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STRONG COEFFICIENT QUANTIZATION PROPERTIES IN BANACH SPACES

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To the memory of Professor Edward Odell

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ABSTRACT. We prove that a dictionary for a Banach space X has the strong coefficient quantization property if it has the same property when it restricted on the unit ball of X. We also obtain the same result for the strong net quantization property.

1. INTRODUCTION AND THE MAIN RESULTS

Dilworth, Odell, Schlumprecht and Zsák [2] introduced and investigated two natural coefficient quantization properties in Banach spaces. For $\varepsilon > 0$ and $\delta > 0$, a dictionary (x_i) for a Banach space X, which means $X = [(x_i)] := \overline{\text{span}}(x_i)$, is said to have the (ε, δ)-*coefficient quantization property* ((ε, δ)-CQP) if

$$\mathcal{F}_\delta((x_i)_{i\in F}) := \Big\{ \sum_{i\in F} n_i \delta x_i : n_i \in \mathbb{Z} \Big\} \text{ is } \varepsilon\text{-dense in } [(x_i)_{i\in F}]$$

for every finite $F \subset \mathbb{N}$. We say that (x_i) has the CQP if (x_i) has the (ε, δ)-CQP for some $\varepsilon > 0$ and $\delta > 0$. The following second notion is more general than the CQP. A dictionary (x_i) is said to have the (ε, δ)-*net quantization property*

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(((ε,δ)-NQP) if

$$\mathcal{F}_\delta((x_i)) := \Big\{ \sum_{i\in E} n_i\delta x_i : E\subset\mathbb{N} \text{ finite},\ n_i\in\mathbb{Z}\Big\} \text{ is } \varepsilon\text{-dense in } [(x_i)_{i\in F}]$$

for every finite $F\subset\mathbb{N}$. We say that (x_i) has the NQP if (x_i) has the (ε,δ)-NQP for some $\varepsilon>0$ and $\delta>0$. The difference of the two concepts is the support of the approximants.

One of the main results in [2] is to relax the definitions above by only requiring that one can approximate each element of the unit ball instead of the whole space. In the paper, we extend the result to strong quantizatioin versions which were also introduced in [2]. A dictionary (x_i) is said to have the (ε,δ)-*strong coefficient quantization property* (((ε,δ)-SCQP) if

$$\mathcal{F}_{\widehat{D}}((x_i)_{i\in F}) := \Big\{ \sum_{i\in F} d_i x_i : d_i\in D_i\Big\} \text{ is } \varepsilon\text{-dense in } [(x_i)_{i\in F}]$$

for every sequence $\widehat{D} := (D_i)$ of δ-nets for $\mathbb{R}$ with $0\in D_i$ and for every finite $F\subset\mathbb{N}$. The (ε,δ)-*strong net quantization property* (((ε,δ)-SNQP) is similarly defined by replacing the set of the approximants by

$$\mathcal{F}_{\widehat{D}}((x_i)) := \Big\{ \sum_{i\in E} d_i x_i : E\subset\mathbb{N} \text{ finite},\ d_i\in D_i\Big\}.$$

The (ε,δ)-SCQP (resp. (ε,δ)-SNQP) implies the $(\varepsilon,2\delta)$-CQP (resp. $(\varepsilon,2\delta)$-NQP) because $2\delta\mathbb{Z}$ is a δ-net. But it is open whether or not the CQP (resp. NQP) implies the SCQP (resp. SNQP) [2, Problem 2.14(1), Question 7.1(2)].

We need the similar notations as in [2] for the SCQP and the SNQP to state our results. Suppose that (x_i) has the SCQP (resp. SNQP). The function ε_{sc} (resp. ε_{sn}) : $\mathbb{R}^+\to\mathbb{R}^+$ is defined by

$$\varepsilon_{sc}(\delta) \text{ (resp. } \varepsilon_{sn}(\delta)) = \inf\{\gamma : (x_i) \text{ has the } (\gamma,\delta)\text{-SCQP (resp. SNQP)}\}.$$

Then by the analogue for the SCQP and the SNQP of [2, Proposition 2.3(a)] the functions ε_{sc} and ε_{sn} are well defined.

