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WEYL TYPE THEOREM AND SPECTRUM FOR (p,k)-QUASIPOSINORMAL OPERATORS

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ABSTRACT. Let T be a (p,k)-quasiposinormal operator on a complex Hilbert space $\mathcal{H}$, i.e $T^{*k}(c^2(T^*T)^p - (TT^*)^p)T^k \geq 0$ for a positive integer $0 < p \leq 1$, some $c > 0$ and a positive integer k. In this paper, we prove that the spectral mapping theorem for Weyl spectrum holds for (p,k) - quasiposinormal operators. We show that the Weyl type theorems holds for (p,k)- quasiposinormal. We prove that if T^* is (p,k)-quasiposinormal, then generalized a-Weyl's theorem holds for T. Also we prove that $\sigma_{jp}(T) - \{0\} = \sigma_{ap}(T) - \{0\}$ holds for (p,k)-quasiposinormal operator.

1. INTRODUCTION AND PRELIMINARIES

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. For a positive operators A and B, write $A \geq B$ if $A - B \geq 0$. If A and B are invertible and positive operators, it is well known that $A \geq B$ implies that log $A \geq$ log B. However [2], log $A \geq$log B does not necessarily imply $A \geq B$. A result due to Ando [6] states that for invertible positive operators A and B, log $A \geq$ log B if and only if $A^r \geq (A^{\frac{r}{2}} B^r A^{\frac{r}{2}})^{\frac{1}{2}}$ for all $r \geq 0$. For an operator T, let $U|T|$ denote the polar decomposition of T, where U is a partially isometric operator, $|T|$ is a positive square root of T^*T and ker $(T) =$ ker $(U) =$ ker $(|T|)$, where ker (S) denotes the kernel of operator S.

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An operator $T \in B(\mathcal{H})$ is positive, $T \geq 0$, if $(Tx,x) \geq 0$ for all $x \in \mathcal{H}$, and posinormal if there exists a positive $\lambda \in B(\mathcal{H})$ such that $TT^* = T^*\lambda T$. Here λ is called interrupter of T. In other words, an operator T is called posinormal if $TT^* \leq c^2 T^*T$, where T^* is the adjoint of T and $c > 0$ [15]. An operator T is said to be heminormal if T is hyponormal and T^*T commutes with TT^*. An operator T is said to be p-posinormal if $(TT^*)^p \leq c^2(T^*T)^p$ for some $c > 0$. It is clear that 1-posinormal is posinormal. An operator T is said to be p-hyponormal, for $p \in (0,1)$, if $(T^*T)^p \geq (TT^*)^p$. An 1-hyponormal operator is hyponormal which has been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [30]. Furuta et al [19], have characterized class A operator as follows. An operator T belongs to class A if and only if $(T^*|T|T)^{\frac{1}{2}} \geq T^*T$.

An operator T is called normal if $T^*T = TT^*$ and (p,k)-quasihyponormal if ${T^*}^k((T^*T)^p - (TT^*)^p)T^k \geq 0$ $(0 < p \leq 1, k \in \mathbb{N})$. In this paper, we investigate (p,k)-quasiposinormal operator T, i.e., ${T^*}^k(c^2(T^*T)^p - (TT^*)^p)T^k \geq 0$ $(0 < p \leq 1, k \in \mathbb{N}$ and $c > 0)$. Aluthge [1], Gupta [11], S.C. Arora and P. Arora [3] introduced p - hyponormal, p-quasihyponormal and k-quasihyponormal operators, respectively.

Aluthge [1] studied p-hyponormal operators for $0 < p \leq 1$. In particular he defined the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ which is called the Aluthge transformation and the operator $\widetilde{\widetilde{T}} = |\widetilde{T}|^{\frac{1}{2}}\widetilde{U}|\widetilde{T}|^{\frac{1}{2}}$, where $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ is the polar decomposition of $\widetilde{T}$. An operator T is said to be w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T^*}|$. Then we have

$$p\text{-hyponormal} \subset p\text{-posinormal} \subset (p,k)\text{-quasiposinormal},$$

$$p\text{-hyponormal} \subset p\text{ -quasihyponormal} \subset (p,k)\text{ - quasihyponormal} \subset (p,k)\text{-quasiposinormal}$$

and

$$\text{hyponormal} \subset k\text{-quasihyponormal} \subset (p,k)\text{-quasihyponormal} \subset (p,k)\text{ - quasiposinormal}$$

for a positive integer k and a positive number $0 < p \leq 1$.

