Banach J. Math. Anal. 8 (2014), no. 2, 16–29

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

MAXIMAL IDEAL SPACE OF SOME BANACH ALGEBRAS AND RELATED PROBLEMS

SUNA SALTAN[1]* AND YASEMİN ÖZEL[2]

Communicated by M. Abel

ABSTRACT. Let $C_A^{(n)} := C_A^{(n)}(\mathbb{D}\times\mathbb{D})$ denote the subspace of functions in the Banach space $C^{(n)}\left(\overline{\mathbb{D}\times\mathbb{D}}\right)$ which are analytic in the bi-disc $\mathbb{D}\times\mathbb{D}$. We consider the subspace B_{zw} consisting from the functions $f \in C_A^{(n)}$ which can be represented in the form $f(z,w) = g(zw)$, where g is a single variable function from the disc algebra $C_A(\mathbb{D})$. We prove that B_{zw} is a Banach algebra under the Duhamel multiplication

$$(f \circledast g)(zw) = \frac{\partial^2}{\partial z \partial w}\int_0^z\int_0^w f((z-u)(w-v))\, g(uv)\, dv du$$

and describe its maximal ideal space. We also consider the Hardy type operator $f \to xy\int_0^x\int_0^y f(t\tau)\, d\tau dt$ and discuss its some properties.

1. Introduction

Let $\mathcal{B}$ be a Banach algebra. Recall that (see Rickart [9]) the radical $\mathcal{R}$ of an algebra $\mathcal{B}$ is equal to the intersection of the kernel of all (strictly) irreducible representations of $\mathcal{B}$. If $\mathcal{R} = \{0\}$, then $\mathcal{B}$ is said to be semi-simple and, if $\mathcal{R} = \mathcal{B}$, then $\mathcal{B}$ is called a radical algebra. Equivalently, $\mathcal{B}$ is a radical Banach algebra, if for every element $b \in \mathcal{B}$ the multiplication operator M_b, $M_b a := ba$ $(a \in \mathcal{B})$, is a quasinilpotent operator on $\mathcal{B}$ (i.e., $\sigma(M_b) = \{0\}$).

Date: Received: Apr. 5, 2013; Revised: Jun. 18, 2013; Accepted: Aug. 7, 2013.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 47B47; Secondary 47B38, 46E35.

Key words and phrases. Banach algebra, radical Banach algebra, Duhamel multiplication, quasinilpotent operator, invariant subspace.

The classical example to the radical Banach algebras is the disc algebra $\mathcal{A}(\mathbb{D})$ under a different multiplication defined in terms of a convolution

$$(f * g)(z) := \int_0^z f(z-t)g(t)dt,$$

where $|z| \leq 1$ and the integral is taken over any Jordan arc which (except possibly for z) lies entirely within the interior of the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (Recall that the norm of disc algebra $\mathcal{A}(\mathbb{D})$ is defined by $\|f\|_{\mathcal{A}(\mathbb{D})} := \sup_{z \in \overline{\mathbb{D}}} |f(z)|$). With this definition of multiplication, it is not difficult to prove that

$$\|f^{*n}\| \leq \frac{\|f\|^n}{(n-1)!},$$

which implies that $\lim\limits_{n\to\infty} \|f^n\|^{1/n} = 0$, that is f is quasinilpotent, and therefore $\mathcal{A}(\mathbb{D})$ is a radical algebra under convolution multiplication $*$ (see, for example, Rickart [9, p. 316] and Hille–Phillips [4, p. 701]).

The next example is the class $L^1(0,1)$ of all complex-valued functions f which are absolutely continuous on $[0,1]$. Under the ordinary definitions of addition and multiplication by scalars and the norm

$$\|f\| = \int_0^1 |f(x)|\, dx,$$

$L^1(0,1)$ is a Banach space. It becomes a Banach algebra under the convolution multiplication

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt.$$

As in the preceding example, the following is true,

$$\|f^{*n}\| \leq \frac{\max \|f(x)\|^n}{n!},$$

so that $L^1(0,1)$ is a radical algebra (more detailly see for instance, Rickart [9] and Gelfand, Raikov and Shilov [3, p. 118]).

As is known (see, for example, Rickart [9], Dunford and Schwartz [2], Gelfand, Raikov and Shilov [3]), description of the maximal ideal space is well studied for the radical Banach algebras. However, for the "non-radical Banach algebras" with respect to the Duhamel multiplication the same questions, apparently, are not completely investigated. Some particular results are contained in Karaev [5, 6], Karaev and Tuna [7, 8].

