



## CHLODOVSKY TYPE OPERATORS ON PARABOLIC DOMAIN

SEZGİN SUCU\* AND ERTAN İBİKLİ

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**ABSTRACT.** In the present paper, we investigate approximation to analytic function having property  $B$  by complex linear operators sequence in parabolic domain. We apply Gergen, Dressel and Purcell's method to Chlodovsky type complex Szasz operators.

### 1. INTRODUCTION

There are many reasons for studying approximation theory ranging from a need to represent functions in computer calculations to an interest in mathematics of the subject. Over one hundred years ago Weierstrass proved a very significant theorem on uniform approximation of continuous functions by polynomials on intervals of the real line. His study had a incredible impression on the advancement of some parts of the theory of functions besides was an initial point for investigation of analytic approximation on subset in complex space.

Picard, Volterra and Lebesgue gave other proofs of the above mentioned approximation theorem after Weierstrass. One of the most elementary proofs of this theorem is given by Bernstein. Bernstein [1] obtained a probabilistic proof of Weierstrass' approximation theorem by introducing the following polynomials

$$B_n(f; x) := \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}$$

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\* Corresponding author.

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known as Bernstein polynomials. In the case of the function  $f(z)$  is defined and analytic in a certain region involving the interval  $[0, 1]$ , the problem was investigated by Wright [26], Kantorovich [15] and then Bernstein [2]. The degree of approximation for previous mentioned work at first was obtained by Gal [5] on compact disks.

Bernstein–Chlodovsky polynomials were introduced by Chlodovsky [3] as a generalization of Bernstein polynomials to the interval  $[0, \tau_n]$ , where  $\tau_n$  tends to infinity and with the properties  $\frac{\tau_n}{n} \rightarrow 0$  as  $n \rightarrow \infty$ . The polynomials are defined by

$$\tilde{B}_n(f; x) := \sum_{k=0}^n \binom{n}{k} \left(\frac{x}{\tau_n}\right)^k \left(1 - \frac{x}{\tau_n}\right)^{n-k} f\left(\frac{\tau_n k}{n}\right), \quad n \in \mathbb{N}.$$

In his paper, he also proved that the sequence of complex version of Bernstein–Chlodovsky polynomials  $\tilde{B}_n(f; z)$  converges uniformly to analytic function  $f(z)$  in each circle of finite radius under some conditions on  $\tau_n$ .

In 1950, Szasz [22] studied approximations of functions on an unbounded interval. He extended the Bernstein polynomials to an infinite interval as follows

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}. \tag{1.1}$$

Szasz–Chlodovsky operator was defined as

$$C_n(f; x) := e^{-\frac{nx}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nx}{\tau_n}\right)^k \frac{1}{k!} f\left(\frac{\tau_n k}{n}\right) \tag{1.2}$$

and studied by Stypinski [21]. The author investigated convergence and approximation properties of Chlodovsky variant of Szasz operators given by (1.2).

Gergen, Dressel and Purcell [11] proved that for a certain class of analytic functions  $f(z)$  the sequence of complex Szasz operators  $S_n(f; z)$  approximates these functions. Quantitative estimates of this convergence result were obtained by Gal ([4], [6], (see also [5], pp. 104-113 and pp. 114-124, respectively)). Ghorbanalizadeh [12] extended Gergen, Dressel and Purcell’s results for Stancu type generalization of Szasz operators.

In recent years there has been considerable interest in the problem of complex approximation. Convergence properties with quantitative estimations of complex Bernstein type polynomials in compact disks were studied by many authors such as Gal ([4], [5], [6], [7], [8]), Gal–Gupta ([9], [10]), Gupta [13], Mahmudov([16], [17]), Mahmudov–Gupta [18], Ostrovska [19] and Wang–Wu [25].

Considering the parabolic set for  $d > 0$

$$p(d) := \{z = x + iy : |z| < x + 2d^2\}.$$

If there exists to each  $b$  ( $0 < b < d$ ) a positive number  $B(b)$  such that for  $z \in p(b)$

$$|f(z)| \leq B(b) \exp \left\{ \frac{x}{2} - |x|^{1/2} \left[ b^2 - \frac{1}{2} (|z| - x) \right]^{1/2} \right\}, \tag{1.3}$$

then a function  $f(z)$  defined in  $p(d)$  is said to be property  $B$  in  $p(d)$ . We say that a sequence  $\{f_k\}_{k>0}$ , each defined in  $p(d)$ , has property  $B$  uniformly in  $p(d)$

if there corresponds to each  $b$  ( $0 < b < d$ ) a positive constant  $B(b)$  such that (1.3) holds for each  $f_k$  function.