If $T \in B(\mathcal{H})$, we shall write $N(T)$ and $R(T)$ for the null space and the range of T, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T, respectively. An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim \mathcal{H}/R(T) < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_W(T) \subset \sigma_e(T) \cup$ acc $\sigma(T)$ where we write acc K for the set of all accumulation points of $K \subset \mathbb{C}$. If we write iso $K = K\setminus$ acc K, then we let

$$\pi_{00}(T) = \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(T-\lambda) < \infty\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \backslash \sigma_W(T) = \pi_{00}(T).$$

Let $\sigma_p(T)$ denote the point spectrum of T, i.e., the set of its eigenvalues. Let $\sigma_{jp}(T)$ denote the joint point spectrum of T. We note that $\lambda \in \sigma_{jp}(T)$ if and only if there exists a non-zero vector x such that $Tx = \lambda x$, $T^*x = \overline{\lambda}x$. It is evident that $\sigma_{jp}(T) \subset \sigma_p(T)$. It is well known that, if T is normal, then $\sigma_{jp}(T) = \sigma_p(T)$. Let $T = U|T|$ be the polar decomposition of T and $\lambda = |\lambda|e^{i\theta}$ be a complex number, $|\lambda| > 0$, $|e^{i\theta}| = 1$. Then $\lambda \in \sigma_{jp}(T)$ if and only if there exists a non-zero vector x such that $Ux = e^{i\theta}$, $|T|x = |\lambda|x$. Let $\sigma_{ap}(T)$ denote the approximate point spectrum of T, i.e., the set of all complex numbers λ which satisfy the following condition: there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_n \|(T-\lambda)x_n\| = 0$. It is evident that $\sigma_p(T) \subset \sigma_{ap}(T)$. Let $\sigma_{jap}(T)$ be the joint approximate point spectrum of T, i.e., the set of all complex numbers λ which satisfy the following conditions: there exists a sequence $\{x_n\}$ of unit vectors such that $\lim_{n\to\infty} \|(T-\lambda)x_n\| = \lim_{n\to\infty} \|(T^* - \overline{\lambda})x_n\| = 0$. It is evident that $\sigma_{jap}(T) \subset \sigma_{ap}(T)$ for all $T \in B(\mathcal{H})$. It is well known that, for a normal operator T, $\sigma_{jap}(T) = \sigma_{ap}(T) = \sigma(T)$.

In [29], Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators [13], algebraically hyponormal operators [21], p-hyponormal operators [12] and algebraically p-hyponormal operators [17]. More generally, M. Berkani investigated generalized Weyl's theorem which extends Weyl's theorem, and proved that generalized Weyl's theorem holds for hyponormal operators [7, 8, 9]. In a recent paper [25] the author showed that generalized Weyl's theorem holds for (p, k) - quasihyponormal operators. Recently, X. Cao, M. Guo and B. Meng [10] proved Weyl type theorem holds for p-hyponormal operators. In this paper, we prove that Weyl type theorems hold for (p, k)-quasiposinormal operators. Especially we prove that if T^* is (p, k)-quasiposinormal, then generalized a-Weyl's theorem holds for T.

2. Weyl's Theorem for (p, k)- quasiposinormal operators

Mi Young Lee and Sang Hun Lee [22] have introduced (p, k)- quasiposinormal operators and have proved many interesting properties of it.

Lemma 2.1. *([22], [28]) (1) Let T be (p, k)-quasiposinormal. Then*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{\text{ran}(T^k)} \oplus ker(T^{*k})$, where T_1 is p-posinormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.
(2) If $\mathcal{Y} \subset \mathcal{H}$ is an invariant subspace of T, then the restriction $T|_{\mathcal{Y}}$ is also (p, k)-quasiposinormal operator.

