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REFINEMENTS AND REVERSES OF MEANS INEQUALITIES FOR HILBERT SPACE OPERATORS

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Abstract. In this paper we derive some improvements of means inequalities for Hilbert space operators. More precisely, we obtain refinements and reverses of the arithmetic-geometric operator mean inequality. As an application, we also deduce an improved variant for the refined arithmetic–Heinz mean inequality. We also present some eigenvalue inequalities for differences of certain operator means.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathfrak{B}(\mathcal{H})$ be the space of all bounded linear operators on $\mathcal{H}$. Let $\mathfrak{B}_h(\mathcal{H})$ be the semi-space in $\mathfrak{B}(\mathcal{H})$ of all self-adjoint operators. Moreover, let $\mathfrak{B}^+(\mathcal{H})$ and $\mathfrak{B}^{++}(\mathcal{H})$, respectively, denote the sets of all positive and positive invertible operators in $\mathfrak{B}_h(\mathcal{H})$. The weighted operator geometric mean $\sharp_\mu$ and the arithmetic mean ∇_μ, for $\mu \