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THE SOLVABILITY AND THE EXACT SOLUTION OF A SYSTEM OF REAL QUATERNION MATRIX EQUATIONS

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Abstract. In this paper, we establish necessary and sufficient conditions for the solvability of the system of real quaternion matrix equations

$$\begin{cases} A_1X = C_1, \\ YB_1 = D_1, \\ A_2Z = C_2, ZB_2 = D_2, A_3ZB_3 = C_3, \\ A_4X + YB_4 + C_4ZD_4 = E_1. \end{cases}$$

We also present an expression of the general solution to the system. The findings of this paper widely extend the known results in the literature.

1. Introduction and preliminaries

Throughout this paper we denote the set of all $m\times n$ matrices over the quaternion number field $\mathbb{H}$

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$. For a matrix A, A^* and $\mathcal{R}(A)$ stand for the conjugate and the column space of A, respectively. I_n denotes the $n\times n$ identity matrix. The Moore-Penrose inverse $A^\dagger$ of A is defined to be the unique matrix $A^\dagger$, such that

$$\text{(i) } AA^\dagger A = A, \text{ (ii) } A^\dagger AA^\dagger = A^\dagger, \text{ (iii) } (AA^\dagger)^* = AA^\dagger, \text{ (iv) } (A^\dagger A)^* = A^\dagger A.$$

Linear matrix functions and their special cases- linear matrix equations are fundamental subjects of study in matrix theory (e.g. [3]-[8], [21]-[27]). The

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matrix function is a matrix-valve map between two linear spaces. The definition of matrix function and introduction of some matrix functions can be seen in [9]. In matrix theory and applications, many problems can be transformed in equivalent rank problems. In recent years this has been applied in seeking for the solvability for matrix equations (see, e.g. [16, 17, 24, 25]).

It is well known that in engineering and linear models, many problems can be expressed by some matrix functions. The limited conditions can be interpreted in limited matrix equations. With the developments of statistical and other science subjects, more parameters and variables are demanded for the matrix equations. Thus, investigations on some matrix functions with more parameters and variables are necessary for the matrix theory and the practical applications. For instance, Roth [13] developed the Sylvester's matrix equation

$$AX - XB = C,$$

giving a necessary and sufficient condition for the consistency of

$$AX - YB = C. \tag{1.1}$$

In statistics, the growth curve model is consistent if and only if the more generalized matrix equation

$$AY_3B + CY_4D = E \tag{1.2}$$

is consistent [18]. A regression model related to equation (1.2) is $M = AXB + CYD + \varepsilon$, where both X and Y are unknown parameter matrices and ε is a random error matrix. This matrix function is also called the nested growth curve model (see [14, 15]). In general, more limited equations means more complexity because more parameters and variables must be considered. Therefore, we first retrospect the development of some matrix equations and investigate the more complex ones.

There have been many papers discussing the classical system of matrix equations

$$A_1XB_1 = C_1, \; A_2XB_2 = C_2. \tag{1.3}$$

For instance, Mitra [10] first studied the system (1.3) over $\mathbb{C}$. Vander Woude [21] investigated it over a field in 1987. Özgüler and Akar [12] gave a condition for the solvability of the system over a principle domain in 1991. In 2004, Wang [26] gave some necessary and sufficient conditions for the existence of the solution to the system (1.3) and provided the expression of the general solution when it is solvable. Moreover, Wang, Chang and Ning [27] provided some necessary and sufficient conditions for the existence of and an explicit expression for a common solution to the six classical linear quaternion matrix equations

$$A_1X = C_1, XB_2 = C_2, A_2X = C_3, XB_2 = C_4, A_3XB_3 = C_5, A_4XB_4 = C_6. \tag{1.4}$$

Observe that (1.1), (1.3) and (1.4) are special cases of the following system of real quaternion matrix equations

$$\begin{cases} A_1X = C_1, \\ YB_1 = D_1, \\ A_2Z = C_2, ZB_2 = D_2, A_3ZB_3 = C_3, \\ A_4X + YB_4 + C_4ZD_4 = E_1 \end{cases} \tag{1.5}$$

However, to our knowledge, so far there has been little information on the expression of the general solution to (1.5) with more variables and more parameters. This paper aims to give some solvability conditions and the expressions of the general solution to (1.5).

In order to get some necessary and sufficient conditions for the existence of the solution to the system (1.5), we need to derive the maximal and minimal ranks of the real quaternion matrix function with triple variables

$$g(X, Y, Z) = E_1 - A_4X - YB_4 - C_4ZD_4, \tag{1.6}$$

where X, Y and Z satisfy the following consistent matrix equations

$$A_1X = C_1, \ YB_1 = D_1, \ A_2Z = C_2, \ ZB_2 = D_2, \ A_3ZB_3 = C_3. \tag{1.7}$$

The investigation on extremal ranks has been actively ongoing for more than 30 years. It is worthy to say that Professor Yongge Tian made great contributions in the literature. Minimal and maximal ranks and inertias are found to be useful in control theory (e.g. [1], [2]). In 2002, Tian [20] considered the maximal and minimal ranks of the matrix function

$$p(X) = A_1 - B_1XC_1 \tag{1.8}$$

subject to

$$B_2XC_2 = A_2. \tag{1.9}$$

In 2008, Wang, Yu and Lin [22] studied the extremal ranks of the quaternion matrix function

$$f(X) = C_4 - A_4XB_4 \tag{1.10}$$

subject to

$$A_1X = C_1, \ XB_2 = C_2, \ A_3XB_3 = C_3. \tag{1.11}$$

Note that (1.8) and (1.10) are special cases of (1.6). The other goal of this paper is to consider the extremal ranks of (1.6) with more variables.

The remaining of this paper is organized as follows. In Section 2, we consider the extremal ranks of the real quaternion matrix function (1.6) subject to (1.7). In Section 3, we give some necessary and sufficient conditions for the solvability to the system of real quaternion matrix equations (1.5) and present an expression of the general solution to system (1.5).

2. Extremal ranks of (1.6) subject to (1.7) with applications

In this section, we investigate the matrix function (1.6) subject to (1.7). The conclusion extends the known results in [20] and [22]. We begin with the following lemmas.