In this paper, we propose a modification of Szasz operators as follows

$$C_n^{\alpha, \beta}(f; z) := e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} f\left(\frac{k+\alpha}{n+\beta}\tau_n\right), \quad (1.4)$$

where  $\alpha, \beta \geq 0$  and  $(\tau_n)$  is a positive increasing sequence of real with the properties

$$\lim_{n \rightarrow \infty} \tau_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\tau_n}{n} = 0. \quad (1.5)$$

This paper is organized as follows. We shall first introduce basic notion on necessary and sufficient conditions that a function be represented by a convergent Laguerre series in a parabolic domain  $p(d)$  and give some lemmas. By using these lemmas, we show that for analytic function  $f(z)$  having property  $B$  in  $p(d)$  the sequence of Chlodovsky type complex Szasz operators  $C_n^{\alpha, \beta}(f; z)$  converges to the function  $f(z)$  in parabolic domain.

## 2. APPROXIMATION BY COMPLEX CHLODOVSKY TYPE OPERATORS

The problem of finding necessary and sufficient conditions that a function be represented by a Hermite series convergent in a strip has been solved by Hille [14]. Pollard [20] has solved the corresponding problem for Laguerre series of order zero in a strip. On the other hand, Szasz and Yearley [23] investigated the problem for Laguerre series of order  $\theta$  ( $\theta > -1$ ) getting as a region a parabola by way of a strip.

In the first part of this section, we give some background materials needed for proof our main theorem.

**Theorem 2.1.** (Szasz and Yearley [23]) *A necessary and sufficient condition that a function  $f(z)$  be analytic and has property  $B$  in  $p(d)$ ,  $d > 0$ , is that  $f(z)$  possess a Laguerre series (of order 0)*

$$f(z) \sim \sum_{m=0}^{\infty} a_m L_m(z), \quad a_m = \int_0^{\infty} e^{-x} L_m(x) f(x) dx,$$

which converges to in  $p(d)$ .

**Lemma 2.2.** *If  $f(z)$  is polynomial, then  $C_n^{\alpha, \beta}(f; z)$  is polynomial. Besides, the degree of  $C_n^{\alpha, \beta}(f; z)$  is equal to the degree of  $f(z)$ .*

*Proof.* For  $m \in \mathbb{N}_0$  let be  $f(z) = e_m(z) = z^m$ . It is easy to see that the following identity

$$e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} k^m = \sum_{j=0}^m F_j^{(m)} \left(\frac{nz}{\tau_n}\right)^j \quad (2.1)$$

is true, where  $F_j^{(m)}$  are constants such that  $F_m^{(m)} = 1$ . According to the Binomial identity and statement (2.1), we get

$$\begin{aligned} C_n^{\alpha,\beta}(f; z) &= e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} \left(\frac{k+\alpha}{n+\beta}\tau_n\right)^m \\ &= e^{-\frac{nz}{\tau_n}} \left(\frac{\tau_n}{n+\beta}\right)^m \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} \left(\sum_{i=0}^m \binom{m}{i} k^i \alpha^{m-i}\right) \\ &= \sum_{i=0}^m \binom{m}{i} \alpha^{m-i} \left(\frac{\tau_n}{n+\beta}\right)^m \sum_{j=0}^i F_j^{(i)} \left(\frac{nz}{\tau_n}\right)^j. \end{aligned}$$

Using the above equality and (1.5), one can obtain  $C_n^{\alpha,\beta}(e_m; z) \rightarrow z^m$  as  $n \rightarrow \infty$ . This convergence is uniform on each compact subset of complex plane. Due to the fact that  $C_n^{\alpha,\beta}$  operators are linear, the result obtained is verified for arbitrary polynomials.  $\square$

**Lemma 2.3.** *Suppose that  $L_m$  be  $m$ -th Laguerre polynomials of order zero and let be*

$$G_n^{\alpha,\beta,m}(z) := C_n^{\alpha,\beta}(L_m; z).$$

Then

$$|G_n^{\alpha,\beta,m}(z)| \leq (e_{n,\beta}^{\tau_n})^\alpha \exp\left(-\frac{nx}{\tau_n} + \frac{n|z|}{\tau_n} e_{n,\beta}^{\tau_n}\right) \quad (2.2)$$

and for  $|\omega| < 1$

$$\begin{aligned} \sum_{m=0}^{\infty} G_n^{\alpha,\beta,m}(z) \omega^m &= \frac{1}{1-\omega} e^{-\frac{\alpha\omega\tau_n}{(n+\beta)(1-\omega)}} \\ &\quad \times \exp\left\{-\frac{nz}{\tau_n} + \frac{nz}{\tau_n} \exp\left(-\frac{\omega\tau_n}{(n+\beta)(1-\omega)}\right)\right\}, \quad (2.3) \end{aligned}$$

where  $e_{n,\beta}^{\tau_n} = \exp\left(\frac{\tau_n}{2(n+\beta)}\right)$ , are satisfied.

