



(P, Q) -OUTER GENERALIZED INVERSE OF BLOCK MATRICES IN BANACH ALGEBRAS

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ABSTRACT. We investigate additive results for (p, q) -outer generalized inverse of elements in Banach algebra, along with the representation of this inverse in a block matrix in the Banachiewicz–Schur form. Additionally, we investigate the (p, q) -pseudospectrum and (p, q) -condition spectrum of a block matrix $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u$ in a Banach algebra.

1. INTRODUCTION

Let \mathcal{A} be the complex unital Banach algebra with unit 1. The sets of all idempotents and invertible elements of \mathcal{A} will be denoted by \mathcal{A}^\bullet and \mathcal{A}^{-1} , respectively.

An element $a \in \mathcal{A}$ is outer generalized invertible, if there exists some $b \in \mathcal{A}$ satisfying $b = bab$. Such b is called the outer generalized inverse of a . In this case ba and $1 - ab$ are idempotents corresponding to a and b . The set of all outer generalized invertible elements of \mathcal{A} will be denoted with $\mathcal{A}^{(2)}$.

Djordjević and Wei introduced outer generalized inverses with prescribed idempotents in [3] (see also [2]):

Definition 1.1. [3] Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. An element $b \in \mathcal{A}$ satisfying

$$bab = b, ba = p, 1 - ab = q,$$

will be called a (p, q) -outer generalized inverse of a , written $a_{p,q}^{(2)} = b$.

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The uniqueness of $a_{p,q}^{(2)}$ is provided in the following theorem.

Theorem 1.2. [3] *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following statements are equivalent:*

- (1) $a_{p,q}^{(2)}$ exists;
- (2) $(1 - q)a = (1 - q)ap$, and there exists some $b \in \mathcal{A}$ such that $pb = b, bq = 0$ and $ab = 1 - q$.

Moreover, if $a_{p,q}^{(2)}$ exists, then it is unique.

The set of all outer generalized invertible elements of \mathcal{A} with prescribed idempotents $p, q \in \mathcal{A}^\bullet$ will be denoted with $\mathcal{A}_{p,q}^{(2)}$. Outer generalized inverses in rings can be found in [6]. Also, inner generalized inverses with prescribed idempotents were studied in [5].

Let M be a 2×2 block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$ and $D \in \mathbb{C}^{l \times k}$. If A is invertible, then the Schur complement of A in M is defined as

$$S = D - CA^{-1}B.$$

If M is invertible, then S is invertible, too, and M can be decomposed as

$$M = \begin{bmatrix} I_m & 0 \\ CA^{-1} & I_l \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I_m & A^{-1}B \\ 0 & I_l \end{bmatrix},$$

where I_t is the identity matrix of order t . In this case, the inverse of M can be written as

$$\begin{aligned} M^{-1} &= \begin{bmatrix} I_m & -A^{-1}B \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. \end{aligned} \quad (1.1)$$

Result (1.1) is well known as the Banachiewicz–Schur form of M , and it has been used in dealing with inverses of block matrices; see [1, 8, 10].

Analogously, we can represent an element of Banach algebra in a block matrix form as follows.

Let $u \in \mathcal{A}$ be an idempotent. Then we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_u,$$

where $a_{11} = uau$, $a_{12} = ua(1 - u)$, $a_{21} = (1 - u)au$, $a_{22} = (1 - u)a(1 - u)$.

Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$. If $a \in (u\mathcal{A}u)^{-1}$ and the Schur complement $s = d - ca^{-1}b \in ((1 - u)\mathcal{A}(1 - u))^{-1}$, then the inverse of x has the Banachiewicz–Schur form

$$x^{-1} = \begin{bmatrix} a^{-1} + a^{-1}bs^{-1}ca^{-1} & -a^{-1}bs^{-1} \\ -s^{-1}ca^{-1} & s^{-1} \end{bmatrix}.$$

If $a \in (u\mathcal{A}u)$ is not invertible but has the outer generalized inverse with prescribed idempotents $p_1, q_1 \in (u\mathcal{A}u)^\bullet$, we can observe the generalized Schur complement $s = d - ca_{p_1, q_1}^{(2)}b$.