Lemma 2.1. [23] *Let $A_1 \in \mathbb{H}^{m\times n_1}, B_1 \in \mathbb{H}^{p_1\times q}, C_3 \in \mathbb{H}^{m\times n_2}, D_3 \in \mathbb{H}^{p_2\times q}, C_4 \in \mathbb{H}^{m\times n_3}, D_4 \in \mathbb{H}^{p_3\times q}$, and $E_1 \in \mathbb{H}^{m\times q}$ be given. Set*

$$\begin{aligned}&A = R_{A_1}C_3, B = D_3L_{B_1}, C = R_{A_1}C_4, D = D_4L_{B_1},\\ &E = R_{A_1}E_1L_{B_1}, M = R_AC, N = DL_B, S = CL_M.\end{aligned}$$

Then the following statements are equivalent:
(1) Equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \tag{2.1}$$

is consistent.
(2)

$$R_AE = MM^\dagger E, \qquad EL_B = EN^\dagger N, \qquad R_AEL_D = 0, \qquad R_CEL_B = 0.$$

(3)

$$r\begin{bmatrix}E_1 & C_4 & C_3 & A_1\\ B_1 & 0 & 0 & 0\end{bmatrix} = r\begin{bmatrix}C_4 & C_3 & A_1\end{bmatrix} + r(B_1),\ r\begin{bmatrix}E_1 & A_1\\ D_4 & 0\\ D_3 & 0\\ B_1 & 0\end{bmatrix} = r\begin{bmatrix}D_4\\ D_3\\ B_1\end{bmatrix} + r(A_1),$$

$$r\begin{bmatrix}E_1 & C_3 & A_1\\ B_1 & 0 & 0\\ D_4 & 0 & 0\end{bmatrix} = r\begin{bmatrix}C_3 & A_1\end{bmatrix} + r\begin{bmatrix}D_4\\ B_1\end{bmatrix},\ r\begin{bmatrix}E_1 & C_4 & A_1\\ B_1 & 0 & 0\\ D_3 & 0 & 0\end{bmatrix} = r\begin{bmatrix}C_4 & A_1\end{bmatrix} + r\begin{bmatrix}D_3\\ B_1\end{bmatrix}.$$

In this case, the general solution of (2.1) can be expressed as

$$\begin{aligned}X_1 &= A_1^\dagger(E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^\dagger W_2B_1 + L_{A_1}W_1,\\ X_2 &= R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^\dagger + A_1A_1^\dagger W_2 + W_3R_{B_1},\\ X_3 &= A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SV_4R_NDB^\dagger + L_AV_3 + V_4R_B,\\ X_4 &= M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_ML_SU_1 + L_MV_4R_N + V_5R_D,\end{aligned}$$

where $V_1, V_2, V_3, V_4, V_5, W_1, W_2, W_3$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Lemma 2.2. [11] *Let $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{m\times k}, C \in \mathbb{H}^{l\times n}, D \in \mathbb{H}^{m\times p}, E \in \mathbb{H}^{q\times n}, Q \in \mathbb{H}^{m_1\times k}$, and $P \in \mathbb{H}^{l\times n_1}$ be given. Then*

(1) $r(A) + r(R_AB) = r(B) + r(R_BA) = r\begin{bmatrix}A & B\end{bmatrix}$.

(2) $r(A) + r(CL_A) = r(C) + r(AL_C) = r\begin{bmatrix}A\\ C\end{bmatrix}$.

(3) $r(B) + r(C) + r(R_B A L_C) = r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$.

(4) $r(P) + r(Q) + r \begin{bmatrix} A & BL_Q \\ R_P C & 0 \end{bmatrix} = r \begin{bmatrix} A & B & 0 \\ C & 0 & P \\ 0 & Q & 0 \end{bmatrix}$.

(5) $r \begin{bmatrix} R_B A L_C & R_B D \\ E L_C & 0 \end{bmatrix} + r(B) + r(C) = r \begin{bmatrix} A & D & B \\ E & 0 & 0 \\ C & 0 & 0 \end{bmatrix}$.

Lemma 2.3. [22] *Let A_1 and C_1 be given. Then the equation $A_1X_1 = C_1$ is consistent if and only if $r \begin{bmatrix} A_1 & C_1 \end{bmatrix} = r(A_1)$. In this case, the general solution to $A_1X_1 = C_1$ can be expressed as*

$$X_1 = A_1^{\dagger} C_1 + L_{A_1} U_1,$$

where U_1 is an arbitrary matrix over $\mathbb{H}$ with appropriate size.

Lemma 2.4. [22] *Let B_1 and D_1 be given. Then the equation $X_2B_1 = D_1$ is consistent if and only if $r \begin{bmatrix} B_1 \\ D_1 \end{bmatrix} = r(B_1)$. In this case, the general solution to $X_2B_1 = D_1$ can be expressed as*

$$X_2 = D_1 B_1^{\dagger} + U_2 R_{B_1},$$

where U_2 is an arbitrary matrix over $\mathbb{H}$ with appropriate size.

Lemma 2.5. [22] *Let $A_2, B_2, C_2, D_2, A_3, B_3$ and C_3 be given. Set*

$$A_5 = A_3 L_{A_2},\ B_5 = R_{B_2} B_3,\ C_5 = C_3 - A_3(A_2^{\dagger} C_2 + L_{A_2} D_2 B_2^{\dagger}) B_3.$$

Then the following statements are equivalent:
(1) *System of real quaternion matrix equations*

$$A_2 Z = C_2,\ Z B_2 = D_2,\ A_3 Z B_3 = C_3 \tag{2.2}$$

is consistent.
(2)

$$R_{A_2} C_2 = 0,\ D_2 L_{B_2} = 0,\ R_{A_5} C_5 = 0,\ C_5 L_{B_5} = 0,\ A_2 D_2 = C_2 B_2.$$

(3)

$$r \begin{bmatrix} A_2 & C_2 \end{bmatrix} = r(A_2),\ \begin{bmatrix} D_2 \\ B_2 \end{bmatrix} = r(B_2),\ A_2 D_2 = C_2 B_2,$$

$$r \begin{bmatrix} A_3 & C_3 \\ A_2 & C_2 B_3 \end{bmatrix} = r \begin{bmatrix} A_3 \\ A_2 \end{bmatrix},\ r \begin{bmatrix} C_3 & A_3 D_2 \\ B_3 & B_2 \end{bmatrix} = r \begin{bmatrix} B_3 & B_2 \end{bmatrix}.$$

In this case, the general solution to (2.2) can be expressed as

$$Z = A_2^{\dagger} C_2 + L_{A_2} D_2 B_2^{\dagger} + L_{A_2} A_5^{\dagger} C_5 B_5^{\dagger} R_{B_2} + L_{A_2} L_{A_5} U_3 R_{B_2} + L_{A_2} U_4 R_{B_5} R_{B_2},$$

where U_3 and U_4 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

The next Lemma is due to Tian.