*Proof.* With the help of the following identity  $|L_m(x)| \leq \exp\left(\frac{x}{2}\right)$  ( $x \geq 0$ ), we deduce

$$\begin{aligned} |G_n^{\alpha,\beta,m}(z)| &= \left| e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} L_m\left(\frac{k+\alpha}{n+\beta}\tau_n\right) \right| \\ &\leq e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{n|z|}{\tau_n}\right)^k \frac{1}{k!} \exp\left(\frac{k+\alpha}{2(n+\beta)}\tau_n\right) \\ &= (e_{n,\beta}^{\tau_n})^\alpha \exp\left(-\frac{nx}{\tau_n} + \frac{n|z|}{\tau_n} e_{n,\beta}^{\tau_n}\right). \end{aligned}$$

On the other hand, using the below generating relation for Laguerre polynomials of order zero

$$\sum_{m=0}^{\infty} L_m(z) \omega^m = \frac{1}{1-\omega} \exp\left(-\frac{\omega z}{1-\omega}\right), \quad |\omega| < 1$$

we get

$$\begin{aligned}
\sum_{m=0}^{\infty} G_n^{\alpha, \beta, m}(z) \omega^m &= \sum_{m=0}^{\infty} \left\{ e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left( \frac{nz}{\tau_n} \right)^k \frac{1}{k!} L_m \left( \frac{k + \alpha}{n + \beta} \tau_n \right) \right\} \omega^m \\
&= \sum_{k=0}^{\infty} e^{-\frac{nz}{\tau_n}} \left( \frac{nz}{\tau_n} \right)^k \frac{1}{k!} \left( \sum_{m=0}^{\infty} L_m \left( \frac{k + \alpha}{n + \beta} \tau_n \right) \omega^m \right) \\
&= \frac{1}{1 - \omega} e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left( \frac{nz}{\tau_n} \right)^k \frac{1}{k!} \exp \left( -\frac{(k + \alpha) \omega}{(n + \beta)(1 - \omega)} \tau_n \right) \\
&= \frac{1}{1 - \omega} e^{-\frac{\alpha \omega \tau_n}{(n + \beta)(1 - \omega)}} \\
&\quad \times \exp \left\{ -\frac{nz}{\tau_n} + \frac{nz}{\tau_n} \exp \left( -\frac{\omega \tau_n}{(n + \beta)(1 - \omega)} \right) \right\}.
\end{aligned}$$

Because of that the double series in the first statement of the above equality is absolutely convergent for  $n, z, \omega$  ( $|\omega| < 1$ ), we change the order of summation.  $\square$

**Lemma 2.4.** *Let be*

$$H_n^\beta(z, \omega) = \Re \left\{ -\frac{nz}{\tau_n} + \frac{nz}{\tau_n} \exp \left( -\frac{\omega \tau_n}{(n + \beta)(1 - \omega)} \right) \right\}.$$

Then

$$H_n^\beta(z, \omega) \leq \eta r \frac{|z| - rx}{1 - r^2}, \quad |\omega| = r < 1 \quad (2.4)$$

where  $\eta = \eta(r, n) = \exp \left( \frac{r}{(n + \beta)(r + 1)} \tau_n \right)$ .

*Proof.* Inequality (2.4) can be easily verified for  $z = 0$  or  $\omega = 0$ . Suppose that for the proportion  $n, z, \omega$  such that  $z \neq 0$  and  $0 < |\omega| = r < 1$ . Representation for the convenience, let us define

$$z = |z| e^{i\phi}, \quad \rho = \frac{r}{1 - r^2}, \quad e^{i\theta} = \frac{\omega(1 - \bar{\omega})}{r(1 - \omega)}$$

$$a = \frac{\tau_n}{n + \beta}, \quad \Phi = \phi - a\rho \sin \theta.$$

It is sufficient to show that the function  $T(\theta, \phi)$  defined by

$$T(\theta, \phi) = (a\rho\eta\tau_n r - (\tau_n - a\beta)) \cos \phi + (\tau_n - a\beta) e^{-a\rho(r + \cos \theta)} \cos \Phi$$

satisfies for  $|\theta| \leq \pi, |\phi| \leq \pi$

$$T(\theta, \phi) \leq a\rho\eta\tau_n. \quad (2.5)$$

From the fact that the function  $T(\theta, \phi)$  is symmetric about the origin, we shall prove the inequality (2.5) on the following defined set

$$\mathcal{D} := \{(\theta, \phi) : 0 \leq \theta \leq \pi, -\pi \leq \phi \leq \pi\}.$$