Accordingly, we investigate equivalent conditions under which $x_{p, q}^{(2)}$ has the generalized Banachiewicz–Schur form in a Banach algebra.

We use the following auxiliary results.

Lemma 1.3. *Let p, q be idempotents in a Banach algebra \mathcal{A} . The following statements are equivalent:*

- (i) $p + q \in \mathcal{A}^\bullet$,
- (ii) $pq = qp = 0$.

Proof. (i) \Rightarrow (ii): Suppose $p + q \in \mathcal{A}^\bullet$. We have

$$(p + q)^2 = p + q \Rightarrow pq + qp = 0 \Rightarrow pq = -qp.$$

Since the following holds

$$pq = p^2q^2 = p(pq)q = p(-qp)q = -pq(pq) = pqqp = pqp = -ppq = -pq,$$

we obtain $pq = 0$. The analogous proof holds for $qp = 0$.

(ii) \Rightarrow (i): Let $p, q \in \mathcal{A}^\bullet$ such that $pq = qp = 0$. Then

$$(p + q)^2 = p^2 + pq + qp + q^2 = p + q,$$

so $p + q \in \mathcal{A}^\bullet$. □

If $u \in \mathcal{A}^\bullet$, then the product of arbitrary elements from algebra $u\mathcal{A}u$ and $(1 - u)\mathcal{A}(1 - u)$ is equal to 0, i.e. for all $a \in u\mathcal{A}u$ and for all $b \in (1 - u)\mathcal{A}(1 - u)$, we have $ab = 0$.

Now, as a corollary of Lemma 1.3, we state the following result.

Lemma 1.4. *Let $u \in \mathcal{A}^\bullet$. If $p_1 \in (u\mathcal{A}u)^\bullet$ and $p_2 \in ((1 - u)\mathcal{A}(1 - u))^\bullet$, then $p = p_1 + p_2 \in \mathcal{A}$ is an idempotent.*

2. (p, q) -OUTER GENERALIZED INVERSE

The first result gives the additive properties of the (p, q) -outer generalized inverse.

Theorem 2.1. *Let $p, q \in \mathcal{A}^\bullet$ and $a, b \in \mathcal{A}_{p, q}^{(2)}$. If*

$$a_{p, q}^{(2)}b + b_{p, q}^{(2)}a + 1 = 0, \quad ab_{p, q}^{(2)} + ba_{p, q}^{(2)} + 1 = 0, \quad (2.1)$$

then $a + b \in \mathcal{A}_{p, q}^{(2)}$ and

$$(a + b)_{p, q}^{(2)} = a_{p, q}^{(2)} + b_{p, q}^{(2)}.$$

Proof. Using the fact that $a, b \in \mathcal{A}_{p,q}^{(2)}$, Theorem 1.2 and conditions (2.1), we have

$$\begin{aligned}
& (a_{p,q}^{(2)} + b_{p,q}^{(2)})(a + b)(a_{p,q}^{(2)} + b_{p,q}^{(2)}) = \\
& = a_{p,q}^{(2)} + pb_{p,q}^{(2)} + a_{p,q}^{(2)}ba_{p,q}^{(2)} + a_{p,q}^{(2)}(1 - q) + b_{p,q}^{(2)}(1 - q) + b_{p,q}^{(2)}ab_{p,q}^{(2)} + pa_{p,q}^{(2)} + b_{p,q}^{(2)} \\
& = a_{p,q}^{(2)} + b_{p,q}^{(2)} + a_{p,q}^{(2)}ba_{p,q}^{(2)} + a_{p,q}^{(2)} + b_{p,q}^{(2)} + b_{p,q}^{(2)}ab_{p,q}^{(2)} + a_{p,q}^{(2)} + b_{p,q}^{(2)} \\
& = a_{p,q}^{(2)} + b_{p,q}^{(2)} + a_{p,q}^{(2)}(ba_{p,q}^{(2)} + 1) + b_{p,q}^{(2)}(1 + ab_{p,q}^{(2)}) + a_{p,q}^{(2)} + b_{p,q}^{(2)} \\
& = a_{p,q}^{(2)} + b_{p,q}^{(2)} + a_{p,q}^{(2)}(-ab_{p,q}^{(2)}) + b_{p,q}^{(2)}(-ba_{p,q}^{(2)}) + a_{p,q}^{(2)} + b_{p,q}^{(2)} \\
& = a_{p,q}^{(2)} + b_{p,q}^{(2)} - pb_{p,q}^{(2)} - pa_{p,q}^{(2)} + a_{p,q}^{(2)} + b_{p,q}^{(2)} \\
& = a_{p,q}^{(2)} + b_{p,q}^{(2)},
\end{aligned}$$