Lemma 2.6. [19] *Let*

$$p(X_1, X_2, X_3, X_4) = A - B_1X_1 - X_2C_2 - B_3X_3C_3 - B_4X_4C_4$$

be a matrix expression over $\mathbb{H}$*, where* $A \in \mathbb{H}^{m\times n}$*. Then the extremal ranks of* $p(X_1, X_2, X_3, X_4)$ *are the following*

$$\max_{\{X_i\}} r\left[p\left(X_1, X_2, X_3, X_4\right)\right] = \min\left\{m, n, r\begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix},\right.$$

$$\left. r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix}\right\},$$

and

$$\min_{\{X_i\}} r\left[p\left(X_1, X_2, X_3, X_4\right)\right] = r\begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix} + r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix} - r(B_1) - r(C_2)$$

$$+ \max\left\{ r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \\ C_4 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix},\right.$$

$$\left. r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \\ C_3 & 0 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix}\right\}.$$

For convenience, we adopt the following notations:

$$J_1 = \left\{X \middle| A_1X = C_1\right\}, J_2 = \left\{Y \middle| YB_1 = D_1\right\}$$

,

$$J_3 = \left\{Z \middle| A_2Z = C_2, ZB_2 = D_2, A_3ZB_3 = C_3\right\}.$$

Theorem 2.7. *Let* $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, A_4, B_4, C_4, D_4$ *and* $E_1 \in \mathbb{H}^{m\times n}$ *be given. Assume that* $J_1 - J_3$ *are not empty sets. Denote that*

$$N_1 = \begin{bmatrix} E_1 & A_4 & D_1 & C_4D_2 \\ B_4 & 0 & B_1 & 0 \\ D_4 & 0 & 0 & B_2 \\ C_1 & A_1 & 0 & 0 \end{bmatrix}, N_2 = \begin{bmatrix} E_1 & A_4 & C_4 & D_1 \\ B_4 & 0 & 0 & B_1 \\ C_1 & A_1 & 0 & 0 \\ C_2D_4 & 0 & A_2 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} E_1 & A_4 & C_4 & D_1 & 0 & 0 \\ B_4 & 0 & 0 & B_1 & 0 & 0 \\ D_4 & 0 & 0 & 0 & B_3 & B_2 \\ C_1 & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & -C_3 & -A_3D_2 \\ C_2D_4 & 0 & A_2 & 0 & 0 & 0 \end{bmatrix}, N_4 = \begin{bmatrix} E_1 & A_4 & C_4 & D_1 & 0 \\ B_4 & 0 & 0 & B_1 & 0 \\ D_4 & 0 & 0 & 0 & B_2 \\ C_1 & A_1 & 0 & 0 & 0 \\ C_2D_4 & 0 & A_2 & 0 & 0 \end{bmatrix},$$

$$N_5 = \begin{bmatrix} E_1 & A_4 & C_4 & D_1 & 0 & 0 \\ B_4 & 0 & 0 & B_1 & 0 & 0 \\ D_4 & 0 & 0 & 0 & B_3 & B_2 \\ C_1 & A_1 & 0 & 0 & 0 & 0 \\ C_2D_4 & 0 & A_2 & 0 & 0 & 0 \end{bmatrix}, N_6 = \begin{bmatrix} E_1 & A_4 & C_4 & D_1 & C_4D_2 \\ B_4 & 0 & 0 & B_1 & 0 \\ D_4 & 0 & 0 & 0 & B_2 \\ C_1 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \end{bmatrix}.$$

Then we have the following:
(a) *The maximal rank of (1.6) subject to (1.7) is*

$$\max_{X\in J_1, Y\in J_2, Z\in J_3} r\left[g(X,Y,Z)\right] = \min\Bigg\{ m, n, r(N_1) - r(A_1) - r(B_1) - r(B_2),$$
$$r(N_2) - r(A_1) - r(B_1) - r(A_2), r(N_3) - r(A_1) - r(B_1) - r\begin{bmatrix} A_2 \\ A_3 \end{bmatrix} - r\begin{bmatrix} B_2 & B_3 \end{bmatrix} \Bigg\}. \tag{2.3}$$

(b) *The minimal rank of (1.6) subject to (1.7) is*

$$\min_{X\in J_1, Y\in J_2, Z\in J_3} r\left[g(X,Y,Z)\right] = r(N_1) + r(N_2) - r\begin{bmatrix} A_1 \\ A_4 \end{bmatrix} - r\begin{bmatrix} B_1 & B_4 \end{bmatrix} + \max\left\{ r(N_3) - r(N_5) - r(N_6), -r(N_4) \right\}. \tag{2.4}$$