(i) Consider the case of  $a\eta r\rho \geq 1$  first. From the well-known inequality  $e^t \leq 1 + te^t$  ( $t \geq 0$ ), one gets

$$\begin{aligned}
T(\theta, \phi) &= (a\rho\eta\tau_n r - (\tau_n - a\beta)) \cos \phi + (\tau_n - a\beta) e^{-a\rho(r+\cos\theta)} \cos \Phi \\
&\leq a\rho\eta\tau_n r - (\tau_n - a\beta) + (\tau_n - a\beta) e^{-a\rho(r-1)} \\
&\leq a\rho\eta\tau_n r - (\tau_n - a\beta) + (\tau_n - a\beta) [1 - a\rho(r-1) e^{-a\rho(r-1)}] \\
&= a\rho\eta\tau_n r - a\rho(r-1)(\tau_n - a\beta)\eta \\
&= a\rho\eta(\tau_n + a\beta(r-1)) \\
&\leq a\rho\eta\tau_n.
\end{aligned}$$

(ii) Now suppose  $a\eta r\rho < 1$ . Let us denote by  $(\theta, \phi)$  the maximum point of the function  $T$  on  $\mathcal{D}$ . Hence, there are three possible cases:

$$(I) \theta = 0 \quad (II) \theta = \pi \quad (III) 0 < \theta < \pi .$$

(I) Observing now that  $\theta = 0$ , we can write

$$\begin{aligned}
T(\theta, \phi) &= (a\rho\eta\tau_n r - (\tau_n - a\beta)) \cos \phi + (\tau_n - a\beta) e^{-a\rho(r+1)} \cos \phi \\
&= \cos \phi \{a\rho\eta\tau_n r + (\tau_n - a\beta) [-1 + e^{-a\rho(r+1)}]\}. \tag{2.6}
\end{aligned}$$

If  $a\rho\eta\tau_n r + (\tau_n - a\beta) [-1 + e^{-a\rho(r+1)}] \geq 0$  is satisfied, then we deduce from (2.6) that

$$\begin{aligned}
T(\theta, \phi) &= \cos \phi \{a\rho\eta\tau_n r + (\tau_n - a\beta) [-1 + e^{-a\rho(r+1)}]\} \\
&\leq a\rho\eta\tau_n r + (\tau_n - a\beta) [-1 + e^{-a\rho(r+1)}] \\
&\leq a\rho\eta\tau_n r \\
&< a\rho\eta\tau_n.
\end{aligned}$$

If  $a\rho\eta\tau_n r + (\tau_n - a\beta) [-1 + e^{-a\rho(r+1)}] < 0$  is satisfied, then we get from (2.6) and inequality  $e^t \leq 1 + te^t$  ( $t \geq 0$ ) that

$$\begin{aligned}
T(\theta, \phi) &= \cos \phi \{a\rho\eta\tau_n r + (\tau_n - a\beta) [-1 + e^{-a\rho(r+1)}]\} \\
&\leq -a\rho\eta\tau_n r + (\tau_n - a\beta) [1 - e^{-a\rho(r+1)}] \\
&= -a\rho\eta\tau_n r + (\tau_n - a\beta) e^{\frac{a\rho}{r-1}} \left( e^{-\frac{a\rho}{r-1}} - 1 \right) \\
&\leq -a\rho\eta\tau_n r + (\tau_n - a\beta) \frac{a\rho}{1-r} \\
&= a\rho \{-r\eta\tau_n + (1+r)(\tau_n - a\beta)\} \\
&< a\rho\eta\tau_n.
\end{aligned}$$

(II) We now proceed to prove of the inequality (2.5) for  $\theta = \pi$ . Using the  $e^t \leq 1 + te^t$  ( $t \geq 0$ ), we conclude the following

$$\begin{aligned}
T(\theta, \phi) &= \{a\rho\eta\tau_n r - (\tau_n - a\beta) + (\tau_n - a\beta) e^{-a\rho(r-1)}\} \cos \phi \\
&\leq a\rho\eta\tau_n r + (\tau_n - a\beta)(\eta - 1) \\
&\leq a\rho\eta\tau_n r + \frac{a\rho}{r+1}\eta\tau_n \\
&= a\rho\eta\tau_n.
\end{aligned}$$