Lemma 2.2. [28] *Let $T \in B(\mathcal{H})$ be a (p, k)-quasiposinormal operator for $c > 0$ and a positive integer k. If $\lambda \neq 0$ and $(T-\lambda)x = 0$ for some $x \in \mathcal{H}$, then $(T-\lambda)^*x = 0$.*

Lemma 2.3. *Let $T \in B(\mathcal{H})$ be a (p, k)-quasiposinormal operator for $c > 0$. Then T has Bishop's property (β), i.e., if $f_n(z)$ is analytic on D and $(T - z)f_n(z) \to 0$ uniformly on each compact subset of D, then $f_n(z) \to 0$ uniformly on each compact subset of D. Hence T has the single valued extension property.*

Proof. Let $f_n(z)$ be analytic on D and $(T - z)f_n(z) \to 0$ uniformly on each compact subset of D. Then

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_{n1}(z) \\ f_{n2}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_{n1}(z) + T_2 f_{n2}(z) \\ (T_3 - z)f_{n2}(z) \end{pmatrix} \to 0$$

Since $T_3^k = 0$, T_3 has (β) and $f_{n2}(z) \to 0$. Hence $(T_1 - z)f_{n1}(z) \to 0$. Since T_1 has (β) by [16], $(T_1 - z)f_{n1}(z) \to 0$. Thus $f_{n1}(z) \to 0$ and $f_n(z) \to 0$. □

Proposition 2.4. *Weyl's theorem holds for (p, k)-quasiposinormal operator T for $c > 0$, i.e., $\sigma(T)\backslash\sigma_W(T) = \pi_{00}(T)$.*

Proof. Let $\lambda \in \sigma(T)\backslash\sigma_W(T)$. Then $T - \lambda$ is Weyl and not invertible. If λ is an interior point of $\sigma(T)$, there exists an open set G such that $\lambda \in G \subset \sigma(T)\backslash\sigma_W(T)$. Hence dim $N(T - \mu) > 0$ for all $\mu \in G$ and T does not have the single valued extension property by [18, Theorem 9]. This is a contradiction. Hence λ is a boundary point of $\sigma(T)$, and hence an isolated point of $\sigma(T)$ by [14, Theorem XI 6.8]. Thus $\lambda \in \pi_{00}(T)$.

Let $\lambda \in \pi_{00}(T)$ and E_λ be the Riesz idempotent for λ of T. Then $0 <$ dim $N(T - \lambda) < \infty$,

$$T = T|_{E_\lambda \mathcal{H}} \oplus T|_{(I - E_\lambda)\mathcal{H}}$$

and

$$\sigma(T|_{E_\lambda \mathcal{H}}) = \{\lambda\},\ \sigma(T|_{(I - E_\lambda)\mathcal{H}}) = \sigma(T)\backslash\{\lambda\}.$$

We remark $T|_{E_\lambda \mathcal{H}}$ is (p, k)-quasiposinormal by Lemma 2.1.

If $\lambda \neq 0$, then $T|_{E_\lambda \mathcal{H}} = \{\lambda\}$ by [28]. Hence $E_\lambda \mathcal{H} \subset N(T - \lambda)$ and E_λ is of finite rank. Since $(T - \lambda)|_{(I - E_\lambda)\mathcal{H}}$ is invertible, $T - \lambda = 0|_{E_\lambda \mathcal{H}} \oplus (T - \lambda)|_{(I - E_\lambda)\mathcal{H}}$ is Weyl. Hence $\lambda \in \sigma(T)\backslash\sigma_W(T)$.

If $\lambda = 0$, then $(T|_{E_0 \mathcal{H}})^k = 0$ by [28]. Hence $E_0 \mathcal{H} \subset N(T^k)$ and

$$\text{dim } E_0 \mathcal{H} \leq \text{dim } N(T^k) \leq k \text{ dim } N(T) < \infty$$

Then $T|_{E_\lambda \mathcal{H}}$ is compact. Since $T|_{(I - E_0)}$ is invertible, $\lambda \in \sigma(T)\backslash\sigma_W(T)$ by [14, Proposition XI 6.9]. □

Theorem 2.5. *If T ia an n-multicyclic (p, k)-quasiposinormal operator, then the restriction T_1 of T on $\overline{\text{ran}(T^k)}$ is also an n-multicyclic operator.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\text{ran}(T^k)} \oplus ker(T^{*k})$. Since $\sigma(T_1) \subset \sigma(T)$ by Lemma 2.1, $\mathcal{R}(\sigma(T)) \subset \mathcal{R}(\sigma(T_1))$. By hypothesis there exist n vectors, $x_1, \cdots, x_n \in \mathcal{H}$, such that

$$\mathcal{H} = \bigvee\{g(T)x_i \mid i = 1, 2, \cdots, n \text{ and } g \in \mathcal{R}(\sigma(T))\}$$