In the present paper, we consider a concrete non-radical Banach function algebra with multiplication as the Duhamel multiplication and describe its maximal ideals, and thus characterize its maximal ideal space. Namely, we consider a Banach algebra $C^{(n)}(\mathbb{D} \times \mathbb{D})\,(n \geq 2)$ consisting of the complex-valued functions f that are continuous on $\overline{\mathbb{D} \times \mathbb{D}}$ and have nth partial derivatives in $\mathbb{D} \times \mathbb{D}$ which can be extended to functions continuous on $\overline{\mathbb{D} \times \mathbb{D}}$. Let $C_A^{(n)} = C_A^{(n)}(\mathbb{D} \times \mathbb{D})$ denote the subspace of functions in $C^{(n)}\left(\overline{\mathbb{D}\times\mathbb{D}}\right)$ which are analytic in $\mathbb{D}\times\mathbb{D}$, that is $C_A^{(n)} = C^{(n)}\left(\overline{\mathbb{D}\times\mathbb{D}}\right)\cap Hol\left(\mathbb{D}\times\mathbb{D}\right).$ The Duhamel multiplication in the space $C_A^{(n)}$ is defined by

$$\left(f\circledast g\right)(z,w)=\frac{\partial^2}{\partial z\partial w}\int_0^z\int_0^w f\left(z-u,w-v\right)g\left(u,v\right)dvdu. \tag{1.1}$$

It is well known (and easy to verify) that this multiplication $\circledast$ has a closure property (i.e., $f\circledast g\in C_A^{(n)}$ for all $f,g\in C_A^{(n)}$), and also has a commutativity and associativity property. Let B_{zw} denote the subspace consisting from the functions $f\in C_A^{(n)}$, which can be represented in the form $f\left(z,w\right)=g\left(zw\right),$ where g is a single variable function from the disc algebra $C_A\left(\mathbb{D}\right).$ Since B_{zw} is a closed subspace of a Banach space $C_A^{(n)}$, B_{zw} is also a Banach space. The norm in B_{zw} is defined by the formula

$$\left\|f\right\|_n:=2\max\left\{\max_{(z,w)\in\overline{\mathbb{D}\times\mathbb{D}}}\left|\frac{\partial^{|\alpha|}f\left(zw\right)}{\partial z^{\alpha_1}\partial w^{\alpha_2}}\right|:|\alpha|=\alpha_1+\alpha_2=0,1,\cdots,n\right\}. \tag{1.2}$$

We prove that B_{zw} is a Banach algebra under the Duhamel multiplication

$$\left(f\circledast g\right)(zw)=\frac{\partial^2}{\partial z\partial w}\int_0^z\int_0^w f\left(\left(z-u\right)\left(w-v\right)\right)g\left(uv\right)dvdu \tag{1.3}$$

with the unit 1 (and therefore $\left(B_{zw},\circledast\right)$ is a non-radical Banach algebra), and we describe its maximal ideal space.

2. The characterization of invertible elements of the Banach algebra $\left(B_{zw},\circledast\right)$

In this section we investigate the maximal ideals of the Banach algebra $\left(B_{zw},\circledast\right)$. For this purpose, we will give an invertibility criterion for the elements of $\left(B_{zw},\circledast\right)$, which is equivalent to the description of maximal ideals of $\left(B_{zw},\circledast\right)$ (an element f of $\left(B_{zw},\circledast\right)$ is invertible in $\left(B_{zw},\circledast\right)$ if and only if f does not belong to any maximal ideal of $\left(B_{zw},\circledast\right)$) (For the related results, see also [5] and [10]).

Lemma 2.1. *The algebra* $\left(B_{zw},\circledast\right)$ *is a Banach algebra with respect to the norm defined by* (1.2).

Proof. It follows from (1.1) that for every $f,g\in C_A^{(n)}\left(\mathbb{D}\times\mathbb{D}\right)$

$$\begin{aligned}\left(f\circledast g\right)(z,w)=&\int_0^z\int_0^w\frac{\partial^2}{\partial z\partial w}f\left(z-u,w-v\right)g\left(u,v\right)dvdu+\\&+\int_0^z\frac{\partial}{\partial z}f\left(z-u,0\right)g\left(u,w\right)du+\int_0^w\frac{\partial}{\partial w}f\left(0,w-v\right)g\left(z,v\right)dv+\\&+f\left(0,0\right)g\left(z,w\right),\end{aligned}$$

which implies that

$$(F \circledast G)(zw) = \int_0^z \int_0^w \frac{\partial^2}{\partial z \partial w} F((z-u)(w-v)) G(uv)\, dv du + F(0) G(zw)$$

for every $F, G \in B_{zw}$. Then the standard calculation for derivatives shows that