(III) The proof for the case when  $0 < \theta < \pi$ , we use the following way. It is important to note that in this case the both first partial derivatives of the function  $T$  at the point  $(\theta, \phi)$  are zero. With the help of this information, we induce

$$\begin{aligned} T_\theta(\theta, \phi) &= (\tau_n - a\beta) a\rho e^{-a\rho(r+\cos\theta)} \{\cos\theta \sin\Phi + \sin\theta \cos\Phi\} \\ &= (\tau_n - a\beta) a\rho e^{-a\rho(r+\cos\theta)} \sin(\theta + \Phi) \\ &= 0, \end{aligned}$$

so

$$\sin(\theta + \Phi) = 0 \quad (2.7)$$

and

$$\begin{aligned} T_\phi(\theta, \phi) &= -(a\rho\eta\tau_n r - (\tau_n - a\beta)) \sin\phi - (\tau_n - a\beta) \sin\Phi e^{-a\rho(r+\cos\theta)} \\ &= 0 \end{aligned}$$

so

$$(a\rho\eta\tau_n r - (\tau_n - a\beta)) \sin\phi + (\tau_n - a\beta) \sin\Phi e^{-a\rho(r+\cos\theta)} = 0. \quad (2.8)$$

As an immediate consequence of (2.7) and (2.8), we have the following result

$$\begin{aligned} T(\theta, \phi) \sin\theta &= (a\rho\eta\tau_n r - (\tau_n - a\beta)) \cos\phi \sin\theta - \cos\theta \sin\Phi (\tau_n - a\beta) e^{-a\rho(r+\cos\theta)} \\ &= (a\rho\eta\tau_n r - (\tau_n - a\beta)) (\cos\phi \sin\theta + \cos\theta \sin\phi) \\ &= (a\rho\eta\tau_n r - (\tau_n - a\beta)) \sin(\theta + \phi). \end{aligned} \quad (2.9)$$

Combining the solution of (2.7) with (2.9), for  $k \in \mathbb{Z}$  one can get

$$\begin{aligned} |T(\theta, \phi) \sin\theta| &= |a\rho\eta\tau_n r - (\tau_n - a\beta)| |\sin(k\pi + a\rho \sin\theta)| \\ &\leq a\rho |a\rho\eta\tau_n r - (\tau_n - a\beta)| |\sin\theta|. \end{aligned}$$

On condition that  $a\rho\eta\tau_n r - (\tau_n - a\beta) \geq 0$ , then from  $a\eta r \rho < 1$  and  $0 < \theta < \pi$

$$|T(\theta, \phi) \sin\theta| \leq a\rho a\beta \sin\theta \leq a\rho\tau_n \sin\theta$$

and provided that  $a\rho\eta\tau_n r - (\tau_n - a\beta) < 0$ , then under the same restrictions

$$|T(\theta, \phi) \sin\theta| \leq a\rho\eta\tau_n \sin\theta.$$

Hence; the above two results immediately prove our lemma.  $\square$

**Lemma 2.5.** (Gergen, Dressel and Purcell [11]) Assume that  $\lambda, \mu, \kappa$  positive constants such that  $\lambda \leq \mu$  and let us define

$$u(t) = \frac{4\lambda^2}{t} + \frac{t}{t+4}\mu^2.$$

Then the following inequality

$$I(\lambda, \mu, \kappa) = \int_0^\infty \frac{t^{-\frac{3}{2}}}{1-e^{-t}} \exp\left(-u(t) - \frac{4\kappa^2}{t}\right) dt \leq M_1(\kappa) \exp(\lambda^2 - 2\lambda\mu)$$

is valid, where

$$M_1(\kappa) := \frac{e}{e-1} \left(2 + \frac{\sqrt{\pi}}{16\kappa^3}\right).$$

**Lemma 2.6.** (Gergen, Dressel and Purcell [11]) For  $0 < b < c$ , let us define

$$J(b, c, z) = \int_0^\infty \frac{t^{-\frac{3}{2}}}{1 - e^{-t}} \exp \left\{ -\frac{4c^2}{t} + \frac{2e^{-\frac{t}{2}}}{1 - e^{-t}} \left( |z| - xe^{-\frac{t}{2}} \right) \right\} dt.$$

Then we have for  $z \in p(b)$

$$J(b, c, z) \leq M_2(b, c) \exp \left\{ x - 2|x|^{\frac{1}{2}} \left[ b^2 - \frac{1}{2}(|z| - x) \right]^{\frac{1}{2}} \right\}$$

where

$$M_2(b, c) = e^{4b^2} M_1 \left( (c^2 - b^2)^{\frac{1}{2}} \right).$$

**Lemma 2.7.** For  $0 < b < c$  and  $z \in p(b)$ , the image of the  $L_m$  Laguerre polynomials of order zero under the operator  $C_n^{\alpha, \beta}$  satisfies the following inequality

$$\sum_{m=0}^{\infty} \left| G_n^{\alpha, \beta, m} \left( z/e_{n, \beta}^{\tau_n} \right) \right|^2 \exp(-4c\sqrt{m}) \leq M_3(b, c) \exp \left\{ x - 2|x|^{\frac{1}{2}} \left[ b^2 - \frac{1}{2}(|z| - x) \right]^{\frac{1}{2}} \right\},$$

where

$$M_3(b, c) = \sup_{n \in \mathbb{N}} e^{\frac{\alpha \tau_n}{n+\beta}} \frac{2c}{\sqrt{\pi}} M_2(b, c).$$