Proof. It follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5 that the general solutions of

$$A_1X = C_1,\ YB_1 = D_1,\ A_2Z = C_2,\ ZB_2 = D_2,\ A_3ZB_3 = C_3$$

can be expressed as

$$X = X_0 + L_{A_1}U_1,\ Y = Y_0 + U_2R_{B_1},\ Z = Z_0 + L_{A_2}L_{A_5}U_3R_{B_2} + L_{A_2}U_4R_{B_5}R_{B_2}, \tag{2.5}$$

where X_0, Y_0, Z_0 are special solutions of the corresponding matrix equations, $U_1 - U_3$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes. Substituting (2.5) into (1.6) yields

$$g(X,Y,Z) = A - A_4L_{A_1}U_1 - U_2R_{B_1}B_4 - C_4L_{A_2}L_{A_5}U_3R_{B_2}D_4 - C_4L_{A_2}U_4R_{B_5}R_{B_2}D_4, \tag{2.6}$$

where

$$A = E_1 - A_4X_0 - Y_0B_4 - C_4Z_0D_4.$$

Applying Lemma 2.6 to (2.6) gives

$$\max_{X\in J_1,Y\in J_2,Z\in J_3} r\left[g\left(X,Y,Z\right)\right]=\min\{m,n,l_1,l_2,l_3\}. \tag{2.7}$$

$$\min_{X\in J_1,Y\in J_2,Z\in J_3} r\left[g\left(X,Y,Z\right)\right]= \\ l_1+l_2-r(A_4L_{A_1})-r(R_{B_1}B_4)+\max\{t_3-t_5-t_6,-t_4\}, \tag{2.8}$$

where

$$l_1=r\begin{bmatrix}A & A_4L_{A_1}\\ R_{B_1}B_4 & 0\\ R_{B_2}D_4 & 0\end{bmatrix},\ l_2=r\begin{bmatrix}A & A_4L_{A_1} & C_4L_{A_2}\\ R_{B_1}B_4 & 0 & 0\end{bmatrix},$$

$$l_3=r\begin{bmatrix}A & A_4L_{A_1} & C_4L_{A_2}L_{A_5}\\ R_{B_1}B_4 & 0 & 0\\ R_{B_5}R_{B_2}D_4 & 0 & 0\end{bmatrix},\ l_4=r\begin{bmatrix}A & A_4L_{A_1} & C_4L_{A_2}\\ R_{B_1}B_4 & 0 & 0\\ R_{B_2}D_4 & 0 & 0\end{bmatrix},$$

$$l_5=r\begin{bmatrix}A & A_4L_{A_1} & C_4L_{A_2}\\ R_{B_1}B_4 & 0 & 0\\ R_{B_5}R_{B_2}D_4 & 0 & 0\end{bmatrix},\ l_6=r\begin{bmatrix}A & A_4L_{A_1} & C_4L_{A_2}L_{A_5}\\ R_{B_1}B_4 & 0 & 0\\ R_{B_2}D_4 & 0 & 0\end{bmatrix}.$$

By Lemma 2.2 and

$$A_1X_0=C_1,\ Y_0B_1=D_1,\ A_2Z_0=C_2,\ Z_0B_2=D_2,\ A_3Z_0B_3=C_3,$$

we obtain that

$$l_1=r(N_1)-r(A_1)-r(B_1)-r(B_2), \tag{2.9}$$

$$l_2=r(N_2)-r(A_1)-r(B_1)-r(A_2), \tag{2.10}$$

$$l_3=r(N_3)-r(A_1)-r(B_1)-r\begin{bmatrix}A_2\\ A_3\end{bmatrix}-r\begin{bmatrix}B_2 & B_3\end{bmatrix}, \tag{2.11}$$

$$l_4=r(N_4)-r(A_1)-r(B_1)-r(A_2)-r(B_2), \tag{2.12}$$

$$l_5=r(N_5)-r(A_1)-r(B_1)-r(A_2)-r\begin{bmatrix}B_2 & B_3\end{bmatrix}, \tag{2.13}$$

$$l_6=r(N_6)-r(A_1)-r(B_1)-r\begin{bmatrix}A_2\\ A_3\end{bmatrix}-r(B_2). \tag{2.14}$$

Substituting (2.9)-(2.14) into (2.7) and (2.8) yields (2.3) and (2.4). □

In Theorem 2.7, let A_1, B_1, C_1, D_1, A_4 and B_4 vanish. Then we can obtain the extremal ranks of (1.10) subject to (1.11).

Corollary 2.8. *The extremal ranks of the quaternion matrix expression $f(X)=C_4-A_4XB_4$ subject to the consistent system (1.11) are the following:*

$$\max_{\substack{A_1X=C_1\\ XB_2=C_2\\ A_3XB_3=C_3}} r(f(X))=\min\{a,b,c\},$$

where

$$a = r\begin{bmatrix} C_1B_4 & A_1 \\ C_4 & A_4 \end{bmatrix} - r(A_1),$$

$$b = r\begin{bmatrix} B_2 & B_4 \\ A_4C_2 & C_4 \end{bmatrix} - r(B_2),$$

$$c = r\begin{bmatrix} A_1 & 0 & 0 & C_1B_4 \\ A_3 & -A_3C_2 & -C_3 & 0 \\ A_4 & 0 & 0 & C_4 \\ 0 & B_2 & B_3 & B_4 \end{bmatrix} - r\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} - r\begin{bmatrix} B_2 & B_3 \end{bmatrix},$$

$$\min_{\substack{A_1X_1=C_1\\ XB_2=C_2\\ A_3XB_3=C_3}} r(f(X)) = r\begin{bmatrix} C_1B_4 & A_1 \\ C_4 & A_4 \end{bmatrix} + r\begin{bmatrix} B_2 & B_4 \\ A_4C_2 & C_4 \end{bmatrix}$$

$$+ r\begin{bmatrix} A_1 & 0 & 0 & C_1B_4 \\ A_3 & -A_3C_2 & -C_3 & 0 \\ A_4 & 0 & 0 & C_4 \\ 0 & B_2 & B_3 & B_4 \end{bmatrix}$$

$$- r\begin{bmatrix} A_1 & 0 & C_1B_4 \\ A_3 & -A_3C_2 & 0 \\ A_4 & 0 & C_4 \\ 0 & B_2 & B_4 \end{bmatrix} - r\begin{bmatrix} A_1 & 0 & 0 & C_1B_4 \\ A_4 & 0 & 0 & C_4 \\ 0 & B_2 & B_3 & B_4 \end{bmatrix}.$$

Remark 2.9. Corollary 2.8 is Theorem 2.5 in [22].