For each $\delta>0$, we also define $\varepsilon^b_{sc}(\delta)$ (resp. $\varepsilon^b_{sn}(\delta)$) to be the infimum of those $\gamma>0$ such that

$$\mathcal{F}_{\widehat{D}}((x_i)_{i\in F}) \text{ (resp. } \mathcal{F}_{\widehat{D}}((x_i))) \text{ is } \gamma\text{-dense in } \mathrm{Ball}([(x_i)_{i\in F}])$$

for every sequence (D_i) of δ-nets for $\mathbb{R}$ with $0\in D_i$ and for every finite $F\subset\mathbb{N}$.

We now have:

Theorem 1.1. *Let (x_i) be a dictionary for X. The following assertions are equivalent.*
(a) *(x_i) has the SCQP.*
(b) *$\varepsilon^b_{sc}(\delta_0)<1$ for some $\delta_0>0$.*
(c) *There exists a $\delta_0>0$ such that $\varepsilon_{sc}(\delta)=\varepsilon^b_{sc}(\delta)<\infty$ for all $0<\delta\le\delta_0$.*

Theorem 1.2. *Let (x_i) be a dictionary for X. The following assertions are equivalent.*
(a) *(x_i) has the SNQP.*
(b) *$\varepsilon^b_{sn}(\delta_0)<1$ for some $\delta_0>0$.*

(c) *There exists a* $\delta_0 > 0$ *such that* $\varepsilon_{sn}(\delta) = \varepsilon^b_{sn}(\delta) < \infty$ *for all* $0 < \delta \le \delta_0$.

Theorem 1.1, which gives an affirmative answer of [2, Problem 2.14(2)], and Theorem 1.2, respectively, are strong quantization versions of [2, Theorems 2.4 and 5.3]. The basic arguments of the proofs of Theorems 1.1 and 1.2 are the same with the one of [2, Theorem 2.4]. Since the proof of Theorem 1.2 is slightly different from the one of Theorem 1.1, we only prove Theorem 1.2 among them. In view of the proof of Theorem 1.2, we remark that the $\delta_0 > 0$ in (c) of Theorem 1.2 (resp. Theorem 1.1) is the same as the $\delta_0 > 0$ in (b) of that (resp. Theorem 1.1).

The NQP, even the SNQP, does not imply the CQP (see [2, Theorem 5.10]). But, in [2, Theorem 5.11], it was shown that a semi-normalized basis has the CQP if and only if its every subsequence has the NQP for the closed linear span of the subsequence. We obtain the same result for the SCQP and the SNQP.

Theorem 1.3. *Let* (x_i) *be a semi-normalized basis for* X. *Then* (x_i) *has the SCQP if and only if every subsequence* (x_{i_k}) *of* (x_i) *has the SNQP for* $[(x_{i_k})]$.

2. Proofs of Theorems 1.2 and 1.3

We note that $\varepsilon^b_{sn} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function.

Proof of Theorem 1.2. (a)⇒(b) and (c)⇒(a) are clear.

(b)⇒(c) Choose a $t > 0$ such that

$$\varepsilon^b_{sn}(\delta) \le \varepsilon^b_{sn}(\delta_0) < t < 1$$

for all $0 < \delta \le \delta_0$.

Now assume that $0 < \delta, \bar{\delta} \le \delta_0$ satisfy

$$t \le \frac{\delta}{\bar{\delta}} \le \frac{1}{t}.$$

Let $\alpha > 0$ be arbitrary. Let (D_i) be a sequence of δ-nets for $\mathbb{R}$ with $0 \in D_i$ and let $F \subset \mathbb{N}$ be finite. Let $x \in \mathrm{Ball}([(x_i)_{i\in F}])$. Then there exists a $y = \sum_{i\in E} d_i x_i \in \mathcal{F}_{\widehat{D}}((x_i))$ such that $\|x - y\| \le t$. We see that $\|(\bar{\delta}/\delta)(x-y)\| \le 1$ and $(D'_i) = ((\bar{\delta}/\delta)(D_i - d_i))$, where $d_i = 0$ if $i \notin E$, is a sequence of $\bar{\delta}$-nets for $\mathbb{R}$ with $0 \in D'_i$. Then there exists a $z = \sum_{i\in E_0} d'_i x_i \in \mathcal{F}_{\widehat{D'}}((x_i))$ such that $\|(\bar{\delta}/\delta)(x-y) - z\| \le (1+\alpha)\varepsilon^b_{sn}(\bar{\delta})$. It follows that

$$\left\| x - \left(y + \frac{\delta}{\bar{\delta}} z \right) \right\| \le (1+\alpha)\frac{\delta}{\bar{\delta}} \varepsilon^b_{sn}(\bar{\delta}).$$