$$\frac{\partial^{|\alpha|}}{\partial z^{\alpha_1} \partial w^{\alpha_2}} (F \circledast G) = \int_0^z \int_0^w \frac{\partial^{|\alpha|}}{\partial z^{\alpha_1} \partial w^{\alpha_2}} G((z-u)(w-v)) \frac{\partial^2}{\partial u \partial v} F(uv)\, dv du + \tag{2.1}$$
$$+ F(0) \frac{\partial^{|\alpha|}}{\partial z^{\alpha_1} \partial w^{\alpha_2}} G(zw),$$

and thus by using (1.2), it is easy to verify that

$$\|F \circledast G\|_n \leq \|F\|_n \|G\|_n .$$

Obviously, $f \circledast \mathbf{1} = \mathbf{1} \circledast f = f$ for every $f \in (B_{zw}, \circledast)$. The lemma is proven. □

Now let us prove our main lemma. The quasinilpotent operators technique is used for the proof of the lemma. Note that this technique for the proof of invertibility of the analytic functions $f \in C_A^{(n)}(\mathbb{D})$, apparently, was firstly applied by Karaev and Tuna in [7], see also Karaev [6] and Karaev and Tuna [8].

Lemma 2.2. *Let $f \in (B_{zw}, \circledast)$. Then f is invertible if and only if $f|_{zw=0} \neq 0$.*

Proof. Let f is an invertible element of the Banach algebra B_{zw} with respect to the Duhamel multiplication $\circledast$ (see formula (2.1) and Lemma 2.1). Then, there exists a unique element $g \in (B_{zw}, \circledast)$ such that $(f \circledast g)(z) = 1$ for all $z \in \mathbb{D} \times \mathbb{D}$. Then, in particular, $(f \circledast g)(0) = 1$, that is $f(0) \cdot g(0) = 1$, and hence $f(0) \neq 0$.

Conversely, let $f(0) \neq 0$. Let us prove then that f is an invertible element of the Banach algebra $(B_{zw}, \circledast)$. For this purpose, it is sufficient to prove that the corresponding Duhamel operator $\mathcal{D}_f$, $\mathcal{D}_f h = f \circledast h$, $h \in B_{zw}$, is invertible in B_{zw}. Indeed, let us denote $F := f - f(0)$. Clearly, $F(0) = 0$, $f(z) = F(z) + f(0)$, and thus $\mathcal{D}_f = f(0) I + \mathcal{D}_F$, where I denotes the identity operator on B_{zw}. Therefore, since $f(0) \neq 0$, in order to prove the invertibility of the operator $\mathcal{D}_f$, it is enough to prove that $\mathcal{D}_F$ is a quasinilpotent operator (i.e., $\sigma(\mathcal{D}_F) = \{0\}$) on B_{zw}. For this, we will use the classical Gelfand formula for the spectral radius of operators:

$$r(\mathcal{D}_F) = \lim_{k \to \infty} \left\| \mathcal{D}_F^k \right\|^{\frac{1}{k}} .$$

Let us now calculate the value $\left\| \mathcal{D}_F^k \right\|^{\frac{1}{k}}$. For this purpose we obtain that

$$\begin{aligned}
\left(\mathcal{D}_F g\right)(zw) &= \frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w F\left((z-u)(w-v)\right) g(uv)\, dv du \\
&= \int_0^z \int_0^w [F'\left((z-u)(w-v)\right) \\
&+ (z-u)(w-v) F''\left((z-u)(w-v)\right)] g(uv)\, dv du + F(0)\, g(zw) \\
&= \int_0^z \int_0^w [F'\left((z-u)(w-v)\right) + \\
&+ (z-u)(w-v) F''\left((z-u)(w-v)\right)] g(uv)\, dv du \\
&= \left(K_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw)
\end{aligned}$$

where

$$\left(K_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw) = \left(\frac{\partial^2 F}{\partial z \partial w} * g\right)(zw).$$