In Theorem 2.7, let $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_4$ and B_4 vanish. Then we can obtain the extremal ranks of (1.8) subject to (1.9).

Corollary 2.10. *Suppose that the matrix equation $B_2XC_2 = A_2$ is consistent. Then*
(a) The maximal rank of $p(X) = A_1 - B_1XC_1$ subject to $B_2XC_2 = A_2$ is

$$\max_{B_2XC_2=A_2} r(p(X)) = \min\left\{ r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix} - r(B_2) - r(C_2), r\begin{bmatrix} A_1 \\ C_1 \end{bmatrix}, r\begin{bmatrix} A_1 & B_1 \end{bmatrix}\right.$$

(b) The minimal rank of $p(X) = A_1 - B_1XC_1$ subject to $B_2XC_2 = A_2$ is

$$\min_{B_2XC_2=A_2} r(p(X)) = r\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} + r\begin{bmatrix} A_1 & B_1 \end{bmatrix} - r\begin{bmatrix} A_1 & B_1 & 0 \\ C_1 & 0 & C_2 \end{bmatrix}$$

$$-r\begin{bmatrix} A_1 & B_1 \\ C_1 & 0 \\ 0 & B_2 \end{bmatrix} + r\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & -A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix}.$$

Remark 2.11. Corollary 2.10 is Theorem 3.2 in [20].

3. The solvable conditions and the expression of the general solution to (1.5)

Our goal in this section is to give some solvable conditions for (1.5) and to provide an expression of this general solution when the solvability conditions are met.

Theorem 3.1. *Let* $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, A_4, B_4, C_4, D_4$ *and* E_1 *be as in Theorem 2.7. Set*

$$A_5 = A_3 L_{A_2},\ B_5 = R_{B_2} B_3,\ C_5 = C_3 - A_3(A_2^{\dagger} C_2 + L_{A_2} D_2 B_2^{\dagger}) B_3, A_6 = A_4 L_{A_1},$$

$$B_6 = R_{B_1} B_4, C_6 = C_4 L_{A_2} L_{A_5}, D_6 = R_{B_2} D_4, C_7 = C_4 L_{A_2}, D_7 = R_{B_5} R_{B_2} D_4,$$

$$E_2 = E_1 - A_4 A_1^{\dagger} C_1 - D_1 B_1^{\dagger} B_4 - C_4(A_2^{\dagger} C_2 + L_{A_2} D_2 B_2^{\dagger} + L_{A_2} A_5^{\dagger} C_5 B_5^{\dagger} R_{B_2}) D_4,$$

$$A = R_{A_6} C_6, B = D_6 L_{B_6}, C = R_{A_6} C_7, D = D_7 L_{B_6},$$

$$E = R_{A_6} E_2 L_{B_6}, M = R_A C, N = D L_B, S = C L_M.$$

Then the following statements are equivalent:
(a) *System (1.5) is consistent.*
(b)

$$R_{A_i} C_i = 0,\ D_i L_{B_i} = 0,\ i = 1, 2,\ R_{A_5} C_5 = 0,\ C_5 L_{B_5} = 0,\ A_2 D_2 = C_2 B_2,$$

$$R_A E = M M^{\dagger} E,\ E L_B = E N^{\dagger} N,\ R_A E L_D = 0,\ R_C E L_B = 0.$$

(c)

$$r\begin{bmatrix} A_i & C_i \end{bmatrix} = r(A_i),\ \begin{bmatrix} D_i \\ B_i \end{bmatrix} = r(B_i),\ i = 1, 2,\ A_2 D_2 = C_2 B_2,$$

$$r\begin{bmatrix} A_3 & C_3 \\ A_2 & C_2 B_3 \end{bmatrix} = r\begin{bmatrix} A_3 \\ A_2 \end{bmatrix},\ r\begin{bmatrix} C_3 & A_3 D_2 \\ B_3 & B_2 \end{bmatrix} = r\begin{bmatrix} B_3 & B_2 \end{bmatrix},$$

$$r(N_1) = r\begin{bmatrix} A_1 \\ A_4 \end{bmatrix} + r\begin{bmatrix} B_4 & B_1 & 0 \\ D_4 & 0 & B_2 \end{bmatrix}, r(N_2) = r\begin{bmatrix} A_4 & C_4 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} + r\begin{bmatrix} B_4 & B_1 \end{bmatrix},$$

$$r(N_3) = r\begin{bmatrix} A_4 & C_4 \\ A_1 & 0 \\ 0 & A_2 \\ 0 & A_3 \end{bmatrix} + r\begin{bmatrix} B_4 & B_1 & 0 & 0 \\ D_4 & 0 & B_3 & B_2 \end{bmatrix}, r(N_4) = r\begin{bmatrix} B_4 & B_1 & 0 \\ D_4 & 0 & B_2 \end{bmatrix} + r\begin{bmatrix} A_4 & C_4 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

In this case, the general solution of (1.5) can be expressed as

$$X = A_1^{\dagger} C_1 + L_{A_1} U_1, \tag{3.1}$$

$$Y = D_1 B_1^{\dagger} + U_2 R_{B_1}, \tag{3.2}$$

$$Z = A_2^{\dagger} C_2 + L_{A_2} D_2 B_2^{\dagger} + L_{A_2} A_5^{\dagger} C_5 B_5^{\dagger} R_{B_2} + L_{A_2} L_{A_5} U_3 R_{B_2} + L_{A_2} U_4 R_{B_5} R_{B_2},$$

$$U_1 = A_6^{\dagger}(E_2 - C_6U_3D_6 - C_7U_4D_7) - A_7^{\dagger}W_2B_6 + L_{A_6}W_1, \tag{3.3}$$

$$U_2 = R_{A_6}(E_2 - C_6U_3D_6 - C_7U_4D_7)B_6^{\dagger} + A_6A_6^{\dagger}W_2 + W_3R_{B_6}, \tag{3.4}$$

$$U_3 = A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}SV_4R_NDB^{\dagger} + L_AV_1 + V_2R_B, \tag{3.5}$$

$$U_4 = M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_SV_3 + L_MV_4R_N + V_5R_D, \tag{3.6}$$

where $V_1, V_2, V_3, V_4, V_5, W_1, W_2, W_3$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

