$

which implies that $Tp \in \mathcal{C}_{r_1}(K_2)$. Then $T(\mathcal{C}_{r_1}(K_2)) \subseteq \mathcal{C}_{r_1}(K_2)$. Similarly, $T(\mathcal{C}_{r_2}(K_1)) \subseteq \mathcal{C}_{r_2}(K_1)$. That is, T is noncyclic on $\mathcal{C}_{r_1}(K_2) \cup \mathcal{C}_{r_2}(K_1)$. It now follows from the minimality of (K_1, K_2) that $K_1 = \mathcal{C}_{r_1}(K_2)$ and $K_2 = \mathcal{C}_{r_2}(K_1)$. Thereby, $K_2 \subseteq \bigcap_{x \in K_1} \mathcal{B}(Tx; r_1)$. Hence,

$$\delta_y(T(K_1)) \leq r_1, \quad \forall y \in K_2.$$

So, $\delta(T(K_1), K_2) \leq r_1$. By using Lemma 2.8 we obtain

$$\delta(K_1, K_2) = \delta(\overline{cov}(T(K_1)), K_2) = \delta(T(K_1), K_2) \leq r_1,$$

which is a contradiction.

□

Definition 4.4. ([6]) Let (A,B) be a nonempty pair in a metric space (X,d) and $T\colon A\cup B\to A\cup B$ a noncyclic mapping. Then a sequence $(\{x_n\},\{y_n\})$ in $A\times B$ is said to be an *approximate best proximity pair sequence* of the noncyclic mapping T provided that

$$\lim_{n\to\infty} d(x_n,Tx_n)=0,\quad \lim_{n\to\infty} d(y_n,Ty_n)=0\quad\text{and}\quad \lim_{n\to\infty} d(x_n,y_n)=dist(A,B).$$

Lemma 4.5. *Let (A,B) be a nonempty, bounded, closed and convex pair of a convex metric space $(X,d,\mathcal{W})$ such that X has the properties (C) and (D). Suppose that A_0 is nonempty and (A,B) is a proximal compactness pair. Let $T\colon A\cup B\to A\cup B$ be a strongly noncyclic relatively C-nonexpansive mapping. Then T has an approximate best proximity pair sequence in $A\times B$.*

Proof. As in the proof of Theorem 3.4, there exists a pair $(K_1,K_2)\subseteq(A,B)$ which is minimal with respect to being nonempty, closed, convex and T invariant and $dist(K_1,K_2)=dist(A,B)$. Also, there exists $(p,q)\in K_1\times K_2$ such that $d(p,q)=dist(K_1,K_2)$. For each $t\in(0,\frac{1}{2})$ define $T_t:K_1\cup K_2\to K_1\cup K_2$ as follows:

$$T_t(x)=\begin{cases}\mathcal{W}(Tx,p,2t);\ x\in K_1,\\ \mathcal{W}(Tx,q,2t);\ x\in K_2.\end{cases}$$

Note that T is noncyclic on $K_1\cup K_2$. Let $r:=4t-4t^2$. Since $t\in(0,\frac{1}{2})$, we conclude $r<1$. Now for each $(x,y)\in K_1\times K_2$ we have

$$\begin{aligned}
d(T_tx,T_ty)&=d(\mathcal{W}(Tx,p,2t),\mathcal{W}(Ty,q,2t))\\
&\le 2td(Tx,\mathcal{W}(Ty,q,2t)+(1-2t)d(p,\mathcal{W}(Ty,q,2t))\\
&\le 2t[2td(Tx,Ty)+(1-2t)d(Tx,q)]+(1-2t)[2td(p,Ty)+(1-2t)d(p,q)]\\
&\le 2t[2t\min\{d(x,Ty),d(y,Tx)\}+(1-2t)d(Tx,q)]\\
&\quad+(1-2t)[2td(p,Ty)+(1-2t)dist(A,B)]\\
&\le 2t[2t\delta(K_1,K_2)+(1-2t)\delta(K_1,K_2)]+(1-2t)[2t\delta(K_1,K_2)+(1-2t)dist(A,B)]\\
&=(4t-4t^2)\delta(K_1,K_2)+(1-(4t-4t^2))dist(A,B)\\
&=r\delta(K_1,K_2)+(1-r)dist(A,B).
\end{aligned}$$