Now let $Y_i = T^k x_i$, $i = 1, 2, \cdots, n$. Then we have the following

$$\begin{aligned}
&\bigvee\{g(T_1)Y_i \mid i = 1, 2, \cdots, n, g \in \mathcal{R}(\sigma(T_1))\} \\
&\qquad \supset \bigvee\{g(T_1)Y_i \mid i = 1, 2, \cdots, n, g \in \mathcal{R}(\sigma(T))\} \\
&\qquad = \bigvee\{g(T)T^k x_i \mid i = 1, 2, \cdots, n, g \in \mathcal{R}(\sigma(T))\} \\
&\qquad = \bigvee\{T^k g(T)x_i \mid i = 1, 2, \cdots, n, g \in \mathcal{R}(\sigma(T))\} \\
&\qquad = \overline{\operatorname{ran}(T^k)}
\end{aligned}$$

and $Y_1, \cdots, Y_n$ are n-multicyclic vectors of T_1.

□

Lemma 2.6 ([23],Theorem 6)**.** *For a given operators A,B,C $\in B(\mathcal{H})$ there is equality $\sigma_W(A) \cup \sigma_W(B) = \sigma_W(M_c \cup \mathfrak{G})$, where $M_c = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $\mathfrak{G}$ is the union of certain of the holes in $\sigma_W(M_c)$ which happen to be subsets of $\sigma_W(A) \cap \sigma_W(B)$.*

The following theorem shows that the spectral mapping theorem for Weyl spectrum holds for (p, k)-quasiposinormal operators.

Theorem 2.7. *If T is a (p, k)- quasiposinormal operator, then $f(\sigma_W(T)) = \sigma_W(f(T))$ for any analytic function f on a neighborhood of $\sigma(T)$.*

Proof. We need only to prove that $\sigma_W(p(T)) = p(\sigma_W(T))$ for any polynomial p. Since T has the matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 is p-posinormal and $T_3^k = 0$, and the spectral mapping theorem for Weyl spectrum holds for p -posinormal operator, it follows that

$$\begin{aligned}
\sigma_W(p(T)) &= \sigma_W(p(T_1)) \cup \sigma_W(p(T_3)) \\
&= p(\sigma_W(T_1)) \cup p(\sigma_W(T_3)) \\
&= p(\sigma_W(T_1) \cup \sigma_W(T_3)) \\
&= p(\sigma_W(T))
\end{aligned}$$

□

It was known [23] if A and B are isoloid and if Weyl's theorem holds for A and B then
Weyls theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Leftrightarrow \sigma_W \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_W(A) \cup \sigma_W(B)$.
We know that the "spectral picture" [26] of the operator$T \in B(\mathcal{H})$, denote by SP(T), which consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

In general, Weyl's theorem does not hold for operator matrix $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ even though Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. W.Y. Lee showed the following

Lemma (see [24]).

Lemma 2.8. *If either $SP(A)$ or $SP(B)$ has no pseudoholes and if A is an isoloid operator for which Weyl's theorem holds then for every $C \in B(\mathcal{H})$,*
Weyls theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Leftrightarrow \sigma_W \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

The following corollary follows from the above Lemma.

Corollary 2.9. *Weyl's theorem holds for every (p, k)-quasiposinormal operator.*

Proof. Let $T \in B(\mathcal{H})$ be a (p, k)-quasiposinormal operator. Then by Lemma 2.1 T has the following matrix representation:
$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus ker(T^{*k})$, where T_1 is p-posinormal, T_3 is nilpotent operator. Therefore Weyl's theorem holds for $\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$ because Weyl's theorem holds for p-posinormal operator and nilpotent operator and both p-posinormal operator and nilpotent operator are isoloid. Hence by Lemma 2.8 Weyl's theorem holds for $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ because $SP(T_3)$ has no pseudoholes. □

3. Generalized a-Weyl's theorem

More generally, Berkani investigated B-Fredholm theory as follows [4, 7, 8, 9]. An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator

$$T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$$

is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\alpha(T_{[n]}) = \dim N(T_{[n]}) < \infty$ and $\beta(T_{[n]}) = \dim R(T^n)/R(T_{[n]}) < \infty$. Similarly, a B-Fredholm operator T is called B-Weyl if $ind(T_{[n]}) = 0$. The following results is due to Berkani and Sarih [9].