Proof. $(b) \iff (c)$: It follows from Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5 that

$$R_{A_i}C_i = 0 \iff r\begin{bmatrix} A_i & C_i \end{bmatrix} = r(A_i),\ D_iL_{B_i} = 0 \iff r\begin{bmatrix} B_i \\ D_i \end{bmatrix} = r(B_i),\ i = 1, 2,$$

$$R_{A_5}C_5 = 0 \iff r\begin{bmatrix} A_3 & C_3 \\ A_2 & C_2B_3 \end{bmatrix} = r\begin{bmatrix} A_3 \\ A_2 \end{bmatrix},$$

$$C_5L_{B_5} = 0 \iff r\begin{bmatrix} C_3 & A_3D_2 \\ B_3 & B_2 \end{bmatrix} = r\begin{bmatrix} B_3 & B_2 \end{bmatrix},$$

$$R_AE = MM^{\dagger}E \iff r\begin{bmatrix} E_2 & C_7 & C_6 & A_6 \\ B_6 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_7 & C_6 & A_6 \end{bmatrix} + r(B_6)$$

$$\iff r\begin{bmatrix} E_2 & A_4L_{A_1} & C_4L_{A_2} \\ R_{B_1}B_4 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} E_2 & A_4L_{A_1} & C_4L_{A_2} \end{bmatrix} + r(R_{B_1}B_4)$$

$$\iff r(N_2) = r\begin{bmatrix} A_4 & C_4 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} + r\begin{bmatrix} B_4 & B_1 \end{bmatrix},$$

$$EL_B = EN^{\dagger}N \iff r\begin{bmatrix} E_2 & A_6 \\ D_6 & 0 \\ D_7 & 0 \\ B_6 & 0 \end{bmatrix} = r\begin{bmatrix} D_6 \\ D_7 \\ B_6 \end{bmatrix} + r(A_6)$$

$$\iff r\begin{bmatrix} E_2 & A_4L_{A_1} \\ R_{B_1}B_4 & 0 \\ R_{B_2}D_4 & 0 \end{bmatrix} = r\begin{bmatrix} R_{B_1}B_4 \\ R_{B_2}D_4 \end{bmatrix} + r(A_4L_{A_1})$$

$$\iff r(N_1) = r\begin{bmatrix} A_1 \\ A_4 \end{bmatrix} + r\begin{bmatrix} B_4 & B_1 & 0 \\ D_4 & 0 & B_2 \end{bmatrix},$$

$$R_AEL_D = 0 \iff r\begin{bmatrix} E_2 & C_6 & A_6 \\ B_6 & 0 & 0 \\ D_7 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_6 & A_6 \end{bmatrix} + r\begin{bmatrix} D_7 \\ B_6 \end{bmatrix}$$

$$\iff r\begin{bmatrix} E_2 & A_4L_{A_1} & C_4L_{A_2}L_{A_5} \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_5}R_{B_2}D_4 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_4L_{A_1} & C_4L_{A_2}L_{A_5} \end{bmatrix} + \begin{bmatrix} R_{B_1}B_4 \\ R_{B_5}R_{B_2}D_4 \end{bmatrix}$$

$$\iff r(N_3) = r\begin{bmatrix} A_4 & C_4 \\ A_1 & 0 \\ 0 & A_2 \\ 0 & A_3 \end{bmatrix} + r\begin{bmatrix} B_4 & B_1 & 0 & 0 \\ D_4 & 0 & B_3 & B_2 \end{bmatrix},$$

$$R_CEL_B = 0 \iff r\begin{bmatrix} E_2 & C_7 & A_6 \\ B_6 & 0 & 0 \\ D_6 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_7 & A_6 \end{bmatrix} + r\begin{bmatrix} D_6 \\ B_6 \end{bmatrix}$$

$$\iff r\begin{bmatrix} E_2 & A_4L_{A_1} & C_4L_{A_2} \\ R_{B_1}B_4 & 0 & 0 \\ R_{B_2}D_4 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_4L_{A_1} & C_4L_{A_2} \end{bmatrix} + r\begin{bmatrix} R_{B_1}B_4 \\ R_{B_2}D_4 \end{bmatrix}$$

$$\iff r(M_4) = r\begin{bmatrix} A_4 & C_5 \\ A_1 & 0 \\ 0 & A_3 \end{bmatrix} + r\begin{bmatrix} B_4 & B_1 & 0 \\ D_4 & 0 & B_2 \end{bmatrix}.$$

$(a) \implies (c)$: Suppose that (X_0, Y_0, Z_0) is a solution of (1.5). It follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5 that

$$r\begin{bmatrix} A_i & C_i \end{bmatrix} = r(A_i),\ \begin{bmatrix} D_i \\ B_i \end{bmatrix} = r(B_i),\ i = 1, 2,\ A_2D_2 = C_2B_2,$$

$$r\begin{bmatrix} A_3 & C_3 \\ A_2 & C_2B_3 \end{bmatrix} = r\begin{bmatrix} A_3 \\ A_2 \end{bmatrix},\ r\begin{bmatrix} C_3 & A_3D_2 \\ B_3 & B_2 \end{bmatrix} = r\begin{bmatrix} B_3 & B_2 \end{bmatrix}.$$

Applying

$$A_1X_0 = C_1,\ Y_0B_1 = D_1,\ A_2Z_0 = C_2,\ Z_0B_2 = D_2,\ A_3Z_0B_3 = C_3$$

and elementary matrix operations, we obtain

$$\begin{bmatrix} I & -Y_0 & -C_4Z_0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} N_1 \begin{bmatrix} I & 0 & 0 & 0 \\ -X_0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_4 & 0 & 0 \\ B_4 & 0 & B_1 & 0 \\ D_4 & 0 & 0 & B_2 \\ 0 & A_1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} I & -Y_0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} N_2 \begin{bmatrix} I & 0 & 0 & 0 \\ -X_0 & I & 0 & 0 \\ -Z_0D_4 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_4 & C_4 & 0 \\ B_4 & 0 & 0 & B_1 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} I & -Y_0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & A_3Z_0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} N_3 \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -X_0 & I & 0 & 0 & 0 & 0 \\ -Z_0D_4 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_4 & C_4 & 0 & 0 & 0 \\ B_4 & 0 & 0 & B_1 & 0 & 0 \\ D_4 & 0 & 0 & 0 & B_3 & B_2 \\ 0 & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} I & -Y_0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} N_4 \begin{bmatrix} I & 0 & 0 & 0 \\ -X_0 & I & 0 & 0 \\ -Z_0D_4 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_4 & C_4 & 0 & 0 \\ B_4 & 0 & 0 & B_1 & 0 \\ D_4 & 0 & 0 & 0 & B_2 \\ 0 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \end{bmatrix}.$$