*Proof.* Let  $C_r$  designate the circle having radius  $r$  ( $0 < r < 1$ ) with the center at the origin in the  $\omega$  plane. To demonstrate the idea, we first state

$$\int_{C_r} \omega^m \bar{\omega}^j |d\omega| = \begin{cases} 2\pi r^{2m+1} & ; \quad m = j \\ 0 & ; \quad m \neq j \end{cases}.$$

By using the above-mentioned equality, Lemma 2.3 and Lemma 2.4, we find

$$\begin{aligned} \sum_{m=0}^{\infty} \left| G_n^{\alpha, \beta, m}(z) \right|^2 r^{2m} &= \frac{1}{2\pi r} \int_{C_r} \frac{1}{|1 - \omega|^2} \left| \exp \left( -\frac{\alpha \omega \tau_n}{(n + \beta)(1 - \omega)} \right) \right|^2 \\ &\quad \times \left| \exp \left\{ -\frac{nz}{\tau_n} + \frac{nz}{\tau_n} \exp \left( -\frac{\omega \tau_n}{(n + \beta)(1 - \omega)} \right) \right\} \right|^2 |d\omega| \\ &= \frac{1}{2\pi r} \int_{C_r} \frac{1}{|1 - \omega|^2} \exp \left\{ \Re \left( -\frac{2\alpha \omega \tau_n}{(n + \beta)(1 - \omega)} \right) \right\} \\ &\quad \times \exp(2H_n^\beta(z, \omega)) |d\omega| \\ &\leq \left( \sup_{n \in \mathbb{N}} e^{\frac{\alpha \tau_n}{n+\beta}} \right) \frac{1}{2\pi r} \exp \left( 2e_{n, \beta}^{\tau_n} r \frac{|z| - rx}{1 - r^2} \right) \int_{C_r} \frac{1}{|1 - \omega|^2} |d\omega| \\ &= \left( \sup_{n \in \mathbb{N}} e^{\frac{\alpha \tau_n}{n+\beta}} \right) \frac{1}{1 - r^2} \exp \left( 2e_{n, \beta}^{\tau_n} r \frac{|z| - rx}{1 - r^2} \right). \end{aligned}$$

With the proper choice of  $t > 0$ , one derives

$$\sum_{m=0}^{\infty} \left| G_n^{\alpha, \beta, m} \left( z/e_{n, \beta}^{\tau_n} \right) \right|^2 e^{-tm} \leq \sup_{n \in \mathbb{N}} e^{\frac{\alpha \tau_n}{n+\beta}} \frac{1}{1 - e^{-t}} \exp \left( 2e^{-\frac{t}{2}} \frac{|z| - e^{-\frac{t}{2}} x}{1 - e^{-t}} \right).$$



Taking into consideration the fact that

$$\exp(-4c\sqrt{m}) = \frac{2c}{\sqrt{\pi}} \int_0^{\infty} t^{-\frac{3}{2}} \exp\left(-mt - \frac{4c^2}{t}\right) dt$$

we have by Lemma 2.6

$$\begin{aligned} & \sum_{m=0}^{\infty} |G_n^{\alpha,\beta,m}(z/e_{n,\beta}^{\tau_n})|^2 \exp(-4c\sqrt{m}) \\ &= \frac{2c}{\sqrt{\pi}} \sum_{m=0}^{\infty} |G_n^{\alpha,\beta,m}(z/e_{n,\beta}^{\tau_n})|^2 \int_0^{\infty} t^{-\frac{3}{2}} \exp\left(-mt - \frac{4c^2}{t}\right) dt \\ &= \frac{2c}{\sqrt{\pi}} \int_0^{\infty} t^{-\frac{3}{2}} \exp\left(-\frac{4c^2}{t}\right) \left\{ \sum_{m=0}^{\infty} |G_n^{\alpha,\beta,m}(z/e_{n,\beta}^{\tau_n})|^2 \exp(-mt) \right\} dt \\ &\leq \sup_{n \in \mathbb{N}} e^{\frac{\alpha\tau_n}{n+\beta}} \frac{2c}{\sqrt{\pi}} \int_0^{\infty} \frac{t^{-\frac{3}{2}}}{1-e^{-t}} \exp\left\{-\frac{4c^2}{t} + 2e^{-\frac{t}{2}} \frac{|z| - e^{-\frac{t}{2}}x}{1-e^{-t}}\right\} dt \\ &\leq \sup_{n \in \mathbb{N}} e^{\frac{\alpha\tau_n}{n+\beta}} \frac{2c}{\sqrt{\pi}} M_2(b, c) \exp\left\{x - 2|x|^{\frac{1}{2}} \left[b^2 - \frac{1}{2}(|z| - x)\right]^{\frac{1}{2}}\right\}, \end{aligned}$$