To show that $y + (\delta/\bar{\delta})z \in \mathcal{F}_{\widehat{D}}((x_i))$, put

$$z = \sum_{i\in E_0\cap E} d'_i x_i + \sum_{i\in E_0\setminus E} d'_i x_i = \sum_{i\in E_0\cap E} \frac{\bar{\delta}}{\delta}(c_i - d_i) x_i + \sum_{i\in E_0\setminus E} \frac{\bar{\delta}}{\delta} c_i x_i,$$

where $c_i \in D_i$. Then

$$\begin{aligned}
&y + \frac{\delta}{\bar{\bar{\delta}}} z \\
&= \sum_{i\in E} d_i x_i + \frac{\delta}{\bar{\delta}}\Big(\sum_{i\in E_0\cap E} \frac{\bar{\delta}}{\delta}(c_i - d_i)x_i + \sum_{i\in E_0\setminus E} \frac{\bar{\delta}}{\delta} c_i x_i \Big) \\
&= \sum_{i\in E\setminus E_0} d_i x_i + \sum_{i\in E_0} c_i x_i \in \mathcal{F}_{\widehat{D}}((x_i)).
\end{aligned}$$

Thus $\varepsilon^b_{sn}(\delta) \le (1+\alpha)(\delta/\bar{\delta})\varepsilon^b_{sn}(\bar{\delta})$. Since $\alpha > 0$ was arbitrary, we have

$$\frac{\varepsilon^b_{sn}(\delta)}{\delta} \le \frac{\varepsilon^b_{sn}(\bar{\delta})}{\hat{\delta}}.$$

By exchanging the roles of δ and $\bar{\delta}$ we also obtain the opposite inequality. We have shown that

$$\frac{\varepsilon^b_{sn}(\delta)}{\delta} = \frac{\varepsilon^b_{sn}(\bar{\delta})}{\hat{\delta}}$$

for every $0 < \delta, \bar{\delta} \le \delta_0$ satisfying $t \le \delta/\bar{\delta} \le 1/t$. A simple verification shows that ε^b_{sn} is linear on $(0, \delta_0]$.

Now, in order to complete the proof, let $\delta \in (0, \delta_0]$. Let $\alpha > 0$ be arbitrary. Let (D_i) be a sequence of δ-nets for $\mathbb{R}$ with $0 \in D_i$ and let $F \subset \mathbb{N}$ be finite. Let $x \in [(x_i)_{i\in F}]$. If $\|x\| \ge 1$, then $(D_i/\|x\|)$ is a sequence of $\delta/\|x\|$-nets for $\mathbb{R}$ with $0 \in D_i/\|x\|$ and $\delta/\|x\| \le \delta_0$. Then there exists a $\sum_{i\in E}(d_i/\|x\|)x_i \in \mathcal{F}_{\widehat{D}/\|x\|}((x_i))$ such that

$$\Big\| \frac{x}{\|x\|} - \sum_{i\in E} \frac{d_i}{\|x\|} x_i \Big\| \le (1+\alpha)\varepsilon^b_{sn}\Big(\frac{\delta}{\|x\|}\Big) = (1+\alpha)\frac{1}{\|x\|}\varepsilon^b_{sn}(\delta).$$

It follows that

$$\Big\| x - \sum_{i\in E} d_i x_i \Big\| \le (1+\alpha)\varepsilon^b_{sn}(\delta).$$

If $\|x\| \le 1$, then clearly we can find a $(1+\alpha)\varepsilon^b_{sn}(\delta)$-approximant of x. Thus $\varepsilon_{sn}(\delta) \le (1+\alpha)\varepsilon^b_{sn}(\delta)$. Since $\alpha > 0$ was arbitrary, we complete the proof. □

The following corollary is a strong quantization version of [2, Corollary 2.5].

Corollary 2.1. *Let (x_i) be a dictionary for X. Let $0 < \varepsilon_0 < 1$ and $\delta_0 > 0$. If for every sequence (D_i) of δ_0-nets for $\mathbb{R}$ with $0 \in D_i$ and for every finite $F \subset \mathbb{N}$ we have*

$$\mathcal{F}_{\widehat{D}}((x_i)) \text{ is } \varepsilon_0\text{-dense in } Ball([(x_i)_{i\in F}]),$$

then for all $\varepsilon > \varepsilon_0$,

$$\mathcal{F}_{\widehat{D}}((x_i)) \text{ is } \varepsilon\text{-dense in } [(x_i)_{i\in F}]$$

for every sequence (D_i) of δ_0-nets for $\mathbb{R}$ with $0 \in D_i$ and for every finite $F \subset \mathbb{N}$.