$(c) \Longrightarrow (a)$: Suppose that the equalities in (c) hold. It follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5 that the equations in

$$A_1X = C_1,\ YB_1 = D_1,\ A_2Z = C_2,\ ZB_2 = D_2,\ A_3ZB_3 = C_3.$$

are consistent, respectively. On the other hand, by Theorem 2.7, we obtain that

$$\min_{X\in J_1, Y\in J_2, Z\in J_3} r\left(E_1 - A_4X - YB_4 - C_4ZD_4\right) = 0.$$

Hence, the system (1.5) has a solution.

$(a) \Longleftrightarrow (b)$: We separate the equations in system (1.5) into two groups

$$A_1X = C_1,\ YB_1 = D_1,\ A_2Z = C_2,\ ZB_2 = D_2,\ A_3ZB_3 = C_3, \tag{3.7}$$

$$A_4X + YB_4 + C_4ZD_4 = E_1. \tag{3.8}$$

It follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5 that matrix equations in (3.7) are consistent, respectively, if and only if

$$R_{A_i}C_i = 0,\ D_iL_{B_i} = 0,\ i = 1, 2,\ R_{A_5}C_5 = 0,\ C_5L_{B_5} = 0,\ A_2D_2 = C_2B_2.$$

And the general solutions to these matrix equations in (3.7) can be expressed as

$$X = A_1^{\dagger}C_1 + L_{A_1}U_1, \tag{3.9}$$

$$Y = D_1B_1^{\dagger} + U_2R_{B_1}, \tag{3.10}$$

$$Z = A_2^{\dagger}C_2 + L_{A_2}D_2B_2^{\dagger} + L_{A_2}A_5^{\dagger}C_5B_5^{\dagger}R_{B_2} + L_{A_2}L_{A_5}U_3R_{B_2} + L_{A_2}U_4R_{B_5}R_{B_2}, \tag{3.11}$$

Substituting (3.9)-(3.11) into (3.8) gives

$$A_6U_1 + U_2B_6 + C_6U_3D_6 + C_7U_4D_7 = E_2. \tag{3.12}$$

Hence, the system (1.5) is consistent if and only if the matrix equations in (3.7) and (3.12) are consistent, respectively. By Lemma 2.1, we know that the matrix equation (3.12) is consistent if and only if

$$R_AE = MM^{\dagger}E,\ EL_B = EN^{\dagger}N,\ R_AEL_D = 0,\ R_CEL_B = 0.$$

We know by Lemma 2.1 that the general solutions of equation (3.12) can be expressed as (3.3)-(3.6). $\square$

In Theorem 3.1, let A_2, B_2, C_2 and D_2 vanish. We can obtain the general solution to the following system

$$\begin{cases} A_1X = C_1, YB_1 = D_1, \\ A_2ZB_2 = C_2 \\ A_3X + YB_3 + C_3ZD_3 = E_1. \end{cases} \tag{3.13}$$

Corollary 3.2. *Let $A_1, B_1, C_1, D_1, A_2, B_2, C_2, A_3, B_3, C_3, D_3, E_1$ and $N_1 - N_6$ be given. Set*

$$\begin{aligned} A_4 &= A_3L_{A_1}, B_4 = R_{B_1}B_3, C_4 = C_3L_{A_2}, D_4 = R_{B_2}D_3, \\ E_4 &= E_1 - A_3A_1^{\dagger}C_1 - D_1B_1^{\dagger}B_3 - C_3A_2^{\dagger}C_2B_2^{\dagger}D_3, \\ A &= R_{A_4}C_4, B = D_3L_{B_4}, C = R_{A_4}C_3, D = D_4L_{B_4}, \\ E &= R_{A_4}E_4L_{B_4}, M = R_AC, N = DL_B, S = CL_M. \end{aligned}$$

Then the following statements are equivalent:
(a) System (3.13) is consistent.
(b)

$$R_{A_2}C_2 = 0,\ D_2L_{B_2} = 0,\ R_{A_5}C_5 = 0,\ C_5L_{B_5} = 0,\ A_2D_2 = C_2B_2,$$

$$R_AE = MM^{\dagger}E,\ EL_B = EN^{\dagger}N,\ R_AEL_D = 0,\ R_CEL_B = 0.$$

(c)

$$r\begin{bmatrix} A_i & C_i \end{bmatrix} = r(A_i),\ r\begin{bmatrix} B_i \\ D_i \end{bmatrix} = r(B_i),\ i = 1, 2,$$

$$r\begin{bmatrix} E_1 & A_3 & D_1 \\ B_3 & 0 & B_1 \\ D_3 & 0 & 0 \\ C_1 & A_1 & 0 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} + r\begin{bmatrix} B_3 & B_1 \\ D_3 & 0 \end{bmatrix},$$

$$r\begin{bmatrix} E_1 & A_3 & C_3 & D_1 \\ B_3 & 0 & 0 & B_1 \\ C_1 & A_1 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_3 & C_3 \\ A_1 & 0 \end{bmatrix} + r\begin{bmatrix} B_3 & B_1 \end{bmatrix},$$

$$r\begin{bmatrix} E_1 & A_3 & C_3 & D_1 & 0 \\ B_3 & 0 & 0 & B_1 & 0 \\ D_3 & 0 & 0 & 0 & B_2 \\ C_1 & A_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & -C_2 \end{bmatrix} = r\begin{bmatrix} A_3 & C_3 \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix} + r\begin{bmatrix} B_3 & B_1 & 0 \\ D_3 & 0 & B_2 \end{bmatrix},$$

$$r\begin{bmatrix} E_1 & A_3 & C_3 & D_1 \\ B_3 & 0 & 0 & B_1 \\ D_3 & 0 & 0 & 0 \\ C_1 & A_1 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} A_1 & 0 \\ A_3 & C_3 \end{bmatrix} + r\begin{bmatrix} B_3 & B_1 \\ D_3 & 0 \end{bmatrix}.$$