Proposition 3.1. *Let $T \in B(\mathcal{H})$.*
(1) If $R(T^n)$ is closed and $T_{[n]}$ is Fredholm, then $R(T^m)$ is closed and $T_{[m]}$ is Fredholm for every $m \geq n$. Moreover, $ind\, T_{[m]} = ind\, T_{[n]} = ind\, T$.
(2) An operator T is B-Fredholm (B-Weyl) if and only if there exist T-invariant subspaces M and N such that $T = T|_M \oplus T|_N$ where $T|_M$ is Fredholm (Weyl) and $T|_N$ is nilpotent.

The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not B-Weyl}\} \subset \sigma_W(T).$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \backslash \sigma_{BW}(T) = E(T)$$

where $E(T)$ denotes the set of all isolated points of the spectrum which are eigenvalues (no restriction on multiplicity). Note that, if the generalized Weyl's theorem holds for T, then so does Weyl's theorem [8]. Recently in [7] M. Berkani and A. Arroud showed that if T is hyponormal, then generalized Weyl's theorem holds for T.

Proposition 3.2. *Generalized Weyl's theorem holds for (p,k)-quasiposinormal operator T.*

Proof. Let $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$. Then $T-\lambda$ is B-Weyl and not invertible. Then

$$T-\lambda = (T-\lambda)|_M \oplus (T-\lambda)|_N$$

where $(T-\lambda)|_M$ is Weyl and $(T-\lambda)|_N$ is nilpotent by Proposition 3.1. The case $M=\{0\}$ or $N=\{0\}$ is easy, so we may assume $M \neq \{0\}$ and $N \neq \{0\}$.

First we assume $\lambda \in \sigma(T|_M)$. In this case $T|_M$ is (p,k)-quasiposinormal by Lemma 2.1 and

$$\lambda \in \sigma(T|_M)\backslash\sigma_W(T|_M) = \pi_{00}(T|_M)$$

by Proposition 2.4. Hence λ is an isolated point of $\sigma(T|_M)$ and an eigenvalue of $T|_M$. Hence λ is an eigenvalue of T. On the other hand $(T-\lambda)|_N$ is nilpotent, so λ is an isolated point of $\sigma(T)$. Hence $\lambda \in E(T)$.

Secondly we assume $\lambda \notin \sigma(T|_M)$. In this case, $(T-\lambda)|_N$ is nilpotent, and λ is an eigenvalue of $T|_N$ and T. Since $(T-\lambda)|_M$ is invertible, λ is an isolated point of $\sigma(T)$. Hence $\lambda \in E(T)$.

Conversely, let $\lambda \in E(T)$. Since λ is an isolated point of $\sigma(T)$,

$$T-\lambda = (T-\lambda)|_{E_\lambda\mathcal{H}} \oplus (T-\lambda)|_{(I-E_\lambda)\mathcal{H}}$$

where E_λ denotes the Riesz idempotent for λ of T. Then $(T-\lambda)|_{E_\lambda\mathcal{H}}$ is (p,k) - quasiposinormal by Lemma 2.1 and $\sigma(T|_{E_\lambda\mathcal{H}}) = \lambda$.

If $\lambda \neq 0$, $T|_{E_\lambda\mathcal{H}} = \{\lambda\}$ by [28]. Hence

$$T-\lambda = 0|_{E_\lambda\mathcal{H}} \oplus (T-\lambda)|_{(I-E_\lambda)\mathcal{H}}$$

Since $(T-\lambda)|_{(I-E_\lambda)\mathcal{H}}$ is invertible, $T-\lambda$ is B-Weyl by Proposition 3.1. Hence $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$.

If $\lambda = 0$, then $(T|_{E_\lambda\mathcal{H}})^k = 0$ by [28]. Hence $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$ by Proposition 3.1. □

Theorem 3.3. *If T^* is (p,k)-quasiposinormal, then Weyl's theorem holds for T.*

Proof. Proposition 3.2 implies that

$$\sigma(T^*)\backslash\sigma_{BW}(T^*) = E(T^*)$$

It is obvious that

$$(\sigma(T^*)\backslash\sigma_{BW}(T^*))^*, = \sigma(T)\backslash\sigma_{BW}(T)$$

hence we have to prove

$$(E(T^*)^*) = E(T).$$

Let $\lambda^* \in E(T^*)$. Then λ is an isolated point of $\sigma(T)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. If $\lambda^* \neq 0$, then F_{λ^*} is self-adjoint,

$$\{0\} \neq F_{\lambda^*}\mathcal{H} = N((T-\lambda)^*) = N(T-\lambda)$$

by [28]. Hence $\lambda \in E(T)$. If $\lambda^* = 0$, then $T^*|_{F_0}$ is (p,k) - quasiposinormal by Lemma 2.1 and $(T^*|_{F_0\mathcal{H}})^k = 0$ by [28]. Hence $T^{*^k}F_0 = 0$. Let $E_0 = F_0^*$ be the Riesz idempotent for 0 of T. Then $T^kE_0 = (T^{*^k}F_0)^* = 0$. Hence $T|_{E_0\mathcal{H}}$ is nilpotent. Thus $\lambda = 0 \in E(T)$.