Proof. By the assumption

$$\varepsilon^b_{sn}(\delta) \le \varepsilon^b_{sn}(\delta_0) \le \varepsilon_0 < 1$$

for all $0 < \delta \leq \delta_0$. We can apply the proof of Theorem 1.2 replacing $t > 0$ by $\varepsilon_0 < \varepsilon < 1$ to show that $\varepsilon_{sn}(\delta) \leq \varepsilon_{sn}^b(\delta)$ for all $0 < \delta \leq \delta_0$. In particular, $\varepsilon_{sn}(\delta_0) \leq \varepsilon_{sn}^b(\delta_0) \leq \varepsilon_0$. Hence we obtain the desired conclusion. □

Remark 2.2. We can adapt the proof of Theorem 1.2 to show Theorem 1.1 and also obtain the analogue for Theorem 1.1 of Corollary 2.1 by replacing $\mathcal{F}_{\widehat{D}}((x_i))$ by $\mathcal{F}_{\widehat{D}}((x_i)_{i\in F})$.

Proof of Theorem 1.3. Since the SCQP is inherited by subsequences, the "only if" part is clear. In order to show the "if" part, assume that (x_i) does not have the SCQP. We use the argument of the proof in [2, Theorem 5.11] to find a subsequence of (x_i) failing to have the SNQP for its closed linear span.

Let K be the basis constant of (x_i) and we may assume that $\|x_i\| \leq 1$ for all i.

Step 1. For every $\delta > 0$, there exists $M_\delta \subset \mathbb{N}$ such that $(x_i)_{i\in M_\delta}$ does not have the $(1, \delta)$-SNQP.

See the proof of Claim 1 in that of [2, Theorem 5.11] for the proof of Step 1.

Step 2. For every $n \in \mathbb{N}$, there exists a sequence $(D_{i,n})_i$ of $1/n$-nets for $\mathbb{R}$ with $0 \in D_i$ such that for some finite $F_n \subset [n+1, \infty)$ and some $y_n \in [(x_i)_{i\in F_n}]$, we have $\|y - y_n\| > 2K$ for all $y \in \mathcal{F}_{\widehat{D_n}}((x_i)_{i\in F_n})$.

Proof of Step 2. Fix $n \in \mathbb{N}$. By Step 1 there exists $M_n \subset \mathbb{N}$ such that $(x_i)_{i\in M_n}$ does not have the $(2K+1, 1/n)$-SNQP. Thus there exists a sequence (D_i) of $1/n$-nets for $\mathbb{R}$ with $0 \in D_i$ such that for some finite $E_n \subset M_n$ and $z_n = \sum_{i\in E_n} a_i x_i$, we have $\|y - z_n\| > 2K + 1$ for all $y \in \mathcal{F}_{\widehat{D}}((x_i)_{i\in M_n})$.

Put $w_n = z_n|_{[1,n]}$, $y_n = z_n|_{[n+1,\infty)}$, and $F_n = E_n \cap [n+1, \infty)$. For $i \in E_n \cap [1, n]$, choose $d_i \in D_i$ so that $|a_i - d_i| \leq 1/n$. Let $y_0 = \sum_{i\in[1,n]\cap E_n} d_i x_i$. Then $\|w_n - y_0\| \leq 1$, hence we have that for all $y \in \mathcal{F}_{\widehat{D}}((x_i)_{i\in F_n})$

$$\begin{aligned}
&\|y - y_n\| \\
&= \|y_0 + y - z_n + z_n - y_n - y_0\| \\
&\geq \|y_0 + y - z_n\| - \|w_n - y_0\| > 2K + 1 - 1 = 2K.
\end{aligned}$$

For each $k = 0, 1, 2, \cdots$, let $(D_{i,n_k})_i$ be the sequence of $1/n_k$-nets, finite $F_{n_k} \subset [n_k + 1, \infty)$, and $y_{n_k} \in [(x_i)_{i\in F_{n_k}}]$ in Step 2, where $n_0 = 1$ and $\max F_{n_k} = n_{k+1}$. Put $M = \bigcup_{k\geq 0} F_{n_k}$. Then we have

Step 3. $(x_i)_{i\in M}$ does not have the SNQP.