In this case, the general solution of (3.13) can be expressed as

$$X = A_1^\dagger C_1 + L_{A_1}U_1,\ Y = D_1B_1^\dagger + U_2R_{B_1},\ Z = A_2^\dagger C_2B_2^\dagger + L_{A_2}U_3 + U_4R_{B_2},$$

$$U_1 = A_4^\dagger(E_4 - C_4U_3D_3 - C_3U_4D_4) - A_4^\dagger W_2B_4 + L_{A_4}W_1,$$

$$U_2 = R_{A_4}(E_4 - C_4U_3D_3 - C_3U_4D_4)B_4^\dagger + A_4A_4^\dagger W_2 + W_3R_{B_4},$$

$$U_3 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SV_4R_NDB^\dagger + L_AV_1 + V_2R_B,$$

$$U_4 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_ML_SV_3 + L_MV_4R_N + V_5R_D,$$

where $V_1, V_2, V_3, V_4, V_5, W_1, W_2, W_3$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

In Theorem 3.1, let A_1, B_1, C_1, D_1, A_4 and B_4 vanish. We can derive the general solution to the following system

$$A_2Z = C_2, ZB_2 = D_2, A_3ZB_3 = C_3, C_4ZD_4 = E_1. \tag{3.14}$$

Corollary 3.3. *Let $A_2, B_2, C_2, D_2, A_3, B_3, C_3, C_4, D_4$ and E_1 be given. Set*

$$A_5 = A_3L_{A_2},\ B_5 = R_{B_2}B_3,\ C_5 = C_3 - A_3(A_2^\dagger C_2 + L_{A_2}D_2B_2^\dagger)B_3,$$

$$A = C_4L_{A_2}L_{A_5}, B = R_{B_2}D_4, C = C_4L_{A_2}, D = R_{B_5}R_{B_2}D_4,$$

$$E = E_1 - C_4(A_2^\dagger C_2 + L_{A_2}D_2B_2^\dagger + L_{A_2}A_5^\dagger C_5B_5^\dagger R_{B_2})D_4,$$

$$M = R_AC, N = DL_B, S = CL_M.$$

Then the following statements are equivalent:
(a) System (3.14) is consistent.
(b)

$$R_{A_i}C_i = 0,\ D_iL_{B_i} = 0,\ i = 1, 2,\ R_{A_5}C_5 = 0,\ C_5L_{B_5} = 0,\ A_2D_2 = C_2B_2,$$

$$R_AE = MM^\dagger E,\ EL_B = EN^\dagger N,\ R_AEL_D = 0,\ R_CEL_B = 0.$$

(c)

$$r\begin{bmatrix} A_2 & C_2 \end{bmatrix} = r(A_2),\ \begin{bmatrix} D_2 \\ B_2 \end{bmatrix} = r(B_2),\ A_2D_2 = C_2B_2,$$

$$r\begin{bmatrix} A_3 & C_3 \\ A_2 & C_2B_3 \end{bmatrix} = r\begin{bmatrix} A_3 \\ A_2 \end{bmatrix},\ r\begin{bmatrix} C_3 & A_3D_2 \\ B_3 & B_2 \end{bmatrix} = r\begin{bmatrix} B_3 & B_2 \end{bmatrix},$$

$$r\begin{bmatrix} E_1 & C_4 \\ C_2D_4 & A_2 \end{bmatrix} = r\begin{bmatrix} C_4 \\ A_2 \end{bmatrix}, r\begin{bmatrix} E_1 & C_4D_2 \\ D_4 & B_2 \end{bmatrix} = r\begin{bmatrix} B_2 & D_4 \end{bmatrix},$$

$$r\begin{bmatrix} E_1 & C_4 & 0 & 0 \\ D_4 & 0 & B_3 & B_2 \\ 0 & A_3 & 0 & 0 \\ 0 & A_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_4 \\ A_3 \\ A_2 \end{bmatrix} + r\begin{bmatrix} B_3 & D_4 & B_2 \end{bmatrix},$$

$$r\begin{bmatrix} E_1 & C_4 & 0 \\ D_4 & 0 & B_2 \\ 0 & A_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_4 \\ A_2 \end{bmatrix} + r\begin{bmatrix} D_4 & B_2 \end{bmatrix}.$$

In this case, the general solution to (3.14) can be expressed as

$$Z = A_2^{\dagger}C_2 + L_{A_2}D_2B_2^{\dagger} + L_{A_2}A_5^{\dagger}C_5B_5^{\dagger}R_{B_2} + L_{A_2}L_{A_5}U_3R_{B_2} + L_{A_2}U_4R_{B_5}R_{B_2},$$

$$U_3 = A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}SV_4R_NDB^{\dagger} + L_AV_1 + V_2R_B,$$

$$U_4 = M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_SV_3 + L_MV_4R_N + V_5R_D,$$

where V_1, V_2, V_3, V_4, V_5 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Remark 3.4. Our expression of the general solution to system (3.14) is different from the expression in [27].

4. Conclusions

In this paper we have given the extremal ranks of the matrix function (1.6) subject to (1.7), which extend the known results in [20] and [22]. We have derived some solvable conditions for the existence of the general solution to system (1.5), and proved that (3.1)-(3.6) are solutions of system (1.5) when the solvability conditions are met. Using the results on (1.5), we have established some necessary and sufficient conditions for the existence of the general solution to (3.13) and (3.14), respectively. The expressions of such solutions to (3.13) and (3.14) have also been given, respectively. There is no doubt that most of the results in this paper can be extended to the corresponding system for linear operators on a Hilbert space or elements in a ring with involution.

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