It now follows from Remark 4.2 that T_t has a best proximity pair for each $t\in(0,\frac{1}{2})$, that is, there exists $(x_t,y_t)\in K_1\times K_2$ such that

$$x_t=T_t(x_t),\quad y_t=T_t(y_t)\quad\text{and}\quad d(x_t,y_t)=dist(A,B),\ \forall t\in(0,\frac{1}{2}).$$

We now have

$$\begin{aligned}
d(x_t,Tx_t)&=d(T_t(x_t),Tx_t)\\
&=d(\mathcal{W}(Tx_t,p,2t),Tx_t)\le(1-2t)d(p,Tx_t)\le(1-2t)diam(A).
\end{aligned}$$

This implies that $d(x_t,Tx_t)\to 0$ if $t\to\frac{1}{2}^-$. By a similar way, we can see that $d(y_t,Ty_t)\to 0$ whenever $t\to\frac{1}{2}^-$ and this completes the proof. □

Here, we state the main result of this section.

Theorem 4.6. *Let (A, B) be a nonempty, bounded, closed and convex pair of a convex metric space $(X, d, \mathcal{W})$ such that X has the properties (C) and (D), A_0 is nonempty and (A, B) is a proximal compactness pair. Let $T\colon A\cup B \to A\cup B$ be a strongly noncyclic relatively C-nonexpansive mapping. Suppose that $(K_1, K_2) \subseteq (A, B)$ is a minimal, closed and convex pair which is T invariant and such that $dist(K_1, K_2) = dist(A, B)$ and let $(\{x_n\}, \{y_n\})$ be an approximate best proximity pair sequence in $A \times B$. Then for each $(p, q) \in K_1 \times K_2$ with $d(p, q) = dist(A, B)$ we have*

$$\limsup_{n\to\infty} d(Tx_n, q) = \limsup_{n\to\infty} d(p, Ty_n) = \delta(K_1, K_2).$$

Proof. Lemma 4.5 guarantees that the noncyclic mapping T has an approximate best proximity pair sequence in $A \times B$. By the fact that (A, B) is proximal compactness, there exists a subsequence $(\{x_{n_k}\}, \{y_{n_k}\})$ of the sequence $(\{x_n\}, \{y_n\})$ such that $x_{n_k} \to x^\star$ and $y_{n_k} \to y^\star$ for some $(x^\star, y^\star) \in K_1 \times K_2$. So,

$$d(x^\star, y^\star) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = dist(A, B).$$

It follows from Lemma 4.3 that $(x^\star, y^\star)$ is a diametral pair. Let $(p, q) \in K_1 \times K_2$ be such that $d(p, q) = dist(A, B)$. Put,

$$r_1 := \limsup_{n\to\infty} d(Tx_n, q), \quad r_2 := \limsup_{n\to\infty} d(p, Ty_n).$$

We assert that

$$r_1 = r_2 = \delta(K_1, K_2).$$

Suppose that $r_2 < \delta(K_1, K_2)$ and set

$$L_1 := \{x \in K_1 : \limsup_{n\to\infty} d(x, Ty_n) \le r_1\},$$
$$L_2 := \{y \in K_2 : \limsup_{n\to\infty} d(Tx_n, y) \le r_2\}.$$

Then $(p, q) \in L_1 \times L_2$ and it is easy to see that (L_1, L_2) is a closed pair in X. Moreover, if $x_1, x_2 \in L_1$ and $\alpha \in (0, 1)$, then

$$\limsup_{n\to\infty} d(\mathcal{W}(x_1, x_2, \alpha), Ty_n) \le \limsup_{n\to\infty}[\alpha d(x_1, Ty_n) + (1-\alpha)d(x_2, Ty_n)] \le r_1.$$