Conversely, let $\lambda \in E(T)$. Then λ^* is an isolated point of $\sigma(T^*)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. If $\lambda \neq 0$, then F_{λ^*} is self-adjoint and

$$\{0\} \neq F_{\lambda^*}\mathcal{H} = N((T-\lambda)^*) = N(T-\lambda)$$

by [28]. Hence $\lambda^* \in E(T^*)$. Let $\lambda = 0$. Since $T^*|_{F_0}\mathcal{H}$ is (p,k)-quasiposinormal and $\sigma(T^*|_{F_0}\mathcal{H}) = \{0\}$, we have $(T^*|_{F_0}\mathcal{H})^k = 0$ by [28]. This implies that $T^*|_{F_0}\mathcal{H}$ is nilpotent. Thus $\lambda^* = 0 \in E(T^*)$. □

Next we investigate a-Weyl's theorem [4].
We define $T \in SF_+^-$ if $R(T)$ is closed, dim $N(T) < \infty$ and ind $T \leq 0$. Let $\pi_{00}^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 <$ dim $\ker(T - \lambda) < \infty$. Let $\sigma_{SF_+^-}(T) = \{\lambda | T - \lambda \notin SF_+^-\} \subset \sigma_W(T)$.

We say that a-Weyl's theorem holds for T if

$$\sigma_a(T) \backslash \sigma_{SF_+^-}(T) = \pi_{00}^a(T).$$

Rakocevic [27, Corollary 2.5] proved that if a-Weyl's theorem holds for T, then Weyl's theorem holds for T.

Theorem 3.4. *If T^* is (p,k)-quasiposinormal, then a-Weyl's theorem holds for T.*

Proof. Since T^* has the single valued extension property by Lemma 2.3, we have $\sigma(T) = \sigma_a(T)$ and $\pi_{00}(T) = \pi_{00}^a(T)$ [4, Corollary 2.45].

Let $\lambda \in \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$. If λ is an interior point of $\sigma_a(T)$, then there exists an open set G such that $\lambda \in G \subset \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$. Since T^* has the single valued extension property, ind $(T - \mu)^* \leq 0$ for all $\mu \in \mathbb{C}$ by [4, Corollary 3.19]. Let $\mu \in G$. Then $T - \mu \in SF_+^-$ and ind $(T - \mu) = 0$. On the other hand, $R(T - \mu)$ is closed, $T - \mu$ is not invertible and $0 <$ dim $N(T - \mu) < \infty$. Hence $0 <$ dim $N((T - \mu)^*) < \infty$ and T^* does not have a single valued extension property by [18, Theorem 9]. This is a contradiction. Hence we may assume that λ is a boundary point of $\sigma(T)$. Since $T - \lambda \in SF_+^-$, λ is an isolated point of $\sigma(T)$ by [14, Theorem XI 6.8]. Thus $\lambda \in \pi_{00}(T) = \pi_{00}^a(T)$.

Conversely, $\lambda \in \pi_{00}^a(T) = \pi_{00}(T)$. Then λ^* is an isolated point of $\sigma(T^*)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. If $\lambda^* \neq 0$, then F_{λ^*} is self-adjoint and

$$F_{\lambda^*}\mathcal{H} = N((T - \lambda)^*) = N(T - \lambda)$$

by [28]. Since dim $N(T - \lambda) < \infty$, F_{λ^*} is compact. We decompose

$$(T - \lambda)^* = 0|_{F_{\lambda^*}\mathcal{H}} \oplus (T - \lambda)^*|_{(I - F_{\lambda^*})\mathcal{H}}$$

Then $(T - \lambda)^*|_{(I - F_{\lambda^*})\mathcal{H}}$ is invertible and

$$T - \lambda = 0|_{F_{\lambda^*}\mathcal{H}} \oplus (T - \lambda)|_{(I - F_{\lambda^*})\mathcal{H}}$$

Hence $R(T - \lambda) = (I - F_{\lambda^*})\mathcal{H}$ is closed and ind $(T - \lambda) = 0$. Thus $\lambda \in \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$.