for  $z \in p(b)$ . This proves the claim.  $\square$

**Lemma 2.8.** *Assume that function  $f(z)$  is analytic and has property  $B$  in  $p(d)$ . Then, the following statements hold:*

- (i) *For each  $n \in \mathbb{N}$ ,  $C_n^{\alpha,\beta}(f; z)$  is entire function of  $z \in \mathbb{C}$ .*
- (ii) *The sequence  $\{C_n^{\alpha,\beta}(f; z/e_{n,\beta}^{\tau_n})\}$  has property  $B$  uniformly in  $p(d)$ .*

*Proof.* (i) Due to  $f(z)$  has property  $B$  in  $p(d)$ , there exists a positive constant  $A$  such that

$$|f(x)| \leq Ae^{\frac{x}{2}}, \quad x \geq 0.$$

This means that the following series

$$e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} f\left(\frac{k+\alpha}{n+\beta}\tau_n\right)$$

is convergent for  $n \in \mathbb{N}$  and  $z \in \mathbb{C}$ . Therefore, the first assertion is immediate.

(ii) Since the function  $f(z)$  is analytic and has property  $B$  in  $p(d)$ , from the work of Szasz–Yeardley [23] stated previously in Theorem 2.1, one deduces

$$f(z) = \sum_{m=0}^{\infty} a_m L_m(z) \quad \text{for } z \in p(d),$$

where

$$a_m = \int_0^{\infty} e^{-x} L_m(x) f(x) dx \quad \text{and} \quad d = -\limsup_{m \rightarrow \infty} \left( \frac{1}{2\sqrt{m}} \ln |a_m| \right). \quad (2.10)$$

So, from the last equality, for arbitrary  $\epsilon > 0$  there exists an appropriate  $A_\epsilon > 0$  such that

$$|a_m| \leq A_\epsilon \exp(2\sqrt{m}(-d + \epsilon)), \quad m = 1, 2, \dots \quad (2.11)$$

Considering the (2.11), we get

$$\sum_{m=0}^{\infty} |a_m| < \infty \quad (2.12)$$

and for  $0 < c < d$

$$M(c, f) = \sum_{m=0}^{\infty} |a_m|^2 \exp(4c\sqrt{m}) < \infty. \quad (2.13)$$

Taking account of  $|L_m(x)| \leq \exp\left(\frac{x}{2}\right)$  ( $x \geq 0$ ) and the statement (2.12), the series in the first line of following is absolutely convergent for  $n \in \mathbb{N}$  and  $z \in \mathbb{C}$ , because of that one can write

$$\begin{aligned} C_n^{\alpha, \beta}(f; z) &= e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} f\left(\frac{k + \alpha}{n + \beta} \tau_n\right) \\ &= e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} \left(\sum_{m=0}^{\infty} a_m L_m\left(\frac{k + \alpha}{n + \beta} \tau_n\right)\right) \\ &= \sum_{m=0}^{\infty} a_m \left(e^{-\frac{nz}{\tau_n}} \sum_{k=0}^{\infty} \left(\frac{nz}{\tau_n}\right)^k \frac{1}{k!} L_m\left(\frac{k + \alpha}{n + \beta} \tau_n\right)\right) \\ &= \sum_{m=0}^{\infty} a_m G_n^{\alpha, \beta, m}(z). \end{aligned} \quad (2.14)$$

Applying Cauchy–Schwarz inequality to (2.14), we deduce

$$|C_n^{\alpha, \beta}(f; z)| \leq \sqrt{M(c, f)} \left\{ \sum_{m=0}^{\infty} |G_n^{\alpha, \beta, m}(z)|^2 \exp(-4c\sqrt{m}) \right\}^{1/2}. \quad (2.15)$$

Hence; for  $0 < b < c < d$  and  $z \in p(b)$  the statement (2.15) and Lemma 2.7 yield

$$|C_n^{\alpha, \beta}(f; z/e_{n, \beta}^{\tau_n})| \leq \sqrt{M(c, f) M_3(b, c)} \exp \left\{ \frac{x}{2} - |x|^{\frac{1}{2}} \left[ b^2 - \frac{1}{2} (|z| - x) \right]^{\frac{1}{2}} \right\}.$$