$$\begin{aligned}
(a_{p,q}^{(2)} + b_{p,q}^{(2)})(a + b) &= a_{p,q}^{(2)}a + a_{p,q}^{(2)}b + b_{p,q}^{(2)}a + b_{p,q}^{(2)}b \\
&= p + pa_{p,q}^{(2)}b + pb_{p,q}^{(2)}a + p \\
&= p + p(a_{p,q}^{(2)}b + b_{p,q}^{(2)}a + 1) \\
&= p,
\end{aligned}$$

and also

$$\begin{aligned}
(a + b)(a_{p,q}^{(2)} + b_{p,q}^{(2)}) &= aa_{p,q}^{(2)} + ba_{p,q}^{(2)} + ab_{p,q}^{(2)} + bb_{p,q}^{(2)} \\
&= (1 - q) + ba_{p,q}^{(2)} + ab_{p,q}^{(2)} + (1 - q) \\
&= (1 - q) + ba_{p,q}^{(2)}(1 - q) + ab_{p,q}^{(2)}(1 - q) + (1 - q) \\
&= (1 - q) + (ba_{p,q}^{(2)} + ab_{p,q}^{(2)} + 1)(1 - q) \\
&= (1 - q).
\end{aligned}$$

Thus, we proved $(a + b)_{p,q}^{(2)} = a_{p,q}^{(2)} + b_{p,q}^{(2)}$. □

The following theorem gives us equivalent conditions under which $x_{p,q}^{(2)}$ has the generalized Banachiewicz–Schur form in a Banach algebra.

Theorem 2.2. Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Let $a \in (u\mathcal{A}u)_{p_1, q_1}^{(2)}$ and let $s = d - ca_{p_1, q_1}^{(2)} b \in ((1-u)\mathcal{A}(1-u))_{p_2, q_2}^{(2)}$ be the generalized Schur complement of a in x . Then the following statements are equivalent:

(i) $x \in \mathcal{A}_{p,q}^{(2)}$ and $x_{p,q}^{(2)} = r$, where

$$r = \begin{bmatrix} a_{p_1, q_1}^{(2)} + a_{p_1, q_1}^{(2)} b s_{p_2, q_2}^{(2)} c a_{p_1, q_1}^{(2)} & -a_{p_1, q_1}^{(2)} b s_{p_2, q_2}^{(2)} \\ -s_{p_2, q_2}^{(2)} c a_{p_1, q_1}^{(2)} & s_{p_2, q_2}^{(2)} \end{bmatrix}$$

(ii) $ca_{p_1, q_1}^{(2)} a = ss_{p_2, q_2}^{(2)} c$ and $aa_{p_1, q_1}^{(2)} b = bs_{p_2, q_2}^{(2)} s$.

Proof. By Lemma 1.4 we obtain that p and q are idempotents.

Using the assumptions $a \in (u\mathcal{A}u)_{p_1, q_1}^{(2)}$ and $s \in ((1-u)\mathcal{A}(1-u))_{p_2, q_2}^{(2)}$, we verify $rxr = r$.