Then, we obtain the following:

$$\begin{aligned}
\left| \left(K_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw) \right| &= \left| \int_0^z \int_0^w [F'\left((z-u)(w-v)\right) + \right. \\
&(z-u)(w-v) F''\left((z-u)(w-v)\right)] g(uv)\, dv du | \\
&\leq \int_0^z \int_0^w |[F'\left((z-u)(w-v)\right) + \\
&+ (z-u)(w-v) F''\left((z-u)(w-v)\right)]| \, |g(uv)| \, |dv| \, |du| \\
&\leq \|F\|_n \, \|g\|_n \, |zw| \text{, and}
\end{aligned}$$

$$\begin{aligned}
&\left|\left(K^{2}_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw)\right| \\
&= \left|\left[K_{\frac{\partial^2 F}{\partial z \partial w}}\left(K_{\frac{\partial^2 F}{\partial z \partial w}} g\right)\right](zw)\right| \\
&= \left|\int_0^z\int_0^w [F'((z-u)(w-v)) + \right. \\
&\left. + (z-u)(w-v)F''((z-u)(w-v))]K_{\frac{\partial^2 F}{\partial z \partial w}} g(uv)\,dvdu\right| \\
&= \left|\int_0^z\int_0^w [F'((z-u)(w-v)) + \right. \\
&+ (z-u)(w-v)F''((z-u)(w-v))]\left\{\int_0^u\int_0^v [F'((u-t)(v-\tau)) + \right. \\
&+ \ ((u-t)(v-\tau))F''((u-t)(v-\tau))]\,g(t\tau)\,d\tau dt\}\,dvdu| \\
&\le \int_0^z\int_0^w |F'((z-u)(w-v)) + \\
&+ (z-u)(w-v)F''((z-u)(w-v))|\left\{\int_0^u\int_0^v |F'((u-t)(v-\tau))| + \right. \\
&+ |(u-t)(v-\tau)F''((u-t)(v-\tau))|\,|g(t\tau)|\,|d\tau|\,|dt|\}\,|dv|\,|du| \\
&\le \|F\|_n^2\,\|g\|_n\,\frac{|zw|^2}{2!}.
\end{aligned}$$

Thus, by induction we get

$$\left|\left(K^{k}_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw)\right| \le \|F\|_n^k\,\|g\|_n\,\frac{|zw|^k}{k!}. \tag{2.2}$$

On the other hand,

$$\left|\frac{\partial^2}{\partial z \partial w}\left(K_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw)\right|$$

$$= \left| \frac{\partial^2}{\partial z \partial w} \left(\int_0^z \int_0^w [F'((z-u)(w-v)) + \right.\right.$$
$$\left.\left. + (z-u)(w-v) F''((z-u)(w-v))] g(uv)\, dvdu \right) \right|$$
$$= \left| \frac{\partial^2}{\partial z \partial w} \left(\int_0^z \int_0^w g((z-u)(w-v)) \right) [F'(uv) + uvF''(uv)\, dvdu] \right|$$
$$= \left| \int_0^z \int^w [g'((z-u)(w-v)) + (z-u)(w-v) g''((z-u)(w-v))](F'(uv) + \right.$$
$$\left. + uvF''(uv))dvdu + g(0)(F'(zw) + zwF''(zw)) \right|$$
$$= \left| \int_0^z \int_0^w [F'((z-u)(w-v)) + (z-u)(w-v) F''((z-u)(w-v))] [g'(uv) \right.$$
$$\left. + uvg''(uv)]dvdu + g(0)(F'(zw) + zwF''(zw)) \right|$$
$$\leq \int_0^z \int_0^w |F'((z-u)(w-v)) + (z-u)(w-v) F''((z-u)(w-v))| \, |g'(uv) +$$
$$+ uvg''(uv)| \, |dv| \, |du| + |g(0)| \, |F'(zw) + zwF''(zw)|$$
$$\leq \|F\|_n \|g\|_n |zw| + \|F\|_n \|g\|_n$$
$$= \|F\|_n \|g\|_n (|zw| + 1).$$

Same calculus shows that

$$\left| \frac{\partial^2}{\partial z \partial w} \left(K^2_{\frac{\partial^2 F}{\partial z \partial w}} g \right)(zw) \right|$$
$$= \left| \frac{\partial^2}{\partial z \partial w} \int_0^z \int_0^w [F'((z-u)(w-v)) + \right.$$
$$+ (z-u)(w-v) F''((z-u)(w-v))] \left\{ \int_0^u \int_0^v [F'((u-t)(v-\tau)) + \right.$$
$$\left. \left. + (u-t)(v-\tau) F''((u-t)(v-\tau))] g(t\tau)\, d\tau dt \right\} dvdu \right|$$
$$= \left| \int_0^z \int_0^w [F'((z-u)(w-v)) + \right.$$