If $\lambda^* = 0$, then

$$T^{*^k}|_{F_0\mathcal{H}} = (T^*|_{F_0\mathcal{H}})^k = 0$$

by [28]. Since $E_0 = F_0^*$ is the Riesz idempotent for 0 of T and $T^k E_0 = (T^{*^k} F_0)^* = 0$, we have $E_0\mathcal{H} \subset N(T^k)$. Then

$$\dim E_0\mathcal{H} \leq \dim N(T^k) \leq k \dim N(T) < \infty.$$

This implies E_0 is compact. We decompose

$$T = T|_{E_0\mathcal{H}} \oplus T|_{(I - E_0)\mathcal{H}}.$$

Since $T|_{(I - E_0)\mathcal{H}}$ is invertible, $R(T) = R(T|_{E_0\mathcal{H}}) \oplus (I - E_0)\mathcal{H}$ is closed, $N(T) \subset E_0\mathcal{H}$ and ind $T = 0$. Thus $0 \in \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$. □

Next we investigate generalized a-Weyl's theorem [4].
We define $T \in SBF_+^-$ if there exists a positive integer n such that $R(T^n)$ is closed,

$T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, dim $N(T_{[n]}) =$ dim $N(T) \cap R(T^n) < \infty$) and $0 \geq$ ind $T_{[n]}$(=ind T) [9]. We define $\sigma_{SBF_+^-}(T) = \{\lambda | T - \lambda \notin SBF_+^-\} \subset \sigma_{SF_+^-}(T)$. Let $E^a(T)$ denote the set of all isolated points λ of $\sigma_a(T)$ with $0 <$ dim $\ker(T - \lambda)$. We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \backslash \sigma_{SBF_+^-}(T) = E^a(T).$$

Berkani and Koliha [8] proved that if generalized a-Weyl's theorem holds for T, then a-Weyl's theorem holds for T.

Theorem 3.5. *If T^* is (p,k)-quasiposinormal, then generalized a-Weyl's theorem holds for T.*

Proof. Since T^* has the single valued extension property by Lemma 2.3, we have $\sigma(T) = \sigma_a(T)$, $\pi_{00}(T) = \pi_{00}^a(T)$ and $E(T) = E^a(T)$ [4, Corollary 2.45].

Let $\lambda_0 \in \sigma_a(T) \backslash \sigma_{SBF_+^-}(T)$. If λ_0 is an interior point of $\sigma_a(T)$, then there exists an open set G such that $\lambda_0 \in G \subset \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$. Let $\lambda \in G$. Then $T - \lambda \in SBF_+^-$ i.e., there exists a positive integer n such that $R((T-\lambda)^n)$ is closed, dim $N(T_n - \lambda) < \infty$ and ind $(T - \lambda) =$ ind $(T_n - \lambda) \leq 0$. Then there exists a positive number ϵ such that if $0 < |\lambda - \mu| < \epsilon$ then $T - \mu$ is upper semi-Fredholm, ind $(T - \mu) =$ ind $(T - \lambda) \leq 0$ and $\mu \in G$ by [9, Theorem 3.1]. Since T^* has a single valued extension property, ind $(T - \mu)^* \leq 0$ by [4, Corollary 3.19]. Hence ind $(T - \mu) = 0$. If $0 =$ dim $N(T - \mu)$, then $T - \mu$ is invertible. This is a contradiction. Hence $0 <$ dim $N(T - \mu) < \infty$, and $0 <$dim $N((T - \mu)^*) < \infty$. Then T^* does not have the single valued extension property by [18]. This is a contradiction.