The equation $rx = p$ is equivalent to the equations:

$$s_{p_2, q_2}^{(2)} c = s_{p_2, q_2}^{(2)} c a_{p_1, q_1}^{(2)} a \quad \text{and} \quad a_{p_1, q_1}^{(2)} b = a_{p_1, q_1}^{(2)} b s_{p_2, q_2}^{(2)} s.$$

On the other hand, $1 - xr = q$ is equivalent to:

$$bs_{p_2, q_2}^{(2)} = aa_{p_1, q_1}^{(2)} bs_{p_2, q_2}^{(2)} \quad \text{and} \quad ca_{p_1, q_1}^{(2)} = ss_{p_2, q_2}^{(2)} ca_{p_1, q_1}^{(2)}.$$

Therefore, x has (p, q) -outer generalized inverse if and only if

$$\begin{aligned} s_{p_2, q_2}^{(2)} c &= s_{p_2, q_2}^{(2)} ca_{p_1, q_1}^{(2)} a, & a_{p_1, q_1}^{(2)} b &= a_{p_1, q_1}^{(2)} bs_{p_2, q_2}^{(2)} s, \\ bs_{p_2, q_2}^{(2)} &= aa_{p_1, q_1}^{(2)} bs_{p_2, q_2}^{(2)}, & ca_{p_1, q_1}^{(2)} &= ss_{p_2, q_2}^{(2)} ca_{p_1, q_1}^{(2)}, \end{aligned}$$

which are equivalent to

$$ca_{p_1, q_1}^{(2)} a = ss_{p_2, q_2}^{(2)} c, \quad bs_{p_2, q_2}^{(2)} s = aa_{p_1, q_1}^{(2)} b.$$

□

As a corollary, we formulate the following result.

Corollary 2.3. *Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Let $a \in (u\mathcal{A}u)_{p_1, q_1}^{(2)}$ and let $s = d - ca_{p_1, q_1}^{(2)} b \in ((1-u)\mathcal{A}(1-u))_{p_2, q_2}^{(2)}$. The following statements are equivalent:*

- (i) $ca_{p_1, q_1}^{(2)} = a_{p_1, q_1}^{(2)} b = bs_{p_2, q_2}^{(2)} = s_{p_2, q_2}^{(2)} c = 0$,
- (ii) $ca_{p_1, q_1}^{(2)} a = ss_{p_2, q_2}^{(2)} c$, $aa_{p_1, q_1}^{(2)} b = bs_{p_2, q_2}^{(2)} s$,
 $a_{p_1, q_1}^{(2)} bs_{p_2, q_2}^{(2)} = s_{p_2, q_2}^{(2)} ca_{p_1, q_1}^{(2)} = 0$.

If one of these conditions is satisfied, then $x \in \mathcal{A}_{p, q}^{(2)}$ and

$$x_{p, q}^{(2)} = \begin{bmatrix} a_{p_1, q_1}^{(2)} + a_{p_1, q_1}^{(2)} bs_{p_2, q_2}^{(2)} ca_{p_1, q_1}^{(2)} & -a_{p_1, q_1}^{(2)} bs_{p_2, q_2}^{(2)} \\ -s_{p_2, q_2}^{(2)} ca_{p_1, q_1}^{(2)} & s_{p_2, q_2}^{(2)} \end{bmatrix}.$$

3. (p, q) -CONDITION SPECTRUM AND (p, q) -PSEUDOSPECTRUM

The pseudospectrum and the condition spectrum were studied in [4], [7] and [9].

Definition 3.1. [9] (Pseudospectrum)

Let $\epsilon > 0$. The ϵ -pseudospectrum of an element $a \in \mathcal{A}$ is defined as

$$\Lambda_\epsilon(a) = \{z \in \mathbb{C} \mid a - z \text{ is not invertible or } \|(a - z)^{-1}\| \geq \epsilon\}.$$

Definition 3.2. [4] (Condition spectrum)

Let $0 < \epsilon < 1$. The ϵ -condition spectrum of an element $a \in \mathcal{A}$ is defined as

$$\sigma_\epsilon(a) = \left\{ z \in \mathbb{C} \mid a - z \text{ is not invertible or } \|(a - z)^{-1}\| \cdot \|a - z\| \geq \frac{1}{\epsilon} \right\}.$$