$$
\begin{aligned}
&(z-u)(w-v)F''((z-u)(w-v))]\left\{\int_0^u\int_0^v [F'((u-t)(v-\tau))+\right.\\
&+\ (u-t)(v-\tau)F''((u-t)(v-\tau))]\,[g'(t\tau)+t\tau g''(t\tau)\,d\tau dt]+\\
&+\ g(0)\,[F'(uv)+uvF''(uv)]\}\,dvdu|\\
&\le \int_0^z\int_0^w |F'((z-u)(w-v))+\\
&+\ (z-u)(w-v)F''((z-u)(w-v))|\left\{\int_0^u\int_0^v |F'((u-t)(v-\tau))|+\right.\\
&+|(u-t)(v-\tau)F''((u-t)(v-\tau))|\,|g'(t\tau)+t\tau g''(t\tau)|\,|d\tau|\,|dt|+\\
&+\ |g(0)|\,|F'(uv)+uvF''(uv)|\}\,|dv|\,|du|\\
&= \int_0^z\int_0^w |F'((z-u)(w-v))+
\end{aligned}
$$

$$
\begin{aligned}
&+(z-u)(w-v)F''((z-u)(w-v))]\left\{\int_0^u\int^v [F'((u-t)(v-\tau))+\right.\\
&+\ (u-t)(v-\tau)F''((u-t)(v-\tau))]g(t\tau)\,d\tau dt\}\,dvdu\\
&+\ (z-u)(w-v)F''((z-u)(w-v))|\left(\int_0^u\int_0^v |F'((u-t)(v-\tau))+\right.\\
&+\ (u-t)(v-\tau)F''((u-t)(v-\tau))|\,|g'(t\tau)+t\tau g''(t\tau)|\,|d\tau|\,|dt|\,|dv|\,|du|)\\
&+\int_0^z\int_0^w F'((z-u)(w-v))+\\
&+\ (z-u)(w-v)F''((z-u)(w-v))|\,|g(0)|\,|F'(uv)+uvF''(uv)|\,|dv|\,|du|\\
&\le \|F\|_n^2\,\|g\|_n\,\frac{|zw|^2}{2}+\|F\|_n^2\,\|g\|_n\,|zw|\\
&= \|F\|_n^2\,\|g\|_n\left(\frac{|zw|^2}{2}+|zw|\right)\\
&\le \|F\|_n^2\,\|g\|_n\,\frac{\left(|zw|^2+1\right)^2}{2!}.
\end{aligned}
$$

By induction we get

$$
\left|\frac{\partial^2}{\partial z\partial w}\left(K^k_{\frac{\partial^2 F}{\partial z\partial w}}g\right)(zw)\right|\le \|F\|_n^k\,\|g\|_n\,\frac{\left(|zw|^2+1\right)^k}{k!}. \tag{2.3}
$$

On the other hand,

$$
\begin{aligned}
&\left|\frac{\partial^4}{\partial z^2 \partial w^2}\left(K_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw)\right| \\
&= \left|\frac{\partial^4}{\partial z^2 \partial w^2} \int\limits_0^z \int\limits_0^w [F'((z-u)(w-v)) + \right. \\
&+ (z-u)(w-v) F''((z-u)(w-v))] g(uv)\, dvdu| \\
&= \left|\frac{\partial^2}{\partial z \partial w}\left(\int\limits_0^z \int\limits_0^w [F'((z-u)(w-v)) + \right.\right. \\
&+ (z-u)(w-v) F''((z-u)(w-v))] [g'(uv) + uvg''(uv)]\, dvdu \\
&+ \ g(0)(F'(zw) + zwF''(zw)))| \\
&= \left|\int\limits_0^z \int\limits_0^w [F'((z-u)(w-v)) + \right. \\
&+ (z-u)(w-v) F''((z-u)(w-v))][g^{(2)}(uv) + \\
&+ 4uvg^{(3)}(uv) + (uv)^2 g^{(4)}(uv)] dvdu \\
&+ (F'(zw) + zwF''(zw)) g'(0) + (F'(zw) + zwF''(zw)) g(0)| \\
&\leq \|F\|_n \|g\|_n |zw| + \|F\|_n \|g\|_n + \|F\|_n \|g\|_n \\
&= \|F\|_n \|g\|_n (|zw| + 2).
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
&\left|\frac{\partial^4}{\partial z^2 \partial w^2}\left(K^2_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw)\right| \\
&= \left|\frac{\partial^2}{\partial z \partial w}\left(\frac{\partial^2}{\partial z \partial w} K^2_{\frac{\partial^2 F}{\partial z \partial w}} g\right)(zw)\right| \\
&= \left|\frac{\partial^2}{\partial z \partial w}\left(\int\limits_0^z \int\limits_0^w [F'((z-u)(w-v)) + \right.\right. \\
&+ (z-u)(w-v) F''((z-u)(w-v))] \left\{\int\limits_0^u \int\limits_0^v [F'((u-t)(v-\tau)) + \right. \\
&+ [(u-t)(v-\tau) F''((u-t)(v-\tau))] [g'(t\tau) + t\tau g''(t\tau)]\, d\tau dt + \\
&+ \ g(0)[F'(uv) + uvF''(uv)]\}\, dvdu)|) \\
&= \left|\frac{\partial^2}{\partial z \partial w}\left(\int\limits_0^z \int\limits_0^w [F'((z-u)(w-v)) + \right.\right.
\end{aligned}
$$