For a fixed  $b$  such that  $0 < b < d$ , by choosing  $c = \frac{b+d}{2}$ , we conclude

$$|C_n^{\alpha, \beta}(f; z/e_{n, \beta}^{\tau_n})| \leq B(b) \exp \left\{ \frac{x}{2} - |x|^{\frac{1}{2}} \left[ b^2 - \frac{1}{2} (|z| - x) \right]^{\frac{1}{2}} \right\},$$

where  $B(b) = \sqrt{M\left(\frac{b+d}{2}, f\right) M_3\left(b, \frac{b+d}{2}\right)}$ . This completes the proof.  $\square$

After these preparations we are in a position to state our main theorem.

**Theorem 2.9.** *Let the sequence  $(\tau_n)$  satisfies condition (1.5). Under the assumptions of Lemma 2.8, the sequence of linear operators  $\{C_n^{\alpha,\beta}(f; z)\}$  converges to the function  $f(z)$  as  $n \rightarrow \infty$  in  $p(d)$ . This convergence being uniform on each compact subset of  $p(d)$ .*

*Proof.* We may assume that  $\Omega$  is a compact subset of  $p(d)$ . Now, we will show the uniform convergence of a sequence of functions  $C_n^{\alpha,\beta}(f; z)$  to  $f(z)$  on  $\Omega$ .  $U(b, x_0)$  denotes the set of

$$U(b, x_0) := \{z = x + iy : |z| < x + 2b^2, \quad x < x_0\}, \quad (2.16)$$

where  $b, x_0 > 0$ . Let us consider  $b_1, b_2, b_3$  ( $0 < b_1 < b_2 < b_3 < d$ );  $x_1, x_2, x_3$  ( $0 < x_1 < x_2 < x_3$ ) such that  $\Omega \subset U(b_1, x_1)$ . Under favor of Lemma 2.8 (ii), for  $z \in U(b_3, x_3)$  there exists  $\tilde{M} > 0$  such that

$$|C_n^{\alpha,\beta}(f; z/e_{n,\beta}^{\tau_n})| \leq \tilde{M}. \quad (2.17)$$

From the definition (2.16), for arbitrary  $n \in \mathbb{N}$  such that  $\frac{n}{\tau_n} > \frac{n_0}{\tau_{n_0}}$  and  $z \in U(b_2, x_2)$  we have  $ze_{n,\beta}^{\tau_n} \in U(b_3, x_3)$ , where

$$\frac{n_0}{\tau_{n_0}} = \max \left\{ \left( 4 \ln \left( \frac{b_3}{b_2} \right) \right)^{-1}, \left( 2 \ln \left( \frac{x_3}{x_2} \right) \right)^{-1} \right\}.$$

By using (2.17), for suitable natural number  $n$  and  $z \in U(b_2, x_2)$  we get

$$|C_n^{\alpha,\beta}(f; z)| = |C_n^{\alpha,\beta}(f; ze_{n,\beta}^{\tau_n}/e_{n,\beta}^{\tau_n})| \leq \tilde{M}. \quad (2.18)$$

On the other hand, in view of Korovkin theorem we have  $C_n^{\alpha,\beta}(f; x) \rightarrow f(x)$  as  $n \rightarrow \infty$  for  $x$  on  $(0, x_1)$ . From (2.18) and the mentioned conclusion, by Vitali's theorem [24] the sequence  $\{C_n^{\alpha,\beta}(f; z)\}$  is convergent uniformly on  $U(b_1, x_1)$  to function  $F(z)$  which is analytic on  $U(b_1, x_1)$ . Due to the fact that the function  $f(z)$  is analytic on  $U(b_1, x_1)$  and  $F(x) = f(x)$  ( $0 < x < x_1$ ), in view of identity theorem of analytic functions we have  $F(z) = f(z)$  on  $U(b_1, x_1)$ . Through the instrument of the these knowledge, above-stated convergence is uniform on  $\Omega$ .

As a consequence of this conclusion, we get

$$\lim_{n \rightarrow \infty} C_n^{\alpha,\beta}(f; z) = f(z)$$

on  $p(d)$ . So, the desired results are obtained.  $\square$

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ANKARA UNIVERSITY FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, TANDOĞAN TR-06100, ANKARA, TURKEY.

*E-mail address:* [ssucu@ankara.edu.tr](mailto:ssucu@ankara.edu.tr)

*E-mail address:* [ibikli@ankara.edu.tr](mailto:ibikli@ankara.edu.tr)