We generalize the pseudospectrum and the condition spectrum, and we formulate (p, q) -pseudospectrum and (p, q) -condition spectrum as follows:

Definition 3.3. ((p, q)-pseudospectrum)

Let $\epsilon > 0$. The (p, q) – ϵ -pseudospectrum of an element $a \in \mathcal{A}$ is defined as

$$\Lambda_\epsilon(a) = \{z \in \mathbb{C} \mid a - z \notin \mathcal{A}_{p,q}^{(2)} \text{ or } \|(a - z)_{p,q}^{(2)}\| \geq \epsilon\}.$$

Definition 3.4. ((p, q)-condition spectrum)

Let $0 < \epsilon < 1$. The (p, q) – ϵ -condition spectrum of an element $a \in \mathcal{A}$ is defined as

$$\sigma_{(p,q)-\epsilon}(a) = \left\{ z \in \mathbb{C} \mid a - z \notin \mathcal{A}_{p,q}^{(2)} \text{ or } \|(a - z)_{p,q}^{(2)}\| \cdot \|a - z\| \geq \frac{1}{\epsilon} \right\}.$$

Notice that the uniqueness of $a_{p,q}^{(2)}$ allows us to consider the (p, q)-pseudospectrum and (p, q)-condition spectrum.

If $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, then the norm of x can be define as

$$\|x\| = \max\{\|a\|, \|b\|\}.$$

Now, we state an auxiliary result.

Lemma 3.5. Let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then $x \in \mathcal{A}_{p,q}^{(2)}$ if and only if $a \in (u\mathcal{A}u)_{p_1,q_1}^{(2)}$ and $b \in ((1-u)\mathcal{A}(1-u))_{p_2,q_2}^{(2)}$. If $x \in \mathcal{A}_{p,q}^{(2)}$, then

$$x_{p,q}^{(2)} = \begin{bmatrix} a_{p_1,q_1}^{(2)} & 0 \\ 0 & b_{p_2,q_2}^{(2)} \end{bmatrix}_u.$$

Proof. By Lemma 1.4 we obtain that p and q are idempotents.

If $a \in (u\mathcal{A}u)_{p_1,q_1}^{(2)}$ and $b \in ((1-u)\mathcal{A}(1-u))_{p_2,q_2}^{(2)}$, by Theorem 2.2, we obtain $x \in \mathcal{A}_{p,q}^{(2)}$.

If $x \in \mathcal{A}_{p,q}^{(2)}$, there exists the element $y = \begin{bmatrix} a_1 & c \\ d & b_1 \end{bmatrix}_u \in \mathcal{A}$ such that $y = x_{p,q}^{(2)}$. The equation $xy = y$ is equivalent to equations:

$$\begin{aligned} a_1aa_1 + cbd &= a_1 \\ a_1ac + cbb_1 &= c \\ daa_1 + b_1bd &= d \\ dac + b_1bb_1 &= b_1. \end{aligned}$$

Also, $yx = p$ is equivalent to:

$$\begin{aligned} a_1a &= p_1 \\ cb &= 0 \\ da &= 0 \\ b_1b &= p_2, \end{aligned}$$

and $1 - xy = q$ is equivalent to:

$$\begin{aligned} u - aa_1 &= q_1 \\ ac &= 0 \\ bd &= 0 \\ (1 - u) - bb_1 &= q_2. \end{aligned}$$

The equations $a_1ac + cbb_1 = c$, $cb = 0$ and $ac = 0$ imply $c = 0$. Analogously, $daa_1 + b_1bd = d$, $da = 0$ and $bd = 0$ imply $d = 0$. Now, we have the equations:

$$\begin{aligned} a_1aa_1 &= a_1 \\ a_1a &= p_1 \\ u - aa_1 &= q_1, \end{aligned}$$

and

$$\begin{aligned} b_1bb_1 &= b_1 \\ b_1b &= p_2 \\ (1 - u) - bb_1 &= q_2 \end{aligned}$$

proving $a_1 = a_{p_1, q_1}^{(2)}$ and $b_1 = b_{p_2, q_2}^{(2)}$.

