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(X_d, X_d^*)-BESSEL MULTIPLIERS IN BANACH SPACES

MOHAMMAD HASAN FAROUGHI[1], ELNAZ OSGOOEI[1] AND ASGHAR RAHIMI[2]3[*]

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ABSTRACT. Multipliers have recently been introduced as operators for Bessel sequences and frames in Hilbert spaces. In this paper, we define the concept of (X_d, X_d^*) and (l^∞, X_d, X_d^*)-Bessel multipliers in Banach spaces and investigate the compactness of these multipliers. Also, we study the possibility of invertibility of (l^∞, X_d, X_d^*)-Bessel multiplier depending on the properties of its corresponding sequences and its symbol. Furthermore, we prove that every (X_d, X_d^*)-Bessel multiplier is a λ-nuclear operator.

1. INTRODUCTION AND PRELIMINARIES

In 1991 Gröchenig [11] generalized frames to Banach spaces and called them atomic decompositions. He also defined a more general notion for Banach spaces called a Banach frame. Further work on atomic decompositions via group representations appeared in 1996 by Christensen [7], and perturbation theory for atomic decompositions was presented by Christensen and Heil [8]. For further studies on Banach frames and atomic decompositions, we refer to [5, 6, 12, 14, 26, 27, 28, 30].

In [21], Schatten provided a detailed study of ideals of compact operators of the form $\sum m_k \phi_k \otimes \overline{\psi_k}$, where $\{\phi_k\}$ and $\{\psi_k\}$ are orthonormal families, Balazs replaced these orthonormal families with Bessel sequences to define Bessel multipliers for Hilbert spaces [2, 3, 4]. For Bessel sequences, the investigation of the operator $M = \sum m_k \phi_k \otimes \psi_k$ is very natural and useful and there are numerous

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[*] Corresponding author.
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applications of this kind of operators. In [22, 23, 24, 25], Stoeva and Balazs investigated the unconditional convergence of Bessel multipliers and characterized a complete set of conditions for the invertibility of them. Bessel multipliers for p-Bessel sequences in Banach spaces and for g-Bessel sequences in Hilbert spaces were introduced in [16] and [17], respectively. In this paper, we introduce Bessel multipliers for X_d-Bessel sequences in Banach spaces.

This paper is organized as follows: In Section 1, we present properties of α-dual (Köthe-dual) and β-dual of a BK-space and recall some basic properties of X_d-frames in Banach spaces. In Section 2, we define the concept of (X_d, X_d^*) and (l^∞, X_d, X_d^*)-Bessel multipliers in Banach spaces and show that as operators, they are well defined and bounded. In Section 3, we investigate the compactness of these multipliers and study the invertibility of (l^∞, X_d, X_d^*)-Bessel multiplier. The dependency of the multipliers on their parameters is investigated in Section 4. Finally, in Section 5, we prove that every (X_d, X_d^*)-Bessel multiplier is a λ-nuclear operator.

Throughout this paper, X is a Banach space, $B(X)$ is the space of bounded linear operators from X into X and X_d is a complex sequence space; that is, a vector space whose elements are sequences of complex numbers. All sequence spaces will be assumed to include ϕ, the set of finitely nonzero sequences [13]. A sequence space X_d is called a BK-space, if it is a Banach space and all of the coordinate functionals $\{a_k\} \to a_k$ are continuous. A sequence space X_d is called solid if whenever $\{a_k\}$ and $\{b_k\}$ are sequences with $\{b_k\} \in X_d$ and $|a_k| \leq |b_k|$, for each $k \in \mathbb{N}$, then it follows that $\{a_k\} \in X_d$ and $\|\{a_k\}\|_{X_d} \leq \|\{b_k\}\|_{X_d}$. A sequence space X_d is called an AK-space if it is a topological vector space and $\{a_k\} = \lim_n p_n(\{a_k\})$ for each $\{a_k\} \in X_d$, where $p_n(\{a_k\}) = \{a_1, a_2, \cdots, a_n, 0, \cdots\}$.

In [13], Köthe has assigned for each sequence space X_d two sequence spaces X_d^α, α-dual (Köthe-dual) of X_d, with the following definition:

$$X_d^\alpha = \left\{ \{a_k\} : \sum_{k=1}^{\infty} |a_k b_k| < \infty, \quad \forall \{b_k\} \in X_d \right\},$$

and X_d^β, β-dual of X_d, with this definition:

$$X_d^\beta = \left\{ \{a_k\} : \sum_{k=1}^{\infty} a_k b_k \ converges, \quad \forall \{b_k\} \in X_d \right\}.$$

It is evident that $X_d^\alpha \subseteq X_d^\beta$. We note that α-dual and β-dual of a BK-space X_d are BK-spaces with respect to the norms

$$\|\{a_k\}\|_\alpha = \sup_{\|\{b_k\}\|_{X_d} \leq 1} \sum_{k=1}^{\infty} |a_k b_k|, \tag{1.1}$$

and

$$\|\{a_k\}\|_\beta = \sup_{\|\{b_k\}\|_{X_d} \leq 1} |\sum_{k=1}^{\infty} a_k b_k|, \tag{1.2}$$

respectively. Also if X_d is a solid BK-space, then $X_d^\alpha = X_d^\beta$ [15, 29].

Remark 1.1. We note that if X_d is a solid BK-space, the norms defined in (1.1) and (1.2) are equivalent by the open mapping theorem.

It is proved in [15, 29], that the spaces X_d^* and X_d^β are isometrically isomorphic with the norm defined in (1.2), when X_d is a BK-AK-space. So by Remark 1.1, we deduce that if X_d is a solid BK-AK-space, then the spaces X_d^* and X_d^α are isomorphic with the norm defined in (1.1) and there exist $K, K' > 0$ such that

$$K'\|\{a_k\}\|_{X_d^*} \leq \|\{a_k\}\|_\alpha \leq K\|\{a_k\}\|_{X_d^*}, \quad \{a_k\} \in X_d^* \simeq X_d^\alpha, \tag{1.3}$$

where K' can be set to 1.

Lemma 1.1. [10] *Let $\{e_k\}$ be a Schauder basis of a normed space X. The canonical projections $P_n : X \to X$, where $P_n(\sum_{i=1}^\infty a_i e_i) = \sum_{i=1}^n a_i e_i$, satisfy:*
(i) $\dim(P_n(X)) = n$;
(ii) $P_n P_m = P_m P_n = P_{\min(m,n)}$;
(iii) $\lim_{n\to\infty} P_n(x) = x$, *for every* $x \in X$.

We note that sequence $\{m_k\}$ is called semi-normalized, if

$$0 < \inf_k |m_k| \leq \sup_k |m_k| < \infty.$$

A sequence $\{\phi_k\} \subseteq X$ is called norm bounded above, in short NBA (resp. norm bounded below, in short NBB), if $\sup_k \|\phi_k\| < \infty$ (resp. $\inf_k \|\phi_k\| > 0$).

Definition 1.2. Let X be a Banach space and X_d be a BK-space. A countable sequence $\{g_k\}_{k=1}^\infty$ in the dual X^* is called an X_d-frame for X if
(i) $\{g_k(f)\} \in X_d, \quad f \in X$;
(ii) the norms $\|f\|_X$ and $\|\{g_k(f)\}\|_{X_d}$ are equivalent i.e., there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq \|\{g_k(f)\}\|_{X_d} \leq B\|f\|_X, \quad f \in X. \tag{1.4}$$

The constants A and B are called lower and upper X_d-frame bounds, respectively. If (i) and the upper condition in (1.4), are satisfied, then $\{g_k\}$ is called an X_d-Bessel sequence for X with bound B. We call $\{g_k\}$ a tight X_d-frame if $A = B$ and a Parseval X_d-frame if $A = B = 1$.

Definition 1.3. [1] Let $\{g_k\}$ be a sequence of elements in X^* and $\{m_k\} \subseteq \mathbb{C}$. We call $\{g_k\}$ a weighted X_d-frame for X, if the sequence $\{m_k g_k\}$ is an X_d-frame for X.

Proposition 1.4. [5] *Suppose that X_d is a BK-space for which the canonical unit vectors $\{e_k\}$ form a Schauder basis. Then $\{g_k\} \subseteq X^*$ is an X_d^*-Bessel sequence for X with bound B if and only if the operator*

$$T : \{d_k\} \to \sum_{k=1}^\infty d_k g_k,$$

is well defined (hence bounded) from X_d into X^* and $\|T\| \leq B$.

Definition 1.5. A sequence $\{f_k\} \subseteq X$ is called an X_d-Riesz basis for X, if it is complete in X and there exist constants $A, B > 0$ such that

$$A\|\{c_k\}\|_{X_d} \leq \|\sum_{k=1}^{\infty} c_k f_k\| \leq B\|\{c_k\}\|_{X_d}, \quad \{c_k\} \in X_d. \tag{1.5}$$

The constants A and B are called lower and upper X_d-Riesz basis bounds, respectively. If $\{f_k\}$ is an X_d-Riesz basis for $\overline{span}_k\{f_k\}$, then $\{f_k\}$ is called an X_d-Riesz sequence.

Proposition 1.6. [28] *Suppose that X_d is a reflexive BK-space for which the canonical unit vectors $\{e_k\}$ form a Schauder basis. Assume that $\{\psi_k\} \subseteq X^*$ is an X_d^*-Riesz basis for X^* with lower bound A and upper bound B. Then there exists a unique sequence $\{\tilde{\psi}_k\} \subseteq X$, which is an X_d-Riesz basis for X with lower bound $\frac{1}{B}$ and upper bound $\frac{1}{A}$, such that*

$$f = \sum_{k=1}^{\infty} \psi_k(f)\tilde{\psi}_k, \quad f \in X,$$

$$g = \sum_{k=1}^{\infty} g(\tilde{\psi}_k)\psi_k, \quad g \in X^*.$$

This sequence $\{\tilde{\psi}_k\}$ is unique and biorthogonal to $\{\psi_k\}$.

Throughout the following sections, since we use many results of [28, 29], we need to assume that X is a reflexive Banach space and X_d is a solid, reflexive, BK-space such that the canonical unit vectors $\{e_k\}$ form a Schauder basis.

2. MULTIPLIERS FOR X_d-BESSEL SEQUENCES

Motivated by the multipliers for p-Bessel sequences [16], in this section, we will extend multipliers in more general cases, i.e., for X_d-Bessel sequences.

Lemma 2.1. *Suppose that $\{\phi_k\} \subseteq X$ is an X_d^*-Bessel sequence for X^* with bound B'. Then the following statements hold:*
(i) Let $\{\psi_k\} \subseteq X^*$. Suppose that there exists $P > 0$ such that $\|\psi_k\| \leq P$ for each $k \in \mathbb{N}$, and $m = \{m_k\} \in X_d$. Then the operator $M = M_{m,(\phi_k),(\psi_k)} : X \to X$ defined by:

$$M_{m,(\phi_k),(\psi_k)}(f) = \sum_{k=1}^{\infty} m_k \psi_k(f)\phi_k, \quad f \in X,$$

is well defined and bounded with $\|M\| \leq KPB'\|\{m_k\}\|_{X_d}$.
(ii) Let $\{\psi_k\} \subseteq X^*$ be an X_d-Bessel sequence for X with bound B, and $m = \{m_k\} \in l^\infty$. Then the operator $M' = M'_{m,(\phi_k),(\psi_k)} : X \to X$ defined by:

$$M'_{m,(\phi_k),(\psi_k)}(f) = \sum_{k=1}^{\infty} m_k \psi_k(f)\phi_k, \quad f \in X,$$

is well defined and bounded with $\|M'\| \leq KBB'\|\{m_k\}\|_\infty$.

Proof. (i) First, we prove that $\{\sum_{k=1}^{j} m_k\psi_k(f)\phi_k\}_{j=1}^{\infty}$ is Cauchy in X. Consider $i,j\in\mathbb{N}$, $i>j$. Then we have

$$\begin{aligned}\|\sum_{k=j+1}^{i} m_k\psi_k(f)\phi_k\| &= \sup_{g\in X^*,\|g\|\le 1}|\sum_{k=j+1}^{i} m_k\psi_k(f)\phi_k(g)| \\ &\le P\|f\|\sup_{g\in X^*,\|g\|\le 1}\sum_{k=j+1}^{\infty}|m_k\phi_k(g)|. \qquad (2.1)\end{aligned}$$

Since X_d^* and X_d^α are isomorphic, $\{\phi_k(g)\}_{k=j+1}^{\infty}\in X_d^\alpha$. By (1.1) and (2.1), we have

$$\begin{aligned}\|\sum_{k=j+1}^{i} m_k\psi_k(f)\phi_k\| &\le P\|f\|\|\{m_k\}-p_j(\{m_k\})\|_{X_d}\sup_{g\in X^*,\|g\|\le 1}\|\{\phi_k(g)\}_{k=j+1}^{\infty}\|_\alpha \\ &\le P\|f\|\|\{m_k\}-p_j(\{m_k\})\|_{X_d}\sup_{g\in X^*,\|g\|\le 1}\|\{\phi_k(g)\}_{k=1}^{\infty}\|_\alpha,\end{aligned}$$

hence by (1.3), there exists $K>0$ such that

$$\begin{aligned}\|\sum_{k=j+1}^{i} m_k\psi_k(f)\phi_k\| &\le KP\|f\|\|\{m_k\}-p_j(\{m_k\})\|_{X_d}\sup_{g\in X^*,\|g\|\le 1}\|\{\phi_k(g)\}_{k=1}^{\infty}\|_{X_d^*} \\ &\le KPB'\|f\|\|\{m_k\}-p_j(\{m_k\})\|_{X_d}.\end{aligned}$$

Since the canonical unit vectors $\{e_k\}$ form a Schauder basis for X_d, by Lemma 1.1, $\lim_j\|\{m_k\}-p_j(\{m_k\})\|_{X_d}=0$. Therefore $\{\sum_{k=1}^{j} m_k\psi_k(f)\phi_k\}_{j=1}^{\infty}$ is Cauchy in X and so M is well defined.
Now we show that M is bounded. For this we have

$$\begin{aligned}\|M(f)\| &= \|\sum_{k=1}^{\infty} m_k\psi_k(f)\phi_k\| = \sup_{g\in X^*,\|g\|\le 1}|\sum_{k=1}^{\infty} m_k\psi_k(f)\phi_k(g)| \\ &\le P\|f\|\sup_{g\in X^*,\|g\|\le 1}\sum_{k=1}^{\infty}|m_k\phi_k(g)|,\quad f\in X,\end{aligned}$$

hence by (1.1) and (1.3), we have

$$\begin{aligned}\|M(f)\| &\le P\|f\|\|\{m_k\}\|_{X_d}\sup_{g\in X^*,\|g\|\le 1}\|\{\phi_k(g)\}\|_\alpha \\ &\le KPB'\|f\|\|\{m_k\}\|_{X_d},\quad f\in X.\end{aligned}$$

So, $\|M\|\le KPB'\|\{m_k\}\|_{X_d}$.
(ii) Since $\{m_k\}\in l^\infty$, we have

$$|m_k\psi_k(f)|\le\|\{m_k\}\|_\infty|\psi_k(f)|,\quad k\in\mathbb{N},$$

since $\{\psi_k(f)\}\in X_d$ and X_d is a solid BK-space, $\{\frac{m_k\psi_k(f)}{\|\{m_k\}\|_\infty}\}\in X_d$ and we have

$$\|\{m_k\psi_k(f)\}\|_{X_d}\le\|\{m_k\}\|_\infty\|\{\psi_k(f)\}\|_{X_d}. \qquad (2.2)$$

Now we prove that $\{\sum_{k=1}^{j} m_k\psi_k(f)\phi_k\}_{j=1}^{\infty}$ is Cauchy in X. Consider $i, j \in \mathbb{N}$, $i > j$. Since $\{m_k\psi_k(f)\}_{k=j+1}^{\infty} \in X_d$ and $\{\phi_k(g)\}_{k=j+1}^{\infty} \in X_d^{\alpha}$, by (1.1), we have

$$\begin{aligned}\|\sum_{k=j+1}^{i} m_k\psi_k(f)\phi_k\| &= \sup_{g\in X^*,\|g\|\le 1} |\sum_{k=j+1}^{i} m_k\psi_k(f)\phi_k(g)| \\ &\le \|\{m_k\psi_k(f)\} - p_j(\{m_k\psi_k(f)\})\|_{X_d} \sup_{g\in X^*,\|g\|\le 1} \|\{\phi_k(g)\}\|_{\alpha}.\end{aligned}$$

Similar to the proof of (i), we can deduce that $\{\sum_{k=1}^{j} m_k\psi_k(f)\phi_k\}_{j=1}^{\infty}$ is Cauchy in X. Therefore, M' is well defined.
Now we show that M' is bounded. For this by (1.1), we have

$$\begin{aligned}\|M'f\| &= \|\sum_{k=1}^{\infty} m_k\psi_k(f)\phi_k\| = \sup_{g\in X^*,\|g\|\le 1} |\sum_{k=1}^{\infty} m_k\psi_k(f)\phi_k(g)| \\ &\le \|\{m_k\psi_k(f)\}\|_{X_d} \sup_{g\in X^*,\|g\|\le 1} \|\{\phi_k(g)\}\|_{\alpha},\end{aligned}$$

hence by (2.2) and (1.3), we have

$$\begin{aligned}\|M'f\| &\le \|\{m_k\}\|_{\infty}\|\{\psi_k(f)\}\|_{X_d}KB' \\ &\le KBB'\|\{m_k\}\|_{\infty}\|f\|.\end{aligned}$$

So $\|M'\| \le KBB'\|\{m_k\}\|_{\infty}$. □

Remark 2.2. Suppose that the unit vectors $\{E_k\}$, the sequence of coefficient functionals associated to the canonical basis $\{e_k\}$ of X_d, forms a basis for X_d^*. Then by [5, Corollary 3.3], the mapping M in part (i) of Lemma 2.1, is also well defined and bounded, if the $\{\psi_k\}$ is assumed to be an X_d-Bessel sequence for X.

Definition 2.3. Let $\{\phi_k\} \subseteq X$ be an X_d^*-Bessel sequence for X^* with bound B'. Suppose that $\{\psi_k\} \subseteq X^*$ and $\|\psi_k\| \le P$, for each $k \in \mathbb{N}$ and $m = \{m_k\} \in X_d$. The operator $M = M_{m,(\phi_k),(\psi_k)} : X \to X$ defined by:

$$M_{m,(\phi_k),(\psi_k)}(f) = \sum_{k=1}^{\infty} m_k\psi_k(f)\phi_k,$$

is called an (X_d, X_d^*)-Bessel multiplier for sequences $\{\psi_k\}$ and $\{\phi_k\}$. The sequence m is called the symbol of M and $\{\psi_k\}$ and $\{\phi_k\}$ are called its corresponding sequences.

Example 2.4. *Let $X = X_d = l^p$, $1 < p < \infty$. Suppose that $\{E_k\}$ is the sequence of coefficient functionals associated to the canonical basis $\{e_k\}$ of X_d. Denote $\{\psi_k\} = \{\frac{1}{2}E_1, E_2, \frac{1}{2^2}E_1, E_3, \frac{1}{2^3}E_1, \cdots\}$, $\{\phi_k\} = \{e_1, e_2, e_3, e_4, e_5, \cdots\}$ and $\{m_k\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\}$. Then $\|\psi_k\| \le 1$, for each $k \in \mathbb{N}$, $\{\phi_k\} \subseteq l^p$ is a Parseval l^q-frame for l^q and $\{m_k\} \in l^p$. Therefore, $M_{m,(\phi_k),(\psi_k)}$ is a (l^p, l^q)-Bessel multiplier.*

Definition 2.5. Let $\{\phi_k\} \subseteq X$ be an X_d^*-Bessel sequence for X^* with bound B'. Suppose that $\{\psi_k\} \subseteq X^*$ is an X_d-Bessel sequence for X with bound B and

$m = \{m_k\} \in l^\infty$. The operator $M' = M'_{m,(\phi_k),(\psi_k)} : X \to X$ defined by:

$$M'_{m,(\phi_k),(\psi_k)}(f) = \sum_{k=1}^{\infty} m_k \psi_k(f) \phi_k,$$

is called a (l^∞, X_d, X_d^*)-Bessel multiplier for sequences $\{\psi_k\}$ and $\{\phi_k\}$. The sequence m is called the symbol of M' and $\{\psi_k\}$ and $\{\phi_k\}$ are called its corresponding sequences.
We shall denote $M_{m,(\phi_k),(\psi_k)}$ by M and $M'_{m,(\phi_k),(\psi_k)}$ by M'.

Remark 2.6. We note that since X and X_d are reflexive spaces, the definition of (X_d, X_d^*)-Bessel multiplier can be expressed by:

$$M = T_{\phi_k} D_m U,$$

where T_{ϕ_k} from $X_d^{**} \simeq X_d$ into $X^{**} \simeq X$, is the synthesis operator of X_d^*-Bessel sequence $\{\phi_k\}$. The mappings $D_m : l^\infty \to X_d$, $D_m(\{c_k\}) = \{m_k c_k\}$ and $U : X \to l^\infty$, $U(f) = \{\psi_k(f)\}$, are well defined operators. Also, by Definition 2.5, M' can be shown by:

$$M' = T_{\phi_k} D_m U_{\psi_k},$$

where T_{ϕ_k} from $X_d^{**} \simeq X_d$ into $X^{**} \simeq X$, is the synthesis operator of X_d^*-Bessel sequence $\{\phi_k\}$. The mapping $D_m : X_d \to X_d$, $D_m(\{c_k\}) = \{m_k c_k\}$ is a well defined operator and U_{ψ_k} from X into X_d, is the analysis operator of the X_d-Bessel sequence $\{\psi_k\}$. In this case, M' can also be written by:

$$M' = T_{\phi_k} U_{m_k \psi_k},$$

where T_{ϕ_k} is the synthesis operator of X_d^*-Bessel sequence $\{\phi_k\}$, and $U_{m_k\psi_k}$ is the analysis operator of the weighted X_d-Bessel sequence $\{\psi_k\}$, where $\{m_k\}$ is a sequence of weights.

Proposition 2.7. *Suppose that M is an (X_d, X_d^*)-Bessel multiplier. Let $\psi_k \neq 0$, for each $k \in \mathbb{N}$ and $\{\phi_k\} \subseteq X$ be an X_d-Riesz sequence. Then the mapping*

$$m \to M_{m,(\phi_k),(\psi_k)},$$

is injective from X_d into $B(X)$.

Proof. Suppose that $M_{m,(\phi_k),(\psi_k)} = M_{m',(\phi_k),(\psi_k)}$, where $m' = \{m'_k\} \in X_d$. Then $\sum_{k=1}^{\infty} m_k \psi_k(f) \phi_k = \sum_{k=1}^{\infty} m'_k \psi_k(f) \phi_k$, for each $f \in X$. Since $\{\phi_k\}$ is an X_d-Riesz sequence, by [28, Theorem 4.8], $m_k \psi_k(f) = m'_k \psi_k(f)$, for all $f \in X$ and $k \in \mathbb{N}$. Since for each $k \in \mathbb{N}$, there exists $f \in X$ such that $\psi_k(f) \neq 0$, $m_k = m'_k$. □

Corollary 2.8. *Suppose that M' is a (l^∞, X_d, X_d^*)-Bessel multiplier. Let $\psi_k \neq 0$, for each $k \in \mathbb{N}$ and $\{\phi_k\} \subseteq X$ be an X_d-Riesz sequence. Then the mapping*

$$m \to M'_{m,(\phi_k),(\psi_k)},$$

is injective from X_d into $B(X)$.

Proposition 2.9. *Assume that M' is a (l^∞, X_d, X_d^*)-Bessel multiplier for X_d^*-Riesz basis $\{\psi_k\} \subseteq X^*$ with bounds A and B and for X_d-Riesz basis $\{\phi_k\} \subseteq X$ with bounds A' and B'. Then*

$$AA' \|\{m_k\}\|_\infty \leq \|M'\| \leq KBB' \|\{m_k\}\|_\infty.$$

Proof. Part (ii) of Lemma 2.1, gives the upper bound. By Proposition 1.6, $\{\psi_k\}$ has a biorthogonal sequence $\{\tilde{\psi}_k\} \subseteq X$, such that $\{\tilde{\psi}_k\}$ is an X_d-Riesz basis with bounds $\frac{1}{B}$ and $\frac{1}{A}$ and $\psi_k(\tilde{\psi}_i) = \delta_{k,i}$, for all $k, i \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \|M'\| &= \sup_{f\in X} \frac{\|M'f\|}{\|f\|} \geq \sup_{i\in\mathbb{N}} \frac{\|M'\tilde{\psi}_i\|}{\|\tilde{\psi}_i\|} = \sup_{i\in\mathbb{N}} \frac{\|\sum_{k=1}^{\infty} m_k\psi_k(\tilde{\psi}_i)\phi_k\|}{\|\tilde{\psi}_i\|} \\ &= \sup_{i\in\mathbb{N}} \frac{\|m_i\phi_i\|}{\|\tilde{\psi}_i\|} = \sup_{i\in\mathbb{N}} |m_i| \frac{\|\phi_i\|}{\|\tilde{\psi}_i\|}. \end{aligned} \tag{2.3}$$

Since $\{\phi_k\}$ is an X_d-Riesz basis for X with lower bound A' and the canonical unit vectors $\{e_k\}$ form a basis for X_d, by (1.5), we have

$$\|\phi_i\| \geq A', \quad i \in \mathbb{N}. \tag{2.4}$$

Also, since $\{\tilde{\psi}_k\}$ is an X_d-Riesz basis for X with upper bound $\frac{1}{A}$ and the canonical unit vectors $\{e_k\}$ form a basis for X_d, by (1.5), we have

$$\|\tilde{\psi}_i\| \leq \frac{1}{A}, \quad i \in \mathbb{N}. \tag{2.5}$$

Now by (2.4) and (2.5), we have

$$\sup_{i\in\mathbb{N}} |m_i| \frac{\|\phi_i\|}{\|\tilde{\psi}_i\|} \geq AA'\|\{m_k\}\|_\infty.$$

Therefore by (2.3), $\|M'\| \geq AA'\|\{m_k\}\|_\infty$. □

3. Compactness and invertibility of multipliers

The compactness of Bessel multipliers are investigated by Balazs in [2]. Also, in [22, 23, 24, 25], Stoeva and Balazs characterized a complete set of conditions for the invertibility of multipliers.

Lemma 3.1. *With the notations of Definitions 2.3 and 2.5, the following assertions are true:*
(i) If M is an (X_d, X_d^*)-Bessel multiplier, then M is a compact operator.
(ii) If M' is a (l^∞, X_d, X_d^*)-Bessel multiplier and $m = \{m_k\} \in c_0$, then M' is a compact operator.

Proof. (i) We define the finite rank operator

$$M_K(f) = \sum_{k=1}^{K} m_k\psi_k(f)\phi_k,$$

where $\{m_k\}$, $\{\phi_k\}$ and $\{\psi_k\}$ are same as Definition 2.3. Then we have

$$\begin{aligned} \|M - M_K\| &= \sup_{f\in X, \|f\|\leq 1} \sup_{g\in X^*, \|g\|\leq 1} |\sum_{k=K+1}^{\infty} m_k\psi_k(f)\phi_k(g)| \\ &\leq \sup_{f\in X, \|f\|\leq 1} \sup_{g\in X^*, \|g\|\leq 1} \sum_{k=K+1}^{\infty} |m_k\psi_k(f)\phi_k(g)|, \end{aligned} \tag{3.1}$$

since $\{m_k\}_{k=K+1}^{\infty} \in X_d$ and $\{\phi_k(g)\}_{k=K+1}^{\infty} \in X_d^{\alpha}$, by (3.1), (1.1) and (1.3), we have

$$\begin{aligned} \|M - M_K\| &\leq \sup_{f\in X,\|f\|\leq 1} P\|f\|\|\{m_k\} - p_K(\{m_k\})\|_{X_d} \sup_{g\in X^*,\|g\|\leq 1} \|\{\phi_k(g)\}\|_{\alpha} \\ &\leq KPB'\|\{m_k\} - p_K(\{m_k\})\|_{X_d}. \end{aligned}$$

Since the canonical unit vectors $\{e_k\}$ form a Schauder basis for X_d, by Lemma 1.1, $\lim_K \|\{m_k\} - p_K(\{m_k\})\| = 0$ and so M is a compact operator.
(ii) For a given $m \in c_o$, let $m^{(l)} = (m_1, m_2, \cdots, m_l, 0, 0, \cdots)$. Then by part (ii) of Lemma 2.1, we have

$$\begin{aligned} \|M'_{m,(\phi_k),(\psi_k)} - M'_{m^{(l)},(\phi_k),(\psi_k)}\| &= \|M'_{m-m^{(l)},(\phi_k),(\psi_k)}\| \\ &\leq \|m - m^{(l)}\|_{\infty} KBB'. \end{aligned}$$

Since $m \in c_0$, $\lim_l \|m - m^{(l)}\|_{\infty} = 0$, and the proof is evident. □

Here is an example which shows that a (l^{∞}, X_d, X_d^*)-Bessel multiplier may not be a compact operator, if $m = \{m_k\} \notin c_0$.

Example 3.2. *Let $X = X_d = l^p$, $1 < p < \infty$. Suppose that $\{E_k\}$ is the sequence of coefficient functionals associated to the canonical basis $\{e_k\}$ of X_d. Denote $\{\psi_k\} = \{E_k\}$ and $\{\phi_k\} = \{e_k\}$. Then $M'_{1,(\phi_k),(\psi_k)}$ is an (l^{∞}, l_p, l_q)-Bessel multiplier but it is not a compact operator, if X is infinite dimensional.*

Proposition 3.3. *Suppose that M' is a (l^{∞}, X_d, X_d^*)-Bessel multiplier and $m = \{m_k\}$ is semi-normalized. Then the following assertions hold:*
(i) If $\{\phi_k\}$ is an X_d-Riesz basis for X and $\{\psi_k\}$ is an X_d-frame for X, then M' is injective.
(ii) If $\{\psi_k\}$ is an X_d^*-Riesz basis for X^* and $\{\phi_k\}$ is an X_d^*-frame for X^*, then M' is surjective.

Proof. The proof is evident by Remark 2.6 and [28, Propositions 3.4, 4.5]. □

In the following theorem, we investigate the invertibility of M' and determine the formula for $(M')^{-1}$.

Theorem 3.4. *Suppose that M' is a (l^{∞}, X_d, X_d^*)-Bessel multiplier and $m = \{m_k\}$ is semi-normalized. Then the following statements hold:*
(i) If $\{\psi_k\} \subseteq X^*$ is an X_d^*-Riesz basis for X^* and $\{\phi_k\} \subseteq X$ is an X_d^*-Bessel sequence for X^*. Then M' is invertible on X if and only if $\{\phi_k\}$ is an X_d-Riesz basis for X.
(ii) If $\{\phi_k\} \subseteq X$ is an X_d-Riesz basis for X and $\{\psi_k\} \subseteq X^*$ is an X_d-Bessel sequence for X. Then M' is invertible on X if and only if $\{\psi_k\}$ is an X_d^*-Riesz basis for X^*.
In the case that M' is invertible, $(M')^{-1} = M'_{(\frac{1}{m_k}),(\tilde{\psi_k}),(\tilde{\phi_k})}$, where $\{\tilde{\psi_k}\} \subseteq X$ and $\{\tilde{\phi_k}\} \subseteq X^*$ are X_d-Riesz basis for X and X_d^*-Riesz basis for X^*, respectively.

Proof. (i) By Remark 2.6, $M' = T_{\phi_k} D_m U_{\psi_k}$. Suppose that $\{\psi_k\}$ and $\{\phi_k\}$ are X_d^*-Riesz basis for X^* and X_d-Riesz basis for X, respectively. Then by [28, Propositions 3.4, 4.5], T_{ϕ_k} and U_{ψ_k} are invertible and also D_m, since m is semi-normalized. Therefore M' is an invertible operator. By Proposition 1.6, there exist a unique X_d-Riesz basis $\{\tilde{\psi_k}\} \subseteq X$ and a unique X_d^*-Riesz basis $\{\tilde{\phi_k}\} \subseteq X^*$, which are biorthogonal to $\{\psi_k\}$ and $\{\phi_k\}$, respectively. Since m is semi-normalized, $\frac{1}{m} = \{\frac{1}{m_k}\} \in l^\infty$ and we have

$$\begin{aligned} M'_{(\frac{1}{m}),(\tilde{\psi_k}),(\tilde{\phi_k})} \circ M'(f) &= M'_{(\frac{1}{m}),(\tilde{\psi_k}),(\tilde{\phi_k})}(\sum_{k=1}^{\infty} m_k \psi_k(f)\phi_k) \\ &= \sum_{i=1}^{\infty} \frac{1}{m_i} \tilde{\phi}_i (\sum_{k=1}^{\infty} m_k \psi_k(f)\phi_k)\tilde{\psi}_i \\ &= \sum_{i=1}^{\infty} \frac{1}{m_i} \sum_{k=1}^{\infty} m_k \psi_k(f)\tilde{\phi}_i(\phi_k)\tilde{\psi}_i \\ &= \sum_{i=1}^{\infty} \psi_i(f)\tilde{\psi}_i \\ &= f, \quad f \in X. \end{aligned}$$

Conversely, suppose that M' is an invertible operator. Since $\{\psi_k\}$ is an X_d^*-Riesz basis for X^*, by [28, Propositions 3.4, 4.5], U_{ψ_k} and so T_{ϕ_k} are invertible. Therefore $\{\phi_k\}$ is an X_d^*-Riesz basis by [28, Proposition 3.4].
(ii) The proof is similar to the first part. □

Theorem 3.5. *Let M' be an invertible operator on X. Suppose that $\{m_k\} = m \in l^\infty$. Then the following statements hold:*
(i) If $\{\psi_k\} \subseteq X^*$ (resp. $\{m_k\psi_k\}$) is an X_d-Bessel sequence for X with bound B and $\{\phi_k\} \subseteq X$ is an X_d^*-Bessel sequence for X^*, then $\{\phi_k\}$ is an X_d^*-frame for X^*.
(ii) If $\{\phi_k\} \subseteq X^*$ is an X_d^*-Bessel sequence for X^* and $\{\psi_k\} \subseteq X^*$ is an X_d-Bessel sequence for X, then $\{\psi_k\}$ (resp. $\{m_k\psi_k\}$) is an X_d-frame for X.

Proof. (i) By assumption it is enough to find a lower frame bound for $\{\phi_k\}$. Let $g \in X^*$. For $g = 0$, the lower bound condition is trivially fulfilled. Now let $g \neq 0$. Since M' is invertible, by (1.1) and (1.3), we have

$$\begin{aligned} \|g\| &= \|(M'^*)^{-1}M'^*g\| \leq \|M'^{-1}\|\|M'^*g\| \\ &= \|M'^{-1}\| \sup_{f\in X, \|f\|\leq 1} |\langle f, M'^*g\rangle| = \|M'^{-1}\| \sup_{f\in X, \|f\|\leq 1} |\sum_{k=1}^{\infty} m_k\psi_k(f)\phi_k(g)| \\ &\leq \|M'^{-1}\| \sup_{f\in X, \|f\|\leq 1} \|\{m_k\psi_k(f)\}\|_{X_d} \|\{\phi_k(g)\}\|_\alpha \\ &\leq \|M'^{-1}\|\|\{m_k\}\|_\infty \sup_{f\in X, \|f\|\leq 1} \|\{\psi_k(f)\}\|_{X_d} \|\{\phi_k(g)\}\|_\alpha \\ &\leq \|M'^{-1}\|\|\{m_k\}\|_\infty BK\|\{\phi_k(g)\}\|_{X_d^*}. \end{aligned}$$

(ii) The proof is similar to the above argument. □

Theorem 3.6. *Suppose that the canonical unit vectors $\{e_k\}$ and $\{E_k\}$ form bases for X_d and X_d^*, respectively. Let $\{\phi_k\}$ be an X_d-Riesz basis for X and $\{\psi_k\}$ be an X_d-Bessel sequence for X and non-NBB. Assume that $m = \{m_k\} \in l^\infty$ and $m_k \neq 0$, for some $k \in \mathbb{N}$. Then M' is not invertible on X.*

Proof. Suppose that M' is an invertible operator on X. By Remark 2.6, $M' = T_{\phi_k} U_{m_k \psi_k}$. Since $\{\phi_k\}$ is an X_d-Riesz basis, by [28, Propositions 3.4, 4.5], T_{ϕ_k} is invertible and so $\{m_k \psi_k\}$ is an X_d^*-Riesz basis for X^*. Hence, there exists $D > 0$ such that

$$D\|\{c_k\}\| \leq \|\sum_{k=1}^{\infty} c_k m_k \psi_k\|, \quad \{c_k\} \in X_d^*. \tag{3.2}$$

Since the unit vectors $\{E_k\}$ form a basis for X_d^*, by [5, Lemma 3.1] and (3.2), we have

$$D = D\|e_j\| \leq \|m_j \psi_j\|, \quad j \in \mathbb{N}. \tag{3.3}$$

Since $m \in l^\infty$, by (3.3) $\{\psi_k\}$ is NBB, which is a contradiction. □

4. Dependency of parameters

In this section, we investigate the behavior of the Bessel multipliers when the parameters are changing.
We note that X_1 and X_1^* denote the closed unit balls of X and X^*, respectively.

Theorem 4.1. *Let M be an (X_d, X_d^*)-Bessel multiplier for sequences $\{\psi_k\} \subseteq X^*$ and $\{\phi_k\} \subseteq X$, where B' is the upper Bessel bound for $\{\phi_k\}$ and $\|\psi_k\| \leq P$, for each $k \in \mathbb{N}$. Then the operator M depends continuously on $m, \{\psi_k\}$ and $\{\phi_k\}$ in the following sense: Let $\{\psi_k^{(l)}\} \subseteq X^*$ and $\{\phi_k^{(l)}\} \subseteq X$ be sequences indexed by $l \in \mathbb{N}$.*
(i) Let $m^{(l)} \to m$ in X_d. Then for $l \to \infty$, $\|M_{m^{(l)},(\phi_k),(\psi_k)} - M_{m,(\phi_k),(\psi_k)}\| \to 0$.
(ii) Let $\{\psi_k^{(l)}(f)\}$ converges to $\{\psi_k(f)\}$ in l^∞, for each $f \in X_1$. Then for $l \to \infty$, $\|M_{m,(\phi_k),(\psi_k^{(l)})} - M_{m,(\phi_k),(\psi_k)}\| \to 0$.
(iii) Let $\{\phi_k^{(l)}(g)\} \in X_d^*$, for each $g \in X^*$ and $\{\phi_k^{(l)}(g)\}$ converges to $\{\phi_k(g)\}$ in X_d^*, for each $g \in X_1^*$. Then for $l \to \infty$, $\|M_{m,(\phi_k^{(l)}),(\psi_k)} - M_{m,(\phi_k),(\psi_k)}\| \to 0$.
(iv) Let the assumptions in parts (i), (ii) and (iii) hold. Then for $l \to \infty$, $\|M_{m^{(l)},(\phi_k^{(l)}),(\psi_k^{(l)})} - M_{m,(\phi_k),(\psi_k)}\| \to 0$.

Proof. (i) Since $m^{(l)} \to m$ in X_d, for each $\epsilon > 0$ there exists an $N_\epsilon > 0$ such that for all $l \geq N_\epsilon$,

$$\|m^{(l)} - m\|_{X_d} < \epsilon,$$

hence, by part (i) of Lemma 2.1, we have

$$\begin{aligned} \|M_{m^{(l)},(\phi_k),(\psi_k)} - M_{m,(\phi_k),(\psi_k)}\| &= \|M_{m^{(l)}-m,(\phi_k),(\psi_k)}\| \\ &\leq \|m^{(l)} - m\|_{X_d} KPB' \\ &\leq \epsilon KPB'. \end{aligned}$$

(ii) By assumption, for each $\epsilon > 0$ there exists an $N_\epsilon > 0$ such that for all $l \geq N_\epsilon$,

$$\sup_k |\psi_k(f) - \psi_k^{(l)}(f)| < \epsilon\|f\|, \quad f \in X. \tag{4.1}$$

So,

$$|\psi_k^{(l)}(f)| \leq \epsilon\|f\| + |\psi_k(f)| \leq (\epsilon + P)\|f\|, \quad k \in \mathbb{N}, \ f \in X.$$

Therefore, $\|\psi_k^{(l)}\| \leq P + \epsilon$.
Now by (1.1), (4.1) and (1.3) we have

$$\begin{aligned}
\|M_{m,(\phi_k),(\psi_k^{(l)})}(f) - M_{m,(\phi_k),(\psi_k)}(f)\| &= \|\sum_{k=1}^{\infty} m_k(\psi_k^{(l)}(f) - \psi_k(f))\phi_k\| \\
&= \sup_{g\in X^*,\|g\|\leq 1} |\sum_{k=1}^{\infty} m_k(\psi_k^{(l)}(f) - \psi_k(f))\phi_k(g)| \\
&\leq \sup_k |(\psi_k^{(l)} - \psi_k)(f)| \sup_{g\in X^*,\|g\|\leq 1} \|\{\phi_k(g)\}\|_\alpha \|m\|_{X_d} \\
&\leq \sup_k |(\psi_k^{(l)} - \psi_k)(f)| \|m\|_{X_d} KB' \\
&\leq \epsilon\|f\|\|m\|_{X_d} KB', \quad f \in X.
\end{aligned}$$

So, $\|M_{m,(\phi_k),(\psi_k^{(l)})} - M_{m,(\phi_k),(\psi_k)}\| \leq \epsilon\|m\|_{X_d} KB'$.
(iii) By assumption, for each $\epsilon > 0$ there exists an $N_\epsilon > 0$ such that for all $l \geq N_\epsilon$,

$$\|\{\phi_k^{(l)}(g)\} - \{\phi_k(g)\}\|_{X_d^*} < \epsilon\|g\|, \quad g \in X^*. \tag{4.2}$$

Since $\{\phi_k\} \subseteq X$ is an X_d^*-Bessel sequence with bound B', by (4.2), we have

$$\|\{\phi_k^{(l)}(g)\}\|_{X_d^*} \leq \epsilon\|g\| + \|\{\phi_k(g)\}\|_{X_d^*} \leq (\epsilon + B')\|g\|.$$

So, $\{\phi_k^{(l)}\}$ is an X_d^*-Bessel sequence for X^* with bound $B' + \epsilon$.
Now by (4.2), (1.1) and (1.3), we have

$$\begin{aligned}
\|M_{m,(\phi_k^{(l)}),(\psi_k)}(f) - M_{m,(\phi_k),(\psi_k)}(f)\| &= \|\sum_{k=1}^{\infty} m_k\psi_k(f)(\phi_k^{(l)} - \phi_k)\| \\
&= \sup_{g\in X^*,\|g\|\leq 1} |\sum_{k=1}^{\infty} m_k\psi_k(f)(\phi_k^{(l)} - \phi_k)(g)| \\
&\leq KP\|f\|\|m\|_{X_d} \sup_{g\in X^*,\|g\|\leq 1} \|\{\phi_k^{(l)}(g)\} - \{\phi_k(g)\}\|_{X_d^*} \\
&\leq \epsilon K\|m\|_{X_d} P\|f\|.
\end{aligned}$$

So, $\|M_{m,(\phi_k^{(l)}),(\psi_k)} - M_{m,(\phi_k),(\psi_k)}\| \leq \epsilon K\|m\|_{X_d} P$.
(iv) By the above assertions, for l bigger than the maximum N needed for the convergence conditions we have

$$\begin{aligned}
\|M_{m^{(l)},(\phi_k^{(l)}),(\psi_k^{(l)})} - M_{m,(\phi_k),(\psi_k)}\| &\leq \|M_{m^{(l)},(\phi_k^{(l)}),(\psi_k^{(l)})} - M_{m,(\phi_k^{(l)}),(\psi_k^{(l)})}\| \\
&+ \|M_{m,(\phi_k^{(l)}),(\psi_k^{(l)})} - M_{m,(\phi_k),(\psi_k^{(l)})}\| \\
&+ \|M_{m,(\phi_k),(\psi_k^{(l)})} - M_{m,(\phi_k),(\psi_k)}\| \\
&\leq \epsilon KPB' + \epsilon\|m\|_{X_d} KP + \epsilon\|m\|_{X_d} KB'.
\end{aligned}$$

□

Theorem 4.2. *Let M' be a (l^∞, X_d, X_d^*) Bessel multiplier for Bessel sequences $\{\psi_k\} \subseteq X^*$ and $\{\phi_k\} \subseteq X$, where B and B' are the upper Bessel bounds for $\{\psi_k\}$ and $\{\phi_k\}$, respectively. Then the operator M' depends continuously on $m, \{\psi_k\}$ and $\{\phi_k\}$ in the following sense: Let $\{\psi_k^{(l)}\} \subseteq X^*$ and $\{\phi_k^{(l)}\} \subseteq X$ be sequences indexed by $l \in \mathbb{N}$.*
(i) Let $m^{(l)} \to m$ in l^∞. Then for $l \to \infty$, $\|M'_{m^{(l)},(\phi_k),(\psi_k)} - M'_{m,(\phi_k),(\psi_k)}\| \to 0$.
(ii) Let $\{\psi_k^{(l)}(f)\} \in X_d$, for each $f \in X$ and let $\{\psi_k^{(l)}(f)\}$ converges to $\{\psi_k(f)\}$ in X_d, for each $f \in X_1$. Then for $l \to \infty$, $\|M'_{m,(\phi_k),(\psi_k^{(l)})} - M'_{m,(\phi_k),(\psi_k)}\| \to 0$.
(iii) Let $\{\phi_k^{(l)}(g)\} \in X_d^*$, for each $g \in X^*$ and let $\{\phi_k^{(l)}(g)\}$ converges to $\{\phi_k(g)\}$ in X_d^*, for each $g \in X_1^*$. Then for $l \to \infty$, $\|M'_{m,(\phi_k^{(l)}),(\psi_k)} - M'_{m,(\phi_k),(\psi_k)}\| \to 0$.
(iv) Let the assumptions in parts (i), (ii) and (iii) hold. Then for $l \to \infty$, $\|M'_{m^{(l)},(\phi_k^{(l)}),(\psi_k^{(l)})} - M'_{m,(\phi_k),(\psi_k)}\| \to 0$.

Proof. (i) Since $m^{(l)} \to m$ in l^∞, for each $\epsilon > 0$ there exists an $N_\epsilon > 0$ such that for all $l \geq N_\epsilon$,

$$\|m^{(l)} - m\|_\infty < \epsilon,$$

hence, by part (ii) of Lemma 2.1, we have

$$\begin{aligned}
\|M'_{m^{(l)},(\phi_k),(\psi_k)} - M'_{m,(\phi_k),(\psi_k)}\| &= \|M'_{m^{(l)}-m,(\phi_k),(\psi_k)}\| \\
&\leq \|m^{(l)} - m\|_\infty KBB' \\
&\leq \epsilon KBB'.
\end{aligned}$$

(ii) By the assumption, for each $\epsilon > 0$ there exists an $N_\epsilon > 0$ such that for all $l \geq N_\epsilon$,

$$\|\{\psi_k^{(l)}(f)\} - \{\psi_k(f)\}\|_{X_d} < \epsilon\|f\|, \quad f \in X. \tag{4.3}$$

Since $\{\psi_k\} \subseteq X^*$ is an X_d-Bessel sequence with bound B, by (4.3), we have

$$\|\{\psi_k^{(l)}(f)\}\|_{X_d} \leq \epsilon\|f\| + \|\{\psi_k(f)\}\|_{X_d} \leq (\epsilon + B)\|f\|.$$

Now by (1.1), (4.3) and (1.3), we have

$$\begin{aligned}
\|M'_{m,(\phi_k),(\psi_k^{(l)})}(f) - M'_{m,(\phi_k),(\psi_k)}(f)\| &= \sup_{g \in X^*, \|g\| \leq 1} |\sum_{k=1}^{\infty} m_k(\psi_k^{(l)}(f) - \psi_k(f))\phi_k(g)| \\
&\leq \sup_{g \in X^*, \|g\| \leq 1} \|\{m_k(\psi_k^{(l)} - \psi_k)(f)\}\|_{X_d} \|\{\phi_k(g)\}\|_{X_d^\alpha} \\
&\leq \epsilon KB'\|m\|_\infty\|f\|.
\end{aligned}$$

So, $\|M'_{m,(\phi_k),(\psi_k^{(l)})} - M'_{m,(\phi_k),(\psi_k)}\| \leq \epsilon\|m\|_\infty KB'$.
By a similar argument of part (ii) and Theorem 4.1, we can deduce the rest of the proof. □

5. λ-nuclear operators in Banach spaces

The theory of trace-class operators in Hilbert spaces was created in 1936 by Murray and von Neumann. Ruston [19, 20] extended this concept to operators acting in Banach spaces. Trace class operators on Banach spaces are called nuclear operators. Dubinsky and Ramanujan [9] generalized this idea to λ-nuclear operators.

Let E and F be normed linear spaces and λ be a BK-space, whose elements are sequences of complex numbers. Then $\lambda(E)$ denotes the (vector sequence) space of all vectors $x = (x_k)$, $x_k \in E$ such that $\{\langle x_k, a\rangle\} \in \lambda$ for each $a \in E^*$. We define

$$\epsilon_\lambda(x) = \sup_{\|a\|\leq 1} p(\{\langle x_k, a\rangle\}),$$

where p is a norm on λ.

Definition 5.1. [18] Let T be a linear map on the normed space E into another space, F. We define T to be a λ-nuclear map if T admits the representation

$$Tx = \sum_{k=1}^{\infty} \alpha_k \langle x, u_k\rangle y_k, \quad x \in E,$$

where $\|u_k\|$ is bounded above for each $k \in \mathbb{N}$, $\alpha = (\alpha_k) \in \lambda$, $y = (y_k) \in \lambda^*(F)$ and there exists $L > 0$ such that $\epsilon_{\lambda^*}(y) \leq L$. In this case

$$N_\lambda(T) = \inf_\alpha p(\alpha).$$

Let $N_\lambda(E, F)$ denotes the set of all λ-nuclear maps on E into F.

Theorem 5.2. [18] *Let E, F and G be normed linear spaces. Then we have the following assertions:*
(i) If $T \in N_\lambda(E, F)$ and $S \in L(F, G)$, then $S \circ T \in N_\lambda(E, G)$ and $N_\lambda(S \circ T) \leq \|S\| N_\lambda(T)$.
(ii) If $T \in L(E, F)$ and $S \in N_\lambda(F, G)$, then $S \circ T \in N_\lambda(E, G)$ and $N_\lambda(S \circ T) \leq N_\lambda(S)\|T\|$.

Theorem 5.3. [18] *Let $\delta = \{\delta_k\}$ be a fixed member of λ. Then the map $D : l^\infty \to \lambda$ defined by $D(u) = (u_k\delta_k)$ is a λ-nuclear map and $N_\lambda(D) = p(\delta)$.*

Theorem 5.4. [18] *Let $T \in L(E, F)$. The map T is λ-nuclear if and only if it can be factorized as follows:*

$$T = Q \circ D \circ P,$$

where P and Q are continuous linear maps from E into l^∞ and from λ into F, respectively, and D is as defined in Theorem 5.3.

Theorem 5.5. *Let M be an (X_d, X_d^*)-Bessel multiplier for sequences $\{\psi_k\} \subseteq X^*$ and $\{\phi_k\} \subseteq X$, where B' is the upper Bessel bound for $\{\phi_k\}$ and $\|\psi_k\| \leq P$, for each $k \in \mathbb{N}$. Then M is a λ-nuclear operator with*

$$N_\lambda(M) \leq PB'\|m\|_{X_d}.$$

Proof. By Remark 2.6, $M = T_{\phi_k} D_m U$. So by Theorems 5.3, 5.4 and 5.2, we deduce that M is a λ-nuclear operator and $N_\lambda(M) \leq PB'\|m\|_{X_d}$. □

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[1] Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.
E-mail address: mhfaroughi@yahoo.com
E-mail address: osgooei@yahoo.com; osgooei@tabrizu.ac.ir

[2] Department of Mathematics, University of Maragheh, Maragheh, Iran.
E-mail address: rahimi@maragheh.ac.ir; asgharrahimi@yahoo.com