$$+\left(z-u\right)\left(w-v\right)F''\left(\left(z-u\right)\left(w-v\right)\right)]\int\limits_0^u\int\limits_0^v\left[F'\left(\left(u-t\right)\left(v-\tau\right)\right)+\right.$$

$$+\left(u-t\right)\left(v-\tau\right)F''\left(\left(u-t\right)\left(v-\tau\right)\right)\left[g'\left(t\tau\right)+t\tau g''\left(t\tau\right)\right])+$$

$$+\frac{\partial^2}{\partial z\partial w}\int\limits_0^z\int\limits_0^w\left[F'\left(\left(z-u\right)\left(w-v\right)\right)+\left(z-u\right)\left(w-v\right)F''\left(\left(z-u\right)\left(w-v\right)\right)\right]\left[F'\left(uv\right)+\right.$$

$$\left.+uvF''\left(uv\right)\right]g\left(0\right)dvdu|$$

$$=\left|\int\limits_0^z\int\limits_0^w\left[F'\left(\left(z-u\right)\left(w-v\right)\right)+\right.\right.$$

$$+\left(z-u\right)\left(w-v\right)F''\left(\left(z-u\right)\left(w-v\right)\right)]\int\limits_0^u\int\limits_0^v\left[F'\left(\left(u-t\right)\left(v-\tau\right)\right)+\right.$$

$$+\left(u-t\right)\left(v-\tau\right)F''\left(\left(u-t\right)\left(v-\tau\right)\right)]\left[g^{(2)}\left(uv\right)+\right.$$

$$+4uvg^{(3)}\left(uv\right)+\left(uv\right)^2g^{(4)}\left(uv\right)]d\tau dt+$$

$$+g'\left(0\right)\left[F'(zw)+zwF''(zw)\right]\int\limits_0^z\int\limits_0^w\left[F'\left(\left(z-u\right)\left(w-v\right)\right)+F''\left(\left(z-u\right)\left(w-v\right)\right)\right]dvdu$$

$$\left.+g(0)\left[F'\left(zw\right)\cdot F''\left(zw\right)\right]^2\right|$$

$$\leq\left\|F\right\|_n^2\left\|g\right\|_n\frac{\left|zw\right|^2}{2}+\left\|F\right\|_n^2\left\|g\right\|_n\left|zw\right|+\left\|F\right\|_n^2\left\|g\right\|_n$$
$$\leq\left\|F\right\|_n^2\left\|g\right\|_n\frac{\left(\left|zw\right|+2\right)^2}{2!}.$$

Then, by induction we get that

$$\frac{\partial^4}{\partial z^2\partial w^2}\left[\left(K^k_{\frac{\partial^2F}{\partial z\partial w}}\right)(zw)\right]\leq\left\|F\right\|_n^k\left\|g\right\|_n\frac{\left(\left|zw\right|+2\right)^k}{k!}. \tag{2.4}$$

By induction we obtain from (2.2), (2.3) and (2.4) that

$$\left|\frac{\partial^s}{\partial z^n\partial w^m}\left(K^k_{\frac{\partial^2F}{\partial z\partial w}}g\right)(zw)\right|\leq\left\|F\right\|_n^k\left\|g\right\|_n\frac{\left(\left|zw\right|+s\right)^k}{k!}$$

where $s=n+m$. Hence,

$$\left\|K^k_{\frac{\partial^2F}{\partial z\partial w}}g\right\|_n\leq\left\|F\right\|_n^k\left\|g\right\|_n\frac{\left(1+s\right)^k}{k!},$$

that is

$$\left\|K^k_{\frac{\partial^2F}{\partial z\partial w}}\right\|_n\leq\left\|F\right\|_n^k\frac{\left(1+s\right)^k}{k!},$$

or

$$\left\| K^k_{\frac{\partial^2 F}{\partial z \partial w}} \right\|^{1/k} \leq \|F\|_n \frac{1+s}{(k!)^{1/k}} \to 0, \ \ (k \to \infty).$$

Thus, $r\left(K_{\frac{\partial^2 F}{\partial z \partial w}}\right) = 0$, that is $K_{\frac{\partial^2 F}{\partial z \partial w}}$ is a quasinilpotent operator on B_{zw}, which shows that an operator $\mathcal{D}_f = f(0) I + \mathcal{D}_F$ is invertible in B_{zw}.The lemma is proved. □

It is obvious from Lemma 2.2 that $(B_{zw}, \circledast)$ is a non-radical Banach algebra.

Now we can state our main result, which describes the maximal ideal space of the commutative non-radical Banach algebra $(B_{zw}, \circledast)$.

Theorem 2.3. *The maximal ideal space $\mathcal{M}((B_{zw}, \circledast))$ of the Banach algebra $(B_{zw}, \circledast)$ consists of the one homomorphism, the evaluation at the origin: $h(f) = f(0)$.*

Proof. Indeed, by Lemma 2.1, B_{zw} is a (commutative) Banach algebra under the Duhamel multiplication $\circledast$, and by Lemma 2.2, the maximal ideals in $(B_{zw}, \circledast)$ have the form

$$\{f \in B_{zw} : f\,|_{zw=0} = 0\},$$

which shows that $\sigma(f) = \{f(0)\}$, that is the spectrum of every element $f \in (B_{zw}, \circledast)$ consists of the sole point $f(0)$. This shows that the maximal ideal space $\mathcal{M}((B_{zw}, \circledast))$ of $(B_{zw}, \circledast)$ consists of the sole homomorphism, namely, evaluation at the origin, and the Gelfand transform is trivial. The theorem is proved. □

3. Remarks on the Invariant Subspace and Cyclicity of the Hardy type Operator

In the present section, we consider the Hardy type operator H defined on the Lebesgue space $L^p([0,1] \times [0,1])$, $(1 \leq p < +\infty)$ by

$$(Hf)(x,y) := xy \int_0^x \int_0^y f(t,\tau) d\tau dt.$$

It is easy to see that if E is a measurable subset of the unit square $[0,1] \times [0,1]$, which satisfies the condition

$$(x,y) \in E \Rightarrow [0,x] \times [0,y] \subseteq E,$$

then the subspace

$$M_E := \{f \in L^p([0,1] \times [0,1]) : f(x,y) = 0 \text{ a.e. on } E\}$$

is an invariant subspace for H, that is $HM_E \subset M_E$. These subspace are in a sense analogous to the invariant subspaces of the classical Volterra integration operator V, $Vf(x) = \int_0^x f(t)dt$; however, there are many other invariant subspaces:

$$M_+ := \{f \in L^p([0,1] \times [0,1]) : f(x,y) = f(y,x) \text{ a.e. on } [0,1] \times [0,1]\}$$
$$M_- := \{f \in L^p([0,1] \times [0,1]) : f(x,y) = -f(y,x) \text{ a.e. on } [0,1] \times [0,1]\}.$$

Recall that a set $\mathcal{G} \subset L^p([0,1] \times [0,1])$ is said to be a cyclic set for the operator $A : L^p([0,1] \times [0,1]) \to L^p([0,1] \times [0,1])$, if

$$\begin{aligned} span\,\{A^n\mathcal{G} : n = 0,1,2,\cdots\} &= closLinhull\,\{A^n\mathcal{G} : n = 0,1,2,\cdots\} \\ &= L^p([0,1] \times [0,1]). \end{aligned}$$

The spectral multiplicity $\mu(A)$ of the operator A is defined by

$$\mu(A) := \min\{card\mathcal{G} : \mathcal{G} \text{ is a cyclic set for} A\}.$$

A vector $f \in L^p([0,1] \times [0,1])$ is a cyclic vector for A, if

$$span\,\{A^n f : n = 0,1,2,\cdots\} = L^p([0,1] \times [0,1]).$$

In this case, obviously $\mu(A) = 1$, and A is said to be a cyclic operator.

If $\mathcal{G}$ is any finite subset of $L^2([0,1] \times [0,1])$, then by the result of Atzmon and Manos [1, Theorem 1], we get below:

$$span\,\{W^n f : f \in \mathcal{G},\, n = 0,1,2,\cdots\} \neq L^2([0,1] \times [0,1]),$$

that is $span\,\{W^n f : f \in \mathcal{G},\, n \geq 0\}$ is a proper invariant subspace of the double integration operator W defined on $L^2([0,1] \times [0,1])$ by

$$Wf\,(x,y) := \int\limits_0^x \int\limits_0^y f(t,\tau)dtd\tau.$$

This result of Atzmon and Manos shows that $\mu(W) = +\infty$ while $\mu(V) = 1$.

Now it follows from this result that

$$span\,\{H^n f : f \in \mathcal{G},\, n = 0,1,2,\cdots\} \neq L^2([0,1] \times [0,1])$$

for any finite subset $\mathcal{G}$ of $L^2([0,1] \times [0,1])$, that is $\mu(H) = +\infty$. In particular, if $\mathcal{G}$ consists of the single function $f(x,y) \equiv \mathbf{1}$, then it is easily verified that the subspace

$$B_{xy} = span\,\{H^n f : n \geq 0\}$$

consists of all g in $L^2([0,1] \times [0,1])$, which are of the form $g(x,y) = h(xy)$, where h is a measurable function on $[0,1]$.

These examples indicate that the Hardy type operator H has a very rich and varied supply of invariant subspaces, and a characterization of all of them might be a hopeless task (For more informations about invariant subspaces of the double integration operator W on $L^2([0,1] \times [0,1])$, see Atzmon and Manos [1]).

Here we will consider restriction of the Hardy type operator H to its invariant subspaces B_{xy} :

$$H_{xy} = H|B_{xy},$$

that is $H_{xy}h(x,y) = xy\int\limits_0^x\int\limits_0^y h(t\tau)d\tau dt \ (\forall h \in B_{xy})$.

We will show that $\mu(H_{xy}) = 1$, that is H_{xy} is a cyclic operator.

Proposition 3.1. *It is true that* $\mu(H_{xy}) = 1$.

Proof. Indeed, let us calculate $H_{xy}^n\mathbf{1}$ for any $n \geq 0$. Obviously, $H_{xy}^0\mathbf{1} = \mathbf{1}$. Also $H_{xy}\mathbf{1} = (xy)^2$ and

$$H_{xy}^2\mathbf{1}{=}xy\int_0^x\int_0^y \left(H_{xy}\mathbf{1}\right) dt d\tau = xy\int_0^x\int_0^y \tau^2 t^2 d\tau dt$$
$$= xy\int_0^x \left(\int_0^y \tau^2 d\tau\right) t^2 dt$$
$$= xy\int_0^x \frac{y^3}{3} t^2 dt = \frac{(xy)^4}{3^2}.$$

By induction it can be easily verified that

$$H_{xy}^n\ \mathbf{1} {=} \frac{(xy)^{2n}}{(2n-1)^2},\ \ n = 0, 1, 2, \cdots.$$

Therefore, by the Müntz approximation theorem we deduce that

$$span\left\{H_{xy}^n\mathbf{1} : n \geq 0\right\} = span\left\{\frac{(xy)^{2n}}{(2n-1)^2} : n \geq 0\right\} = B_{xy},$$

that is $\mathbf{1}$ is a cyclic vector for H_{xy}, which means that $\mu(H_{xy}) = 1$. The proposition is proved. □

Acknowledgement. The authors would like to thank the referees for giving useful comments and suggestions for the improvement of this paper.

References

1. A. Atzmon and H. Manos, *The integration operator in two variables*, Proc. Amer. Math. Soc. **119** (1993), no. 2, 513–523.
2. N. Dunford and L. Schwartz, *Linear Operators*, Vol.1: General Theory, Springer-Verlag, New York, 1958.
3. I.M. Gelfand, D.A. Raĭkov and G.E. Šilov, *Commutative normed rings*, Sovremennye Problemy Matematiki. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 316 pp., 1960.
4. E. Hille and R. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq., Publ. Vol. 31, Amer. Math. Soc, Providence, R. I., 1957.
5. M.T. Karaev, *Some Applications of the Duhamel Product,* Zap. Nauchn. Semin. POMI **303** (2003), 145–160.
6. M.T. Karaev, *On some applications of ordinary and extended Duhamel products*, (Russian) Sibirsk. Mat. Zh. **46** (2005), no. 3, 553–566; translation in Siberian Math. J. **46** (2005), no. 3, 431–442.
7. M.T. Karaev and H. Tuna, *Description of maximal ideal space of some Banach algebra with multiplication as Duhamel product*, Complex Var. Theory Appl. **49** (2004), no. 6, 449–457.
8. M.T. Karaev and H. Tuna, *On some applications of Duhamel product*, Linear and Multilinear Algebra. **54** (2006), no. 4, 301–311.
9. C.E. Rickart, *General theory of Banach algebras*, The University Series in Higher Mathematics. D. van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 394 pp., 1960.

10. S. Saltan and Y. Özel, *On some applications of a special integrodifferential operators*, J. Funct. Spaces Appl. 2012, Article ID 894527.

[1] Suleyman Demirel University, Department of Mathematics, 32260, Isparta, Turkey.
E-mail address: sunasaltan@sdu.edu.tr

[2] Suleyman Demirel University, Department of Mathematics, 32260, Isparta, Turkey.
E-mail address: jasmineozel@hotmail.com