Furthermore, if $x \in \mathcal{A}_{p, q}^{(2)}$, then

$$x_{p, q}^{(2)} = \begin{bmatrix} a_{p_1, q_1}^{(2)} & 0 \\ 0 & b_{p_2, q_2}^{(2)} \end{bmatrix}_u.$$

□

As a corollary, we have the following result for the invertibility of an element $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$.

Lemma 3.6. *Let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$. Then $x \in \mathcal{A}^{-1}$ if and only if $a \in (u\mathcal{A}u)^{-1}$ and $b \in ((1 - u)\mathcal{A}(1 - u))^{-1}$. If $x \in \mathcal{A}^{-1}$, then*

$$x^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix}_u.$$

Therefore, for the spectrum of an element $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$, the following holds

$$\sigma(x) = \sigma(a) \cup \sigma(b).$$

We investigate whether the similar property holds for the pseudospectrum and condition spectrum. We formulate the following results.

Theorem 3.7. Let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $\epsilon > 0$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then

$$\Lambda_{(p,q)-\epsilon}(x) = \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b).$$

Proof. Let $z \in \Lambda_{(p,q)-\epsilon}(x)$. Then $x - z \notin \mathcal{A}_{p,q}^{(2)}$ or $\|(x - z)_{p,q}^{(2)}\| \geq \epsilon$.

If $x - z = \begin{bmatrix} a - zu & 0 \\ 0 & b - z(1-u) \end{bmatrix}_u \notin \mathcal{A}_{p,q}^{(2)}$, by Lemma 3.5, we obtain that $a - zu \notin (u\mathcal{A}u)_{p_1,q_1}^{(2)}$ or $b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))_{p_2,q_2}^{(2)}$. It implies $z \in \Lambda_{(p_1,q_1)-\epsilon}(a)$ or $z \in \Lambda_{(p_2,q_2)-\epsilon}(b)$, so $z \in \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b)$.

If $x - z = \begin{bmatrix} a - zu & 0 \\ 0 & b - z(1-u) \end{bmatrix}_u \in \mathcal{A}_{p,q}^{(2)}$, we have

$$(x - z)_{p,q}^{(2)} = \begin{bmatrix} (a - zu)_{p_1,q_1}^{(2)} & 0 \\ 0 & (b - z(1-u))_{p_2,q_2}^{(2)} \end{bmatrix}_u$$

and

$$\|(x - z)_{p,q}^{(2)}\| = \max\{\|(a - zu)_{p_1,q_1}^{(2)}\|, \|(b - z(1-u))_{p_2,q_2}^{(2)}\|\} \geq \epsilon.$$

By Lemma 3.5, we conclude that

$$a - zu \in (u\mathcal{A}u)_{p_1,q_1}^{(2)} \text{ and } b - z(1-u) \in ((1-u)\mathcal{A}(1-u))_{p_2,q_2}^{(2)}.$$

The assumption $\max\{\|(a - zu)_{p_1,q_1}^{(2)}\|, \|(b - z(1-u))_{p_2,q_2}^{(2)}\|\} \geq \epsilon$ implies that either $\|(a - zu)_{p_1,q_1}^{(2)}\| \geq \epsilon$ or $\|(b - z(1-u))_{p_2,q_2}^{(2)}\| \geq \epsilon$ holds. It follows that $z \in \Lambda_{(p_1,q_1)-\epsilon}(a)$ or $z \in \Lambda_{(p_2,q_2)-\epsilon}(b)$, so $z \in \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b)$.

We have proved $\Lambda_{(p,q)-\epsilon}(x) \subset \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b)$.