Hence we may assume that λ_0 is a boundary point of $\sigma(T)$. Since $T - \lambda_0 \in SBF_+^-$, $T - \lambda_0$ is topologically uniform descent by [9, Proposition 2.5], and λ_0 is an isolated point of $\sigma(T)$ by [20, Corollary 4.9]. We decompose

$$T - \lambda_0 = (T - \lambda_0)|_M \oplus (T - \lambda_0)|_N$$

where $(T - \lambda_0)|_N$ is nilpotent and $(T - \lambda_0)|_M$ is semi-Fredholm by [9, Theorem 2.6]. If $N = \{0\}$, then

$$\lambda_0 \in \sigma_a(T) \backslash \sigma_{SF_+^-}(T) = \pi_{00}^a(T) = \pi_{00}(T) \subset E(T) = E^a(T)$$

by Theorem 3.4. If $N \neq \{0\}$, then λ_0 is an eigen value of $T|_N$ as $T|_N$ is nilpotent. Hence $\lambda_0 \in E(T) = E^a(T)$. Thus $\sigma_a(T) \backslash \sigma_{SBF_+^-}(T) \subset E^a(T)$.

The converse inclusion is clear because

$$\begin{aligned} E^a(T) &= E(T) \\ &\subset \pi_{00}(T) \\ &= \pi_{00}^a(T) \\ &= \sigma_a(T) \backslash \sigma_{SF_+^-}(T) \\ &\subset \sigma_a(T) \backslash \sigma_{SBF_+^-}(T) \end{aligned}$$

by Theorem 3.4. □

Remark 3.6. (1) If $f(z)$ is an analytic function on $\sigma(T)$, then generalized a-Weyl's theorem holds for T. (The proof is similar to [10, Theorem 3.3]).
(2) Generalized a-Weyl's theorem does not hold for (p,k) - quasiposinormal operator T as seen in [5, Example 2.13]. However if $\ker T \subset \ker T^*$, then generalized a-Weyl's theorem hold for T. (The proof is similar by [28]).

4. Spectra of (p,k)-quasiposinormal operators

Corollary 4.1. *If T is (p,k)-quasiposinormal operator, then $\sigma_{jp}(T)-\{0\} = \sigma_p(T)-\{0\}$.*

Proof. This follows from Lemma 2.2. □

Theorem 4.2. *If T is (p,k)-quasiposinormal operator, then $\sigma_{jp}(T)-\{0\} = \sigma_{ap}(T)-\{0\}$.*

Proof. Let ψ be the representation of Berberian. First, we show that $\psi(T)$ is (p,k)-quasiposinormal.
$(\psi(T))^{*k}[c^2(\psi(T)^*\psi(T))^p - (\psi(T)\psi(T)^*)^p](\psi(T))^k$
$= \psi(T^{*k})[c^2(\psi(T^*)\psi(T))^p - (\psi(T)\psi(T^*))^p]\psi(T^k)$
$= \psi(T^{*k})[c^2(\psi(T^*T))^p - (\psi(TT^*))^p]\psi(T^k)$
$= \psi[T^{*k}(c^2(T^*T)^p - (TT^*)^p)T^k]$

But T is (p,k)-quasiposinormal operator, then $T^{*k}(c^2(T^*T)^p-(TT^*)^p)T^k \geq 0$. So,
$$\psi[T^{*k}(c^2(T^*T)^p - (TT^*)^p)T^k] \geq 0.$$
Thus $\psi(T)$ is (p,k)-quasiposinormal operator. Now,

$$\begin{aligned}
\sigma_a(T)-\{0\} &= \sigma_a(\psi(T))-\{0\}\\
&= \sigma_p(\psi(T))-\{0\}\\
&= \sigma_{jp}(\psi(T))-\{0\} \text{ (by Corollary 4.1)}\\
&= \sigma_{jap}(T)-\{0\}
\end{aligned}$$

□

Corollary 4.3. *If T is an invertible (p,k)-quasiposinormal, then*
$$\sigma_{jap}(T) = \sigma_{ap}(T)$$

Definition 4.4. [14, Exercise 2, Pg. 349] The compression spectrum of T, denoted by $\sigma_c(T)$ is
$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \overline{\lambda} \in \sigma_p(T^*)\}$$

Corollary 4.5. *If T is an (p,k)-quasiposinormal, then*
$$\sigma(T)-\{0\} = \sigma_c(T)-\{0\}$$

Proof. Note that, for any operator $T \in B(\mathcal{H})$ the equality $\sigma(T)-\{0\} = \sigma_p(T) \cup \sigma_c(T)-\{0\}$ holds. If T is (p,k)-quasiposinormal, then Corollary 4.1 implies that $\sigma_{jap}(T)-\{0\} = \sigma_p(T)-\{0\} \subseteq \sigma_c(T)-\{0\}$. Since $\sigma_p(T^*) \subset \sigma_{ap}(T^*)$, the result follows. □

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