Thus, $\mathcal{W}(x_1, x_2, \alpha) \in L_1$, that is, L_1 is convex. Similarly, we can see that L_2 is a convex set. Besides, if $x \in L_1$, then

$$\limsup_{n\to\infty} d(Tx, Ty_n) \le \limsup_{n\to\infty} \min\{d(x, Ty_n), d(y_n, Tx)\}$$
$$\le \limsup_{n\to\infty} d(x, Ty_n) \le r_1,$$

which concludes that $Tx \in L_1$, that is, $T(L_1) \subseteq L_1$. Similarly, we can see that $T(L_2) \subseteq L_2$. Hence, T is noncyclic on $L_1 \cup L_2$. Minimality of (K_1, K_2) deduces that $(K_1, K_2) = (L_1, L_2)$. Then for each $y \in K_2$ we have

$$d(x^\star, Ty) = \lim_{k\to\infty} d(x_{n_k}, Ty)$$
$$\le \limsup_{n\to\infty} d(x_n, Ty) \le \limsup_{n\to\infty}[d(x_n, Tx_n) + d(Tx_n, Ty)]$$
$$= \limsup_{n\to\infty} d(Tx_n, Ty) \le \limsup_{n\to\infty} \min\{d(x_n, Ty), d(y, Tx_n)\}$$

$$\leq \limsup_{n\to\infty} d(y, Tx_n) \leq r_2.$$

Hence, $\delta_{x^\star}(T(K_2)) \leq r_2$. Now, by using Lemma 2.9 we conclude that

$$\delta_{x^\star}(K_2) = \delta_{x^\star}(\overline{cov}(T(K_2))) = \delta_{x^\star}(T(K_2)) \leq r_2 < \delta(K_1, K_2),$$

which is a contradiction by the fact that $(x^\star, y^\star)$ is a diametral pair. By the similar way, we can see that if $r_1 < \delta(K_1, K_2)$, then we get a contradiction.

□

The following corollary is immediate from the proof of Theorem 4.6.

Corollary 4.7. *Under the conditions of Theorem 4.6 if, in addition, the sequence $\{x_n\}$ is converges to $x^\star \in A$, then T has a best proximity pair.*

Proof. By Theorem 4.6 we have

$$\delta(K_1, K_2) = \lim_{n\to\infty} d(Tx_n, y^\star) \leq \lim_{n\to\infty} [d(Tx_n, x_n) + d(x_n, y^\star)]$$

$$= d(x^\star, y^\star) = dist(K_1, K_2).$$

Now, by the fact that the convex metric space X has the property (D), we conclude that K_1 and K_2 are singleton and the result follows.

□

Corollary 4.8. *Let (A, B) be a nonempty, weakly compact and convex pair of a strictly convex Banach space X. Let $T\colon A \cup B \to A \cup B$ be a strongly noncyclic relatively C-nonexpansive mapping. Suppose that $(K_1, K_2) \subseteq (A, B)$ is a minimal, closed and convex pair which is T invariant and such that $dist(K_1, K_2) = dist(A, B)$ and let $(\{x_n\}, \{y_n\})$ be an approximate best proximity pair sequence in $A \times B$. Then for each $(p, q) \in K_1 \times K_2$ with $d(p, q) = dist(A, B)$ we have*

$$\limsup_{n\to\infty} d(Tx_n, q) = \limsup_{n\to\infty} d(p, Ty_n) = \delta(K_1, K_2).$$

The next corollary is similar to the classical *Goebel–Karlovitz Lemma* which is a key lemma in fixed point theory.

Corollary 4.9. *Let A be a nonempty, bounded, closed and convex subset of a convex metric space $(X, d, \mathcal{W})$ and let $T : A \to A$ be a strongly C-nonexpansive mapping. Assume that K is a subset of A which is minimal with respect to being nonempty, closed, convex and T-invariant, and suppose $\{x_n\}$ is a sequence in K such that*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Then, for each $x \in K$, $\lim_{n\to\infty} d(x, Tx_n) = \operatorname{diam}(K)$.

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