Now, let $z \in \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b)$. It follows

$$a - zu \notin (u\mathcal{A}u)_{p_1,q_1}^{(2)} \text{ or } \|(a - zu)_{p_1,q_1}^{(2)}\| \geq \epsilon$$

or

$$b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))_{p_2,q_2}^{(2)} \text{ or } \|(b - z(1-u))_{p_2,q_2}^{(2)}\| \geq \epsilon.$$

If either $a - zu \notin (u\mathcal{A}u)_{p_1,q_1}^{(2)}$ or $b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))_{p_2,q_2}^{(2)}$, by Lemma 3.5, it follows $x - z \notin \mathcal{A}_{p,q}^{(2)}$. So, $z \in \Lambda_{(p,q)-\epsilon}(x)$.

On the other hand, if

$$a - zu \in (u\mathcal{A}u)_{p_1,q_1}^{(2)} \text{ and } b - z(1-u) \in ((1-u)\mathcal{A}(1-u))_{p_2,q_2}^{(2)},$$

it holds either $\|(a - zu)_{p_1,q_1}^{(2)}\| \geq \epsilon$ or $\|(b - z(1-u))_{p_2,q_2}^{(2)}\| \geq \epsilon$. Therefore, $\|(x - z)_{p,q}^{(2)}\| = \max\{\|(a - zu)_{p_1,q_1}^{(2)}\|, \|(b - z(1-u))_{p_2,q_2}^{(2)}\|\} \geq \epsilon$. This proves that $z \in \Lambda_{(p,q)-\epsilon}(x)$.

The inclusion $\Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b) \subset \Lambda_{(p,q)-\epsilon}(x)$ has been proved. \square

Theorem 3.8. Let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $0 < \epsilon < 1$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then

$$\sigma_{(p_1, q_1) - \epsilon}(a) \cup \sigma_{(p_2, q_2) - \epsilon}(b) \subset \sigma_{(p, q) - \epsilon}(x).$$

Proof. Let $z \in \sigma_{(p_1, q_1) - \epsilon}(a) \cup \sigma_{(p_2, q_2) - \epsilon}(b)$. These imply

$$a - zu \notin (u\mathcal{A}u)_{p_1, q_1}^{(2)} \text{ or } \|(a - zu)_{p_1, q_1}^{(2)}\| \cdot \|a - zu\| \geq \frac{1}{\epsilon}$$

or

$$b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))_{p_2, q_2}^{(2)} \text{ or } \|(b - z(1-u))_{p_2, q_2}^{(2)}\| \cdot \|b - z(1-u)\| \geq \frac{1}{\epsilon}.$$

If either $a - zu \notin (u\mathcal{A}u)_{p_1, q_1}^{(2)}$ or $b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))_{p_2, q_2}^{(2)}$, by Lemma 3.5, it follows $x - z \notin \mathcal{A}_{p, q}^{(2)}$. Then, we have $z \in \sigma_{(p, q) - \epsilon}(x)$.

On the other hand, if

$$a - zu \in (u\mathcal{A}u)_{p_1, q_1}^{(2)} \text{ and } b - z(1-u) \in ((1-u)\mathcal{A}(1-u))_{p_2, q_2}^{(2)},$$

it holds either

$$\|(a - zu)_{p_1, q_1}^{(2)}\| \cdot \|a - zu\| \geq \frac{1}{\epsilon} \text{ or } \|(b - z(1-u))_{p_2, q_2}^{(2)}\| \cdot \|b - z(1-u)\| \geq \frac{1}{\epsilon}.$$

Without loss of generality, assume that $\|(a - zu)_{p_1, q_1}^{(2)}\| \cdot \|a - zu\| \geq \frac{1}{\epsilon}$ holds.

Therefore,

$$\begin{aligned} & \|(x - z)_{p, q}^{(2)}\| \|x - z\| = \\ & = \max\{\|(a - zu)_{p_1, q_1}^{(2)}\|, \|(b - z(1-u))_{p_2, q_2}^{(2)}\|\} \cdot \max\{\|a - zu\|, \|b - z(1-u)\|\} \\ & \geq \|(a - zu)_{p_1, q_1}^{(2)}\| \cdot \|a - zu\| \geq \frac{1}{\epsilon}. \end{aligned}$$

This proves that $z \in \sigma_{(p, q) - \epsilon}(x)$. □

The next example shows that the converse inclusion is not true in the previous theorem.

Example 3.9. Let $0 < \epsilon < 1$, $z \in \mathbb{C}$ and $u \in \mathcal{A}^\bullet$ such that $\|u\| < \frac{1}{\sqrt{\epsilon}}$ and

$\|1 - u\| < \frac{1}{\sqrt{\epsilon}}$. Let $x = \begin{bmatrix} (\epsilon^2 + z)u & 0 \\ 0 & (\epsilon + z)(1-u) \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$. Then

$$z \in \sigma_{(1, 0) - \epsilon}(x), \text{ but } z \notin (\sigma_{(u, 0) - \epsilon}((\epsilon^2 + z)u) \cup \sigma_{(1-u, 0) - \epsilon}((\epsilon + z)(1-u))).$$

Proof. For idempotents $u \in \mathcal{A}$ and $1 - u \in \mathcal{A}$, we have $\|u\| \geq 1$ and $\|1 - u\| \geq 1$. There exists the inverse

$$(x - z)_{1, 0}^{(2)} = \begin{bmatrix} \frac{1}{\epsilon^2}u & 0 \\ 0 & \frac{1}{\epsilon}(1-u) \end{bmatrix}_u$$

as well as inverses

$$((\epsilon^2 + z)u - zu)_{u,0}^{(2)} = (\epsilon^2 u)_{u,0}^{(2)} = \frac{1}{\epsilon^2} u$$

and

$$((\epsilon + z)(1 - u) - z(1 - u))_{1-u,0}^{(2)} = (\epsilon(1 - u))_{1-u,0}^{(2)} = \frac{1}{\epsilon}(1 - u).$$

Now, we have

$$\begin{aligned} & \| (x - z)_{1,0}^{(2)} \| \| x - z \| = \\ & = \max \left\{ \left\| \frac{1}{\epsilon^2} u \right\|, \left\| \frac{1}{\epsilon} (1 - u) \right\| \right\} \cdot \max \{ \| \epsilon^2 u \|, \| \epsilon (1 - u) \| \} \\ & = \left\| \frac{1}{\epsilon^2} u \right\| \cdot \| \epsilon (1 - u) \| \geq \left| \frac{1}{\epsilon^2} \right| \cdot |\epsilon| \geq \frac{1}{\epsilon}, \end{aligned}$$

but also

$$\| (\epsilon^2 u)_{u,0}^{(2)} \| \cdot \| \epsilon^2 u \| = \left\| \frac{1}{\epsilon^2} u \right\| \cdot \| \epsilon^2 u \| = \| u \|^2 < \frac{1}{\epsilon},$$

and

$$\| (\epsilon(1 - u))_{1-u,0}^{(2)} \| \cdot \| \epsilon(1 - u) \| = \left\| \frac{1}{\epsilon} (1 - u) \right\| \cdot \| \epsilon(1 - u) \| = \| 1 - u \|^2 < \frac{1}{\epsilon}.$$

Therefore,

$$z \in \sigma_{(1,0)-\epsilon}(x), \text{ but } z \notin (\sigma_{(u,0)-\epsilon}((\epsilon^2 + z)u) \cup \sigma_{(1-u,0)-\epsilon}((\epsilon + z)(1 - u))).$$

□

If $x \in \mathcal{A}$ is invertible, $p = 1$ and $q = 0$, then $x^{-1} = x_{p,q}^{(2)}$.

As corollaries of Theorem 3.7 and Theorem 3.8, we formulate the following results for the pseudospectrum and the condition spectrum.

Theorem 3.10. *Let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$ and $\epsilon > 0$. Then*

$$\Lambda_\epsilon(x) = \Lambda_\epsilon(a) \cup \Lambda_\epsilon(b).$$

Theorem 3.11. *Let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$ and $0 < \epsilon < 1$. Then*

$$\sigma_\epsilon(a) \cup \sigma_\epsilon(b) \subset \sigma_\epsilon(x).$$

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