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SPECIAL OPERATOR CLASSES AND THEIR PROPERTIES

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ABSTRACT. We introduce some special operator classes and study in terms of Berezin symbols their properties. In particular, we give some characterizations of compact operators and Schatten-von Neumann class operators in terms of Berezin symbols. We also consider some classes of compact operators on a Hilbert space H, which are generalizations of the well known Schatten-von Neumann classes of compact operators. Namely, for any number p, $0 < p < \infty$, and the sequence $w := (w_n)_{n\geq 0}$ of complex numbers w_n, $n \geq 0$, we define the following classes of compact operators on H:

$$S_p^w(H) = \left\{ K \in S_\infty(H) : \sum_{n=0}^{\infty} (s_n(K))^p w_n^p \text{ is convergent series} \right\},$$

where $s_n(K)$ denotes the nth singular number of the operator K. The characterizations of these classes are given in terms of Berezin symbols.

1. INTRODUCTION AND BACKGROUND

In this paper we investigate in terms of Berezin symbols some special operator classes. Namely, we consider the following operators, which are called "the weighted model operators":

$$\mathcal{K}_{\varphi,\theta,\Omega} := [T_{\overline{\varphi}\Omega}, T_\theta]\, \varphi\left(M_\theta\right),$$
$$\mathcal{L}_{\varphi,\theta,\Omega} := [T_{\overline{\theta}\Omega}, T_\varphi]\, \varphi\left(M_\theta\right),$$

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where $\Omega \in (\Sigma) \cup \{1\}$, $\varphi \in H^{\infty}(\mathbb{D})$ and $\theta \in (\Sigma)$; here (Σ) denotes the set of all inner functions. When $\Omega = 1$, we shall use the symbols $\mathcal{K}_{\varphi,\theta}$ and $\mathcal{L}_{\varphi,\theta}$ instead of $\mathcal{K}_{\varphi,\theta,1}$ and $\mathcal{L}_{\varphi,\theta,1}$, respectively. Let us denote $\mathcal{K}_{\varphi,\theta,(\Sigma)} := \{\mathcal{K}_{\varphi,\theta,\Omega} : \Omega \in (\Sigma) \cup \{1\}\}$. Recall that the function of model operator is defined as usual by the formula

$$\varphi(M_\theta) f = P_\theta \varphi f$$

for every $f \in K_\theta := H^2 \Theta \theta H^2$, where θ is an inner function.

Here we also consider the classes S_p^w, $0 < p < \infty$, of compact operators and characterize these classes in terms of the boundary behavior of Berezin symbols of the weighted shift operators on the Hardy space $H^2(\mathbb{D})$ associated with s-numbers of the compact operators in S_p^w.

Definition 1.1. Given $0 < p < \infty$ and a sequence $w := \{w_n\}_{n\geq 0}$ of the complex numbers w_n, we define the class $S_p^w := S_p^w(H)$ to be space of all compact operators K on H with the singular numbers $s_n(K)$ for which the series

$$\sum_{n=0}^{\infty} (s_n(K))^p w_n^p$$

is convergent.

It can be easily shown that the classes S_p^w, $0 < p < \infty$, are vector spaces. Also, it is obvious that for $w_n = 1$, $n \geq 0$, our space S_p^w coincides with the usual Schatten-von Neumann space S_p. Generally, if $\{w_n\}_{n\geq 0}$ is a sequence such that

$$C_1 \leq |w_n| \leq C_2 \ (n \geq 0)$$

for some $C_1, C_2 > 0$, then it is easy to see that $S_p^w = S_p$.

Moreover, in this paper we give a compactness criterion for operators on a nonstandard functional Hilbert space contained in a standard functional Hilbert space (see Theorem 2.1).

Before giving our results, let us give the necessary notations and definitions.

By $\mathcal{B}(H)$ we denote the algebra of all bounded linear operators on the infinite dimensional complex Hilbert space H.

Recall that a functional Hilbert space is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that:

(a) the evaluation functional $f \to f(\lambda)$ is continuous for each $\lambda \in \Omega$;

(b) for any $\lambda \in \Omega$ there exists $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

Then by the classical Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $k_{\mathcal{H},\lambda} \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_{\mathcal{H},\lambda} \rangle$ for all $f \in \mathcal{H}$. The function $k_{\mathcal{H},\lambda}$ is called the reproducing kernel of the space $\mathcal{H}$. Let $\widehat{k}_{\mathcal{H},\lambda} = \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}$ denotes the normalized reproducing kernel of the space $\mathcal{H}$ (note that by (b), we surely have $k_\lambda \neq 0$). For a bounded linear operator A on the functional Hilbert space $\mathcal{H}$, its Berezin symbol $\widetilde{A}$ is defined by the formula

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle_{\mathcal{H}} \ (\lambda \in \Omega).$$

It is clear that $\left|\widetilde{A}(\lambda)\right| \leq \|A\|$ for all $\lambda \in \Omega$, that is $\widetilde{A}$ is a bounded function. More informations about reproducing kernels and Berezin symbols, can be found in Aronzajn [1], Berezin [2, 3] and Zhu [11].

A prototypical functional Hilbert space is, for example, the classical Hardy space $H^2 = H^2(\mathbb{D})$, which is the space of all functions analytic on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ having Taylor coefficients that are square summable. It is well known that $k_{\mathcal{H}^2,\lambda}(z) = (1 - \overline{\lambda}z)^{-1}$, $\lambda, z \in \mathbb{D}$.