Proof of Step 3. Suppose that $(x_i)_{i\in M}$ has the (ε, δ)-SNQP for some $\varepsilon > 0$ and $\delta > 0$. Then we see that for some $\delta_0 > 0$ $(x_i)_{i\in M}$ has the $(1, 1/n)$-SNQP for all $n \in \mathbb{N}$ with $1/n < \delta_0$.

Choose k so that $1/n_k < \delta_0$. Then there exists a $y \in \mathcal{F}_{\widehat{D_{n_k}}}((x_i)_{i\in M})$ such that $\|y - y_{n_k}\| \leq 1$. Thus

$$\|P_{F_{n_k}} y - y_{n_k}\| = \|P_{F_{n_k}}(y - y_{n_k})\| \leq 2K,$$

where $P_{F_{n_k}}$ is the projection from X onto $[(x_i)_{i\in F_{n_k}}]$. This is a contradiction because $P_{F_{n_k}} y \in \mathcal{F}_{\overline{D_{n_k}}}((x_i)_{i\in F_{n_k}})$. □

3. Bounded strong coefficient quantizations

In [1, Definition 4.1], a bounded version, which is associated to frames for Banach spaces, of the NQP was introduced. We introduce a stronger notion of that as in the previous concept. Let $\hat{z} := (z_i)$ be a sequence in a Banach space Z. For $\varepsilon > 0$, $\delta > 0$ and $K_{\hat{z}} > 0$, a dictionary (x_i) for X is said to have the $(\varepsilon, \delta, K_{\hat{z}})$-*strong net quantization property* ($(\varepsilon, \delta, K_{\hat{z}})$-SNQP) if for every sequence (D_i) of δ-nets for $\mathbb{R}$ with $0 \in D_i$, for every finite $F \subset \mathbb{N}$ and $x \in [(x_i)_{i\in F}]$, there exists a $\sum_{i\in E} d_i x_i \in \mathcal{F}_{\widehat{D}}((x_i))$ such that

$$\Big\|\sum_{i\in E} d_i z_i\Big\|_Z \leq K_{\hat{z}}\|x\| \quad \text{and} \quad \Big\|x - \sum_{i\in E} d_i x_i\Big\|_X \leq \varepsilon.$$

We similarly define the $(\varepsilon, \delta, K_{\hat{z}})$- *strong coefficient quantization property* ($(\varepsilon, \delta, K_{\hat{z}})$-SCQP) replacing the set $\mathcal{F}_{\widehat{D}}((x_i))$ by $\mathcal{F}_{\widehat{D}}((x_i)_{i\in F})$.

We now obtain a strong quantization version of [1, Proposition 4.2].

Theorem 3.1. *Let (z_i) be a sequence in Z and (x_i) a dictionary for X. For $0 < \varepsilon_0 < 1$, $\delta_0 > 0$, and $K_0 > 0$, if for every sequence (D_i) of δ_0-nets for $\mathbb{R}$ with $0 \in D_i$, for every finite $F \subset \mathbb{N}$ and $x \in$ Ball$([(x_i)_{i\in F}])$, there exists a $\sum_{i\in E} d_i x_i \in \mathcal{F}_{\widehat{D}}((x_i))$ such that*

$$\Big\|\sum_{i\in E} d_i z_i\Big\| \leq K_0 \quad \textit{and} \quad \Big\|x - \sum_{i\in E} d_i x_i\Big\| \leq \varepsilon_0,$$

then (x_i) has the $(1, \delta_0, K_{\hat{z}})$-SNQP, where $K_{\hat{z}} = K_0 \sum_{n=0}^{\infty} \varepsilon_0^n$.

Proof. This proof is very similar to the proof of [1, Proposition 4.2]. First, for every $n \in \mathbb{N}$ and $\delta \leq \varepsilon_0^{n-1}\delta_0$, we assert that for every sequence (D_i) of δ-nets for $\mathbb{R}$ with $0 \in D_i$, for every finite $F \subset \mathbb{N}$ and $x \in \mathrm{Ball}([(x_i)_{i\in F}])$, there exists a $\sum_{i\in E} d_i x_i \in \mathcal{F}_{\widehat{D}}((x_i))$ such that

$$\Big\|\sum_{i\in E} d_i z_i\Big\| \leq K_0 \sum_{k=0}^{n-1} \varepsilon_0^k \quad \text{and} \quad \Big\|x - \sum_{i\in E} d_i x_i\Big\| \leq \varepsilon_0^n.$$

From our assumption, the assertion follows for the case $n = 1$.

Now we assume that the assertion is true for $n \in \mathbb{N}$. Let $\delta \leq \varepsilon_0^n \delta_0$. Let (D_i) be a sequence of δ-nets for $\mathbb{R}$ with $0 \in D_i$ and let $F \subset \mathbb{N}$ be finite and $x \in \mathrm{Ball}([(x_i)_{i\in F}])$. Since $\delta \leq \varepsilon_0^{n-1}\delta_0$, we can choose $\sum_{i\in E} \tilde{d}_i x_i \in \mathcal{F}_{\widehat{D}}((x_i))$ such that

$$\Big\|\sum_{i\in E} \tilde{d}_i z_i\Big\| \leq K_0 \sum_{k=0}^{n-1} \varepsilon_0^k \quad \text{and} \quad \Big\|x - \sum_{i\in E} \tilde{d}_i x_i\Big\| \leq \varepsilon_0^n.$$

Since $\varepsilon_0^{-n}\|x-\sum_{i\in E}\tilde{d}_i x_i\| \le 1$ and $(\varepsilon_0^{-n}(D_i-\tilde{d}_i))$ ($\tilde{d}_i=0$ if $i\notin E$) is a sequence of δ_0-nets for $\mathbb{R}$ with $0\in\varepsilon_0^{-n}(D_i-\tilde{d}_i)$, by our assumption, for some finite $E'\subset\mathbb{N}$ we can find a $d_i'\in\varepsilon_0^{-n}(D_i-\tilde{d}_i)$ for $i\in E'$ ($d_i'=0$ if $i\notin E'$)such that

$$\Big\|\sum_{i\in E'} d_i' z_i\Big\| \le K_0 \quad\text{and}\quad \Big\|\varepsilon_0^{-n}\Big(x-\sum_{i\in E}\tilde{d}_i x_i\Big)-\sum_{i\in E'} d_i' x_i\Big\| \le \varepsilon_0.$$

Hence we have that

$$\Big\|\sum_{i\in E\cup E'}(\tilde{d}_i+\varepsilon_0^n d_i')z_i\Big\| \le K_0\sum_{k=0}^{n}\varepsilon_0^k$$

and

$$\Big\|x-\sum_{i\in E\cup E'}(\tilde{d}_i+\varepsilon_0^n d_i')x_i\Big\| \le \varepsilon_0^{n+1}.$$

As in the proof of Theorem 1.2, we see that $\tilde{d}_i+\varepsilon_0^n d_i'\in D_i$ for $i\in E\cup E'$, which completes our assertion.

In order to complete the proof, let (D_i) be a sequence of δ_0-nets for $\mathbb{R}$ with $0\in D_i$ and let $F\subset\mathbb{N}$ be finite and $x\in[(x_i)_{i\in F}]$.

If $\|x\|\ge 1$, then choose an $n\in\mathbb{N}$ so that $\varepsilon_0^n < 1/\|x\| \le \varepsilon_0^{n-1}$. From our assertion, we can find a $\sum_{i\in E}(d_i/\|x\|)x_i\in\mathcal{F}_{\widehat{D}/\|x\|}((x_i))$ such that

$$\Big\|\sum_{i\in E}\frac{d_i}{\|x\|}z_i\Big\| \le K_0\sum_{k=0}^{n-1}\varepsilon_0^k \le K_z \quad\text{and}\quad \Big\|\frac{x}{\|x\|}-\sum_{i\in E}\frac{d_i}{\|x\|}x_i\Big\| \le \varepsilon_0^n.$$

Hence

$$\Big\|\sum_{i\in E} d_i z_i\Big\| \le K_z\|x\| \quad\text{and}\quad \Big\|x-\sum_{i\in E} d_i x_i\Big\| \le \varepsilon_0^n\|x\| \le 1.$$

If $\|x\|<1$, then take $d_i=0$ for all $i\in F$. □

Remark 3.2. We can also obtain the analogue for the bounded version of SCQP of Theorem 3.1 by replacing $\mathcal{F}_{\widehat{D}}((x_i))$ by $\mathcal{F}_{\widehat{D}}((x_i)_{i\in F})$.

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