Throughout in the paper, for any bounded sequence $\Lambda = \{\lambda_n\}$ of complex numbers the symbol T_Λ will denote the weighted shift operator in the Hardy space H^2 with respect to the standard orthonormal basis $\{z^n\}_{n\geq 0}$ of H^2, i.e.,

$$T_\Lambda z^n = \lambda_n z^{n+1},\ n = 0, 1, 2, ...$$

Recall that the series $\sum_{n=0}^{\infty} a_n$ is Abel convergent if $\sum_{n=0}^{\infty} a_n t^n$ is convergent for each $t \in (0, 1)$ and $\lim\limits_{t\to 1} \sum_{n=0}^{\infty} a_n t^n$ exists and is finite. Finally, note that for any $\varphi \in L^\infty(\mathbb{T})$ the corresponding Toeplitz operator on $H^2 = H^2(\mathbb{D})$ is defined by $T_\varphi f := P_+\varphi f$, where $P_+ : L^2(\mathbb{T}) \to H^2$ is the Riesz projection operator, $\mathbb{T} = \partial\mathbb{D}$. The Hankel operator is defined by $H_\varphi f = (I - P_+)\varphi f$, $f \in H^2$, where $P_- := I - P_+$ is the orthogonal projector of $L^2(\mathbb{T})$ into $H^2_- := \left\{ f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0,\ n > 0 \right\}$.

2. Characterization of Some Operators

In the present section we characterize some Schatten-von Neumann operator ideals in terms of Berezin symbols.

2.1. **Compactness criterion.** Following Nordgren and Rosenthal [9], we say that a functional Hilbert space $\mathcal{H} = \mathcal{H}(Q)$ is standard if the underlying set Q is a subset of a topological space and the boundary ∂Q is non-empty and has the property that $\left\{\widehat{k}_{\mathcal{H},\lambda_n}\right\}$ converges weakly to 0 as $\lambda \to \xi$, for any point $\xi \in \partial Q$. The common functional Hilbert spaces of analytic functions, including $H^2(\mathbb{D})$ (Hardy space) and $L^2_a(\mathbb{D})$ (Bergman space), $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is a unit disc, are standard in this sense.

For any reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ on Q (not necessarily standard), denote $\partial_{\mathcal{H}}Q$ the subset of the boundary of Q defined by (see [4])

$$\partial_{\mathcal{H}}Q := \left\{ \xi \in \partial Q : \widehat{k}_{\mathcal{H},\lambda_n} \to 0 \text{ (weakly) whenever } \lambda \to \xi \right\}.$$

It is clear from the definitions that $\mathcal{H}$ is standard if and only if $\partial_{\mathcal{H}}Q = \partial Q$. In the case where $\partial_{\mathcal{H}}Q \neq \emptyset$, one can obtain an analogue of the main result of the paper by Nordgren and Rosenthal [9, Corollary 2.8], which characterizes compact operators on the standard RKHS in terms of boundary behavior of Berezin symbols of all unitary orbits of operator.

Namely, as is shown in [4] (which completely solves Nordgren and Rosenthal's questions in [9]), the hypothesis of standardness of the Hilbert space $\mathcal{H}(Q)$ in the Corollary 2.8 of the paper [9] can be highly weakened.

Theorem A. (see [4, Theorem 2.2]). Let $\mathcal{H}$ be a RKHS on Q such that $\partial_{\mathcal{H}}Q \neq \emptyset$, and let $T \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent:

(i) T is compact;

(ii) for every point $\xi \in \partial_{\mathcal{H}} Q$ and every unitary operator U on $\mathcal{H}$, we have

$$\lim_{\lambda \to \xi} \widetilde{U^{-1}TU}(\lambda) = 0;$$

(iii) there exists a sequence $(\lambda_n)_{n\geq 1}$ of points in Q, converging to a point $\xi \in \partial_{\mathcal{H}} Q$, such that for every unitary operator U on $\mathcal{H}$, we have

$$\lim_{n \to +\infty} \widetilde{U^{-1}TU}(\lambda_n) = 0.$$

In the following theorem compactness criterion for A is stated in terms of Berezin symbols of unitary orbits $U^{-1}AU$ restricted to the subspaces $U^{-1}\mathcal{H}$.

Theorem 2.1. *Let $\mathcal{K} = \mathcal{K}(Q)$ be a RKHS on some set Q such that $\partial_{\mathcal{K}} Q \neq \emptyset$, $A : \mathcal{K} \to \mathcal{K}$ be a linear bounded operator and $\mathcal{H} \subset \mathcal{K}$ be a closed A-invariant subspace, i.e., $A\mathcal{H} \subset \mathcal{H}$. Then the operator $A|\mathcal{H}$ is compact (i.e., $A \in S_\infty(\mathcal{H})$) if and only if for every $\xi \in \partial_{\mathcal{K}} Q$ and every unitary operator $U \in \mathcal{B}(\mathcal{K})$ we have*

$$\lim_{\lambda \to \xi} \widetilde{P}_{U^{-1}\mathcal{H}}(\lambda)\, \widetilde{U^{-1}AU}^{U^{-1}\mathcal{H}}(\lambda) = 0.$$

Proof. Put $B = AP_{\mathcal{H}}$. It is obvious for arbitrary unitary operator $U \in \mathcal{B}(\mathcal{K})$ that

$$U^{-1}BU = U^{-1}AP_{\mathcal{H}}U = U^{-1}AUU^{-1}P_{\mathcal{H}}U = U^{-1}AUP_{U^{-1}\mathcal{H}}.$$

Since $P_{U^{-1}\mathcal{H}} k_{\mathcal{K},\lambda} = k_{U^{-1}\mathcal{H},\lambda}$ for every $\lambda \in Q$, we have:

$$\begin{aligned}
\widetilde{U^{-1}BU}(\lambda) &= \left\langle U^{-1}BU\widehat{k}_{\mathcal{K},\lambda}, \widehat{k}_{\mathcal{K},\lambda} \right\rangle = \left\langle U^{-1}AUP_{U^{-1}\mathcal{H}}\widehat{k}_{\mathcal{K},\lambda}, \widehat{k}_{\mathcal{K},\lambda} \right\rangle \\
&= \frac{1}{\|k_{\mathcal{K},\lambda}\|^2} \left\langle U^{-1}AUP_{U^{-1}\mathcal{H}}k_{\mathcal{K},\lambda}, k_{\mathcal{K},\lambda} \right\rangle \\
&= \frac{1}{\|k_{\mathcal{K},\lambda}\|^2} \left\langle U^{-1}AUk_{U^{-1}\mathcal{H},\lambda}, P_{U^{-1}\mathcal{H}}k_{\mathcal{K},\lambda} + (I - P_{U^{-1}\mathcal{H}})\, k_{\mathcal{K},\lambda} \right\rangle \\
&= \frac{1}{\|k_{\mathcal{K},\lambda}\|^2} \left[\left\langle U^{-1}AUk_{U^{-1}\mathcal{H},\lambda}, k_{U^{-1}\mathcal{H},\lambda} \right\rangle + \right. \\
&\left. + \left\langle U^{-1}AUk_{U^{-1}\mathcal{H},\lambda}, (I - P_{U^{-1}\mathcal{H}})\, k_{\mathcal{K},\lambda} \right\rangle \right] \\
&= \frac{1}{\|k_{\mathcal{K},\lambda}\|^2} \left\langle U^{-1}AUk_{U^{-1}\mathcal{H},\lambda}, k_{U^{-1}\mathcal{H},\lambda} \right\rangle \\
&= \frac{\|k_{U^{-1}\mathcal{H},\lambda}\|^2}{\|k_{\mathcal{K},\lambda}\|^2} \left\langle U^{-1}AU\widehat{k}_{U^{-1}\mathcal{H},\lambda}, \widehat{k}_{U^{-1}\mathcal{H},\lambda} \right\rangle \\
&= \frac{\|k_{U^{-1}\mathcal{H},\lambda}\|^2}{\|k_{\mathcal{K},\lambda}\|^2} \widetilde{U^{-1}AU}^{U^{-1}\mathcal{H}}(\lambda).
\end{aligned}$$

Thus

$$\widetilde{U^{-1}BU}(\lambda) = \frac{\|k_{U^{-1}\mathcal{H},\lambda}\|^2}{\|k_{\mathcal{K},\lambda}\|^2} \widetilde{U^{-1}AU}^{U^{-1}\mathcal{H}}(\lambda) \quad (\lambda \in Q).$$

On the other hand,

$$\begin{aligned}\|k_{U^{-1}\mathcal{H},\lambda}\|^2 &= \|P_{U^{-1}\mathcal{H}}k_{\mathcal{K},\lambda}\|^2 = \langle P_{U^{-1}\mathcal{H}}k_{\mathcal{K},\lambda}, k_{\mathcal{K},\lambda}\rangle \\ &= \|k_{\mathcal{K},\lambda}\|^2 \left\langle P_{U^{-1}\mathcal{H}}\widehat{k}_{\mathcal{K},\lambda}, \widehat{k}_{\mathcal{K},\lambda}\right\rangle \\ &= \|k_{\mathcal{K},\lambda}\|^2 \widetilde{P}_{U^{-1}\mathcal{H}}(\lambda).\end{aligned}$$

Consequently,

$$\frac{\|k_{U^{-1}\mathcal{H},\lambda}\|^2}{\|k_{\mathcal{K},\lambda}\|^2} = \widetilde{P}_{U^{-1}\mathcal{H}}(\lambda) \quad (\lambda \in Q) \tag{2.1}$$

for all unitary operator $U \in \mathcal{B}(\mathcal{K})$. Therefore

$$\widetilde{U^{-1}BU}(\lambda) = \widetilde{P}_{U^{-1}\mathcal{H}}(\lambda)\,\widetilde{U^{-1}AU}^{U^{-1}\mathcal{H}}(\lambda) \quad (\lambda \in Q). \tag{2.2}$$

for all unitary operator $U \in \mathcal{B}(\mathcal{K})$. It is obvious that $B\mathcal{H} \subset \mathcal{H}$ and $B|\mathcal{H} = A|\mathcal{H}$. Therefore $B \in S_\infty(\mathcal{K})$ if and only if $A \in S_\infty(\mathcal{H})$. Now using this fact, formula (2.2) and Theorem A, we conclude that A is compact in $\mathcal{H}$ if and only if

$$\lim_{\lambda\to\xi\in\partial_{\mathcal{K}}Q}\left(\widetilde{P}_{U^{-1}\mathcal{H}}(\lambda)\,\widetilde{U^{-1}AU}^{U^{-1}\mathcal{H}}(\lambda)\right) = 0$$

for every unitary operator $U \in \mathcal{B}(\mathcal{K})$, which completes the proof. □

Corollary 2.2. *Let $\varphi \in H^\infty$ be a nonconstant function. Then $\varphi(M_\theta) \in S_\infty(K_\theta)$ if and only if*

$$\lim_{\lambda\to\mathbb{T}}\left(\widetilde{P}_{U^{-1}K_\theta}(\lambda)\,\widetilde{U^{-1}T_{\overline{\varphi}}U}^{U^{-1}K_\theta}(\lambda)\right) = 0$$

for every unitary operator $U \in \mathcal{B}(H^2)$.

Proof. Indeed, putting $\mathcal{K} = H^2$, $\mathcal{H} = K_\theta$, $A = T_{\overline{\varphi}}$ in Theorem 2.1, and considering that $\partial_{\mathcal{H}^2}\mathbb{D} = \mathbb{T}$, we conclude that $T_{\overline{\varphi}}|K_\theta$ is compact operator if and only if for every unitary operator $U \in \mathcal{B}(H^2)$

$$\lim_{\lambda\to\mathbb{T}}\left(\widetilde{P}_{U^{-1}K_\theta}(\lambda)\,\widetilde{U^{-1}T_{\overline{\varphi}}U}^{U^{-1}K_\theta}(\lambda)\right) = 0.$$

It now remains only to observe that $\varphi(M_\theta) = (T_{\overline{\varphi}}|K_\theta)^* \in S_\infty(K_\theta) \Leftrightarrow T_{\overline{\varphi}}|K_\theta \in S_\infty(K_\theta)$, consequently,

$$\varphi(M_\theta) \in S_\infty(K_\theta) \Leftrightarrow \lim_{\lambda\to\mathbb{T}}\left(\widetilde{P}_{U^{-1}K_\theta}(\lambda)\,\widetilde{U^{-1}T_{\overline{\varphi}}U}^{U^{-1}K_\theta}(\lambda)\right).$$

This proves the corollary. □

2.2. **S_p-criteria.** Before stating our next result, we introduce the following definition.

Remark 2.3. Formula (2.1), in particular, implies that if $\mathcal{H}_1 = \mathcal{H}_1(Q)$ is a non-standard FHS and $\mathcal{H}_2 = \mathcal{H}_2(Q)$ is a standard FHS such that $\mathcal{H}_1 \subset \mathcal{H}_2$, then

$$\lim_{n\to\infty}\widetilde{P}_{\mathcal{H}_1}(\lambda_n) = 0 \tag{2.3}$$

for some sequence $\{\lambda_n\} \in Q$ tending to a point in ∂Q. In fact, since for every $\mathcal{H}_1$ and $\lambda \in Q$

$$\left\langle f, \widehat{k}_{\mathcal{H}_1,\lambda} \right\rangle = \frac{\|k_{\mathcal{H}_2,\lambda}\|}{\|k_{\mathcal{H}_1,\lambda}\|} \left\langle f, \widehat{k}_{\mathcal{H}_2,\lambda} \right\rangle,$$

we have by formula (2.1) that

$$\left\langle f, \widehat{k}_{\mathcal{H}_1,\lambda} \right\rangle = \left(\widetilde{P}_{\mathcal{H}_1}(\lambda) \right)^{-1/2} \left\langle f, \widehat{k}_{\mathcal{H}_2,\lambda} \right\rangle. \tag{2.4}$$

Since $\mathcal{H}_1$ is nonstandard, there exists $f_0 \in \mathcal{H}_1$ and a sequence $\{\lambda_n\} \in Q$ tending to a boundary point such that

$$\lim_{n\to\infty} \left\langle f_0, \widehat{k}_{\mathcal{H}_1,\lambda_n} \right\rangle \neq 0,$$

and hence, using the condition that $\mathcal{H}_2$ is standard, we assert from (2.4) that $\lim_{n\to\infty} \widetilde{P}_{\mathcal{H}_1}(\lambda_n) = 0$. Thus, (2.3) is a necessary condition for the inclusion $\mathcal{H}_1 \subset \mathcal{H}_2$.

Definition 2.4. Let $\mathcal{H} = \mathcal{H}(Q)$ be a (separable) RKHS on some set Q. We say that $\mathcal{H}$ posses the property (P), if for some orthonormal sequence $\{e_n(z)\}_{n\geq 1}$ of the space $\mathcal{H}$ with infinite codimension (that is $\dim(\mathcal{H}\Theta span(e_n : n \geq 1)) = +\infty$) and for some scalar $\lambda \in Q$ the multiplication operators $\mathcal{M}_{\frac{e_n}{k_{\mathcal{H},\lambda}}}$, $n \geq 1$, are bounded in $\mathcal{H}$.

Since $\{z^n\}_{n\geq 0}$ and $\left\{\sqrt{n+1}z^n\right\}_{n\geq 0}$ are orthonormal bases in H^2 and L_a^2, respectively, and $k_{H^2,\lambda}(z) = \frac{1}{1-\bar{\lambda}z}$ and $k_{L_a^2,\lambda}(z) = \frac{1}{\left(1-\bar{\lambda}z\right)^2}$ are the reproducing kernels of H^2 and L_a^2, respectively, it is clear that the Hardy and Bergman spaces have the property (P).

Our next result is a slight generalization of a result in [6, Theorem 4].

Theorem 2.5. *Let $\mathcal{H} = \mathcal{H}(Q)$ be a FHS with the property (P) with respect to the orthonormal sequence $\{e_n(z)\}_{n\geq 1}$ and the point $\lambda \in Q$. Let $A \in S_\infty(\mathcal{H})$. Then $A \in S_p(\mathcal{H})$ $(p \geq 1)$ if and only if*

$$\sum_{n=1}^{\infty} \left| \left[\mathcal{M}^*_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}} \left(U^{-1}AU\right) \mathcal{M}_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}} \right]^{\sim} (\lambda) \right|^p < +\infty$$

for every unitary operator $U : \mathcal{H} \to \mathcal{H}$.

Proof. It is well-known that (see Zhu [11, Theorem 1.27]) A lies in $S_p(\mathcal{H})$ $(p \geq 1)$ if and only if

$$\sum_{n=1}^{\infty} |\langle Au_n, u_n\rangle|^p < +\infty$$

for all orthonormal sequence $\{u_n\}_{n\geq 1}$. It is not difficult to show that the latter is equivalent to the assertion that

$$\sum_{n\geq 1} |\langle Av_n, v_n\rangle|^p < +\infty$$

for all orthonormal sequences $\{v_n\}_{n\geq 1}$ in $\mathcal{H}$ with infinite codimension. Since $\mathcal{H}$ possesses property (P) with respect to the orthonormal sequence $\{e_n(z)\}_{n\geq 1}$, we have that

$$\dim\left(\mathcal{H}\Theta span\left(e_n(z) : n \geq 1\right)\right) = +\infty.$$

Then there exists a unitary operator U on $\mathcal{H}$ such that $Ue_n = v_n$, $n \geq 1$. Hence we obtain:

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle Av_n, v_n\rangle|^p &= \sum_{n=1}^{\infty} |\langle AUe_n, Ue_n\rangle|^p = \sum_{n=1}^{\infty} \left|\left\langle U^{-1}AUe_n, e_n\right\rangle\right|^p \\
&= \sum_{n=1}^{\infty} \left|\left\langle U^{-1}AU\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}\widehat{k}_{\mathcal{H},\lambda}, \frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|^p \\
&= \sum_{n=1}^{\infty} \left|\left\langle U^{-1}AU\mathcal{M}_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}}\widehat{k}_{\mathcal{H},\lambda}, \mathcal{M}_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}}\widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|^p \\
&= \sum_{n=1}^{\infty} \left|\left\langle \mathcal{M}^*_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}}\left(U^{-1}AU\right)\mathcal{M}_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}}\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle\right|^p \\
&= \sum_{n=1}^{\infty} \left|\left[\mathcal{M}^*_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}}\left(U^{-1}AU\right)\mathcal{M}_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}}\right]^{\sim}(\lambda)\right|^p.
\end{aligned}$$

It now follows from the above assertion that

$$A \in S_p(\mathcal{H}) \Leftrightarrow \sum_{n=1}^{\infty} |\langle Av_n, v_n\rangle|^p < +\infty \Leftrightarrow$$

$$\Leftrightarrow \sum_{n=1}^{\infty} \left|\left[\mathcal{M}^*_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}}\left(U^{-1}AU\right)\mathcal{M}_{\frac{e_n}{\widehat{k}_{\mathcal{H},\lambda}}}\right]^{\sim}(\lambda)\right|^p < +\infty,$$

which proves the theorem, because $\{v_n\}$ is arbitrary, and therefore U is also arbitrary unitary operator. □

3. Weighted model operators $\mathcal{K}_{\varphi,\theta,\Omega}$ and $\mathcal{L}_{\varphi,\theta,\Omega}$

In this section we give some results concerning to the weighted model operators $\mathcal{K}_{\varphi,\theta,\Omega}$ and $\mathcal{L}_{\varphi,\theta,\Omega}$. Let us start with some simple remarks concerning to the operators $\mathcal{K}_{\varphi,\theta,\Omega}$, where $\varphi \in H^\infty$, $\theta \in (\Sigma)$ and $\Omega \in (\Sigma) \cup \{1\}$.

Proposition 3.1. *(a) Each operator $\mathcal{K}_{\varphi,\theta,\Omega}$ is a projection of the operator $T_{\overline{\varphi}}N_{\theta,\Omega}T_\varphi$ in H^2 to the subspace K_θ, i.e.,*

$$\mathcal{K}_{\varphi,\theta,\Omega} = P_\theta\left(T_{\overline{\varphi}}N_{\theta,\Omega}T_\varphi\right)|K_\theta, \tag{3.1}$$

where $N_{\theta,\Omega} := T_{\theta\Omega}P_\theta$ is a nilpotent operator, $N^2_{\theta,\Omega} = 0$.

(b)

$$dist\left(\left[T_{\overline{\theta}}, T_\varphi\right], \Gamma_{(\Sigma)}\right) dist\left(\varphi\overline{\theta}, H^\infty\right) \geq dist\left(\varphi\left(M_\theta\right), K_{\varphi,\theta,(\Sigma)}\right), \tag{3.2}$$

where $\Gamma_{(\Sigma)} := \{T_w : w \in (\Sigma) \cup \{1\}\}$.

(c) If $\varphi \in (\Sigma)$, then the numerical range of the operator $\mathcal{K}_{\varphi,\theta}$ lies in the closed disc $\overline{\mathbb{D}}_{1/2}$.

Proof. (a) Indeed, for each $f \in K_\theta$ we have that

$$\begin{aligned} P_\theta \left(T_{\overline{\varphi}} N_{\theta,\Omega} T_\varphi\right) f &= P_\theta T_{\overline{\varphi}} T_{\theta\Omega} P_\theta \varphi f \\ &= \left(I - T_\theta T_{\overline{\theta}}\right) T_{\overline{\varphi}} T_{\theta\Omega} \varphi \left(M_\theta\right) f \\ &= \left(T_{\overline{\varphi}\Omega} T_\theta - T_\theta T_{\overline{\varphi}\Omega}\right) \varphi \left(M_\theta\right) f \\ &= \left[T_{\overline{\varphi}\Omega}, T_\theta\right] \varphi \left(M_\theta\right) f = \mathcal{K}_{\varphi,\theta,\Omega} f, \end{aligned}$$

which gives (3.1); obviously, $N_{\theta,\Omega}^2 = 0$.

(b) Since for every $\Omega \in (\Sigma)$ the operator T_Ω is an isometry, we have:

$$\begin{aligned} \left\|\varphi\left(M_\theta\right) - \mathcal{K}_{\varphi,\theta,\Omega}\right\| &= \left\|\varphi\left(M_\theta\right) - \left[T_{\overline{\varphi}\Omega}, T_\theta\right] \varphi\left(M_\theta\right)\right\| \\ &= \left\|\left(I - \left[T_{\overline{\varphi}\Omega}, T_\theta\right]\right) \varphi\left(M_\theta\right)\right\| \\ &= \left\|\left(I - \left(T_{\overline{\varphi}} T_\theta - T_\theta T_{\overline{\varphi}}\right) T_\Omega\right) \varphi\left(M_\theta\right)\right\| \\ &= \left\|\left(T_{\overline{\Omega}} T_\Omega - \left[T_{\overline{\varphi}}, T_\theta\right] T_\Omega\right) \varphi\left(M_\theta\right)\right\| \\ &= \left\|\left(T_{\overline{\Omega}} - \left[T_{\overline{\varphi}}, T_\theta\right]\right) T_\Omega \varphi\left(M_\theta\right)\right\| \\ &\le \left\|T_{\overline{\Omega}} - \left[T_{\overline{\varphi}}, T_\theta\right]\right\| \left\|\varphi\left(M_\theta\right)\right\| \\ &= \left\|\left(T_{\overline{\Omega}} - \left[T_{\overline{\varphi}}, T_\theta\right]\right)\right\| \left\|\varphi\left(M_\theta\right)\right\|. \end{aligned}$$

It follows from this that

$$\inf_{\Omega \in (\Sigma) \cup \{1\}} \left\|\varphi\left(M_\theta\right) - \mathcal{K}_{\varphi,\theta,\Omega}\right\| \le \inf_{\Omega \in (\Sigma) \cup \{1\}} \left\|\left(T_{\overline{\Omega}} - \left[T_{\overline{\varphi}}, T_\theta\right]\right)\right\| \left\|\varphi\left(M_\theta\right)\right\|,$$

or, by considering that $\left\|T_{\overline{\Omega}} - \left[T_{\overline{\varphi}}, T_\theta\right]\right\| = \left\|T_\Omega - \left[T_{\overline{\theta}}, T_\varphi\right]\right\|$, we have

$$dist\left(\varphi\left(M_\theta\right), K_{\varphi,\theta,(\Sigma)}\right) \le dist\left(\left[T_{\overline{\theta}}, T_\varphi\right], \Gamma_{(\Sigma)}\right) \left\|\varphi\left(M_\theta\right)\right\|.$$

Now the well-known formula

$$\left\|\varphi\left(M_\theta\right)\right\| = dist\left(\varphi\overline{\theta}, H^\infty\right)$$

implies the inequality (3.2).

(c) Using formula (3.1), we have

$$\begin{aligned} \left\langle \mathcal{K}_{\varphi,\theta} f, f\right\rangle &= \left\langle P_\theta \left(T_{\overline{\varphi}} N_\theta T_\varphi\right) f, f\right\rangle = \left\langle T_{\overline{\varphi}} N_\theta T_\varphi f, f\right\rangle \\ &= \left\langle N_\theta \varphi f, \varphi f\right\rangle \end{aligned}$$

for every $f \in K_\theta$, $\|f\|_2 = 1$; here $N_\theta := T_\theta P_\theta = T_\theta \left(I - T_\theta T_{\overline{\theta}}\right)$. Since φ is an inner function, $\varphi f \in H^2$ and $\|\varphi f\|_2 = \|f\|_2 = 1$, we conclude that

$$\left\langle N_\theta \varphi f, \varphi f\right\rangle \in W\left(N_\theta\right) \text{ (numerical range of } N_\theta).$$

Now using the known result that $W\left(N_\theta\right) = \left\{z \in \mathbb{C} : |z| \le \frac{1}{2}\right\}$ (because $N_\theta^2 = 0$, see [7]), we complete the proof. □

Proposition 3.2. *We have*

$$\left|\widetilde{\mathcal{K}}_{\varphi,\theta,\Omega}(\lambda)\right| = o\left(\frac{1}{1 - |\theta(\lambda)|^2}\right) \text{ as } |\lambda| \to 1^-$$

for every $\Omega \in (\Sigma) \cup \{1\}$.

Proof. By using (3.1) and the following well-known formulas

$$k_\lambda := k_{H^2,\lambda} = \frac{1}{1-\overline{\lambda}z}$$

$$k_{\theta,\lambda} := k_{K_\theta,\lambda} = \frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z}$$

$$\widehat{k_{\theta,\lambda}} = \sqrt{\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}}\frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z}$$

$$T_{fg} - T_f T_g = H^*_{\overline{f}} H_g$$

where $f, g \in L^\infty$, we have:

$$\begin{aligned}
\widetilde{\mathcal{K}}_{\varphi,\theta,\Omega}(\lambda) &= \left\langle P_\theta \left(T_{\overline{\varphi}} N_{\theta,\Omega} T_\varphi\right)\widehat{k}_{\theta,\lambda}, \widehat{k}_{\theta,\lambda}\right\rangle \\
&= \frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\left\langle T_{\theta\Omega} P_\theta \varphi \frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z}, \varphi\frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z}\right\rangle \\
&= \frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\left\langle T_{\theta\Omega} P_\theta \frac{\varphi}{1-\overline{\lambda}z}, \varphi\frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z}\right\rangle \\
&= \frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\left(\left\langle T_{\theta\Omega}\left(I - T_\theta T_{\overline{\theta}}\right)\frac{\varphi}{1-\overline{\lambda}z}, \frac{\varphi}{1-\overline{\lambda}z}\right\rangle - \right. \\
&\left. -\theta(\lambda)\left\langle T_{\theta\Omega}\left(I - T_\theta T_{\overline{\varphi}}\right)\frac{\varphi}{1-\overline{\lambda}z}, \frac{\theta\varphi}{1-\overline{\lambda}z}\right\rangle\right) \\
&= \frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\left(\left\langle T_{|\varphi|^2\theta\Omega}\frac{1}{1-\overline{\lambda}z}, \frac{1}{1-\overline{\lambda}z}\right\rangle - \left\langle T_{\overline{\varphi}\theta^2\Omega} T_{\overline{\theta}\varphi}\frac{1}{1-\overline{\lambda}z}, \frac{1}{1-\overline{\lambda}z}\right\rangle - \right. \\
&- \theta(\lambda)\left\langle T_{|\varphi|^2\Omega}\frac{1}{1-\overline{\lambda}z}, \frac{1}{1-\overline{\lambda}z}\right\rangle + \\
&\left. +\theta(\lambda)\left\langle T_{\overline{\varphi}\theta\Omega} T_{\varphi\overline{\theta}}\frac{1}{1-\overline{\lambda}z}, \frac{1}{1-\overline{\lambda}z}\right\rangle\right) \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left(\left\langle T_{|\varphi|^2\theta\Omega}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle - \left\langle T_{\overline{\varphi}\theta^2\Omega} T_{\overline{\theta}\varphi}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \right. \\
&\left. +\theta(\lambda)\left\langle T_{\overline{\varphi}\theta\Omega} T_{\varphi\overline{\theta}}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle - \theta(\lambda)\left\langle T_{|\varphi|^2\Omega}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right) \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left(\left\langle\left(T_{|\varphi|^2\theta\Omega} - T_{\overline{\varphi}\theta^2\Omega} T_{\overline{\theta}\varphi}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle - \right. \\
&\left. - \theta(\lambda)\left\langle\left(T_{|\varphi|^2\Omega} - T_{\overline{\varphi}\theta\Omega} T_{\varphi\overline{\theta}}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right. \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left(\widetilde{H^*_{\overline{\varphi\theta^2\Omega}} H_{\varphi\overline{\theta}}}(\lambda) - \theta(\lambda)\widetilde{H^*_{\overline{\varphi\theta\Omega}} H_{\varphi\overline{\theta}}}(\lambda)\right).
\end{aligned}$$

Thus

$$\widetilde{\mathcal{K}}_{\varphi,\theta,\Omega}(\lambda) = \frac{1}{1-|\theta(\lambda)|^2}\left(\widetilde{H^*_{\overline{\varphi\theta^2\Omega}} H_{\varphi\overline{\theta}}}(\lambda) - \theta(\lambda)\widetilde{H^*_{\overline{\varphi\theta\Omega}} H_{\varphi\overline{\theta}}}(\lambda)\right)$$

for every $\lambda \in \mathbb{D}$. Consequently, using the fact that

$$\lim_{r \to 1^-} \widetilde{H_{\bar{f}}^* H_g} \left(re^{it} \right) = 0$$

for almost all $t \in [0, 2\pi]$, where $f, g \in L^\infty(\mathbb{T})$, we complete the proof of proposition. □

Our next result characterizes compact operators $\mathcal{L}_{\varphi,\theta}$ ($\varphi \in H^\infty$, $\theta \in (\Sigma)$).

Theorem 3.3. *$\mathcal{L}_{\varphi,\theta} \in S_\infty(K_\theta)$ if and only if*

$$\lim_{\lambda \to \mathbb{T}} \left(U^{-1} \left(H_{\bar{\varphi}}^* H_{\varphi\bar{\theta}} \right) U \right)^\sim (\lambda) = 0$$

for every unitary operator $U \in \mathcal{B}(H^2)$.

Proof. By Nikolski's formula (see Nikolski [8])

$$\varphi(M_\theta) P_\theta = \theta H_{\varphi\bar{\theta}},$$

we have

$$\begin{aligned}
\mathcal{L}_{\varphi,\theta} P_\theta f &= [T_{\bar{\theta}}, T_\varphi] \varphi(M_\theta) P_\theta f = \left(T_{\bar{\theta}\varphi} - T_\varphi T_{\bar{\theta}} \right) \theta H_{\varphi\bar{\theta}} f \\
&= H_{\bar{\varphi}}^* H_{\bar{\theta}} \theta H_{\varphi\bar{\theta}} f = H_{\bar{\varphi}}^* P_- \overline{\theta} \theta P_- \varphi \overline{\theta} f \\
&= H_{\bar{\varphi}}^* P_- \varphi \overline{\theta} f = H_{\bar{\varphi}}^* H_{\varphi\bar{\theta}} f
\end{aligned}$$

for each $f \in H^2$. Thus,

$$\mathcal{L}_{\varphi,\theta} P_\theta = H_{\bar{\varphi}}^* H_{\varphi\bar{\theta}}. \tag{3.3}$$

It follows from formula (3.3) that $\mathcal{L}_{\varphi,\theta} \in S_\infty(K_\theta)$ if and only if $H_{\bar{\varphi}}^* H_{\varphi\bar{\theta}} \in S_\infty(H^2)$. Thus, since $\partial_{\mathcal{H}^2} \mathbb{D} = \mathbb{T}$, Theorem A and Theorem 2.5 together with the formula (3.3) yield the statement of the theorem, as desired. □

4. Characterization of the classes S_p^w, $0 < p < \infty$

The main result of the present section is the following theorem, which gives necessary and sufficient conditions for belonging A to the classes S_p^w, $0 < p < \infty$. Its proof uses some arguments of the papers [6, 10].

Theorem 4.1. *Let H be an infinite dimensional complex Hilbert space, $A \in \mathcal{B}(H)$ be a compact operator with nonincreasing sequence of s-numbers $s_n(A)$, $n \geq 0$, $w := \{w_n\}_{n \geq 0}$ be a bounded sequence of complex numbers, and let $0 < p < \infty$. Then the following assertions are hold:*

(i) if $A \in S_p^w(H)$, then $\frac{\widetilde{T_\Lambda}(\sqrt{t})}{\sqrt{t}} = O(1-t)$ as $t \to 1$, where $\Lambda = ((s_n(A))^p \, w_n^p)_{n \geq 0}$;

(ii) if $\frac{\widetilde{T_\Lambda}(\sqrt{t})}{\sqrt{t}} = O(1-t)$ as $t \to 1$ and $s_n(A) w_n = O\left(n^{-\frac{1}{p}} \right)$ as $n \to \infty$, then $A \in S_p^w(H)$.

Proof. First, let us calculate the Berezin symbol of the weighted shift operator T_Λ acting in H^2 :

$$\begin{aligned}
\widetilde{T_\Lambda}(\lambda) &= \left\langle T_\Lambda \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle_{H^2} = \left\langle T_\Lambda \frac{(1-\overline{\lambda}z)^{-1}}{\left\|(1-\overline{\lambda}z)^{-1}\right\|_{H^2}}, \frac{(1-\overline{\lambda}z)^{-1}}{\left\|(1-\overline{\lambda}z)^{-1}\right\|_{H^2}} \right\rangle_{H^2} \\
&= \left\langle T_\Lambda \frac{(1-\overline{\lambda}z)^{-1}}{(1-|\lambda|^2)^{-\frac{1}{2}}}, \frac{(1-\overline{\lambda}z)^{-1}}{(1-|\lambda|^2)^{-\frac{1}{2}}} \right\rangle_{H^2} \\
&= (1-|\lambda|^2) \left\langle T_\Lambda \sum_{n=0}^{\infty} \overline{\lambda}^n z^n, \sum_{n=0}^{\infty} \overline{\lambda}^n z^n \right\rangle_{H^2} \\
&= (1-|\lambda|^2) \left\langle \sum_{n=0}^{\infty} \overline{\lambda}^n T_\Lambda z^n, \sum_{n=0}^{\infty} \overline{\lambda}^n z^n \right\rangle_{H^2} \\
&= (1-|\lambda|^2) \left\langle \sum_{n=0}^{\infty} \overline{\lambda}^n s_n(A)^p w_n^p z^{n+1}, \sum_{n=0}^{\infty} \overline{\lambda}^n z^n \right\rangle_{H^2} \\
&= (1-|\lambda|^2) \sum_{n=0}^{\infty} s_n(A)^p w_n^p \overline{\lambda}^n \lambda^{n+1} \\
&= \lambda(1-|\lambda|^2) \sum_{n=0}^{\infty} s_n(A)^p w_n^p \, |\lambda|^{2n} ,
\end{aligned}$$

i.e.,

$$\widetilde{T_\Lambda}(\lambda) = \lambda(1-|\lambda|^2) \sum_{n=0}^{\infty} s_n(A))^p w_n^p \, |\lambda|^{2n}$$

for all $\lambda \in \mathbb{D}$. In particular,

$$\widetilde{T_\Lambda}(\sqrt{t}) = \sqrt{t}(1-t) \sum_{n=0}^{\infty} s_n(A)^p w_n^p t^n,$$

or

$$\frac{\widetilde{T_\Lambda}(\sqrt{t})}{\sqrt{t}} = (1-t) \sum_{n=0}^{\infty} s_n(A)^p w_n^p t^n \tag{4.1}$$

for each $t \in (0,1)$.

Formula (4.1), in particular, shows that Abel convergence of the series $\sum_{n=0}^{\infty} s_n(A)^p w_n^p$ is equivalent to the assertion that $\frac{\widetilde{T_\Lambda}(\sqrt{t})}{\sqrt{t}} = O(1-t)$ as $t \to 1$.

(i) Now, if $A \in S_p^w$, then the series $\sum_{n=0}^{\infty} s_n(A)^p w_n^p$ is convergent. Then by the classical Abel theorem (see, for example, Hardy [5]) it is Abel convergent, that is, a finite limit $\lim\limits_{t\to 1} \sum_{n=0}^{\infty} s_n(A)^p w_n^p t^n$ exists. Therefore, it follows from (4.1) that

$$\frac{\widetilde{T_\Lambda}(\sqrt{t})}{\sqrt{t}} = O(1-t) \text{ as } t \to 1.$$

(ii) Conversely, if the conditions in (ii) of the theorem are satisfied, then it follows again from the formula (4.1) that the series $\sum_{n=0}^{\infty} s_n(A)^p w_n^p$ is summable by the

Abel method. On the other hand, since $s_n(A)w_n = O\left(n^{-\frac{1}{p}}\right)$ as $n \to \infty$, obviously, $(s_n(A)w_n)^p = O(\frac{1}{n})$ as $n \to \infty$. Then, by applying the classical Tauberian theorem of Hardy and Littlewood [5] we deduce that the series $\sum_{n=0}^{\infty} s_n(A)^p w_n^p$ is convergent, which implies that A belongs to the class S_p^w. The proof of the theorem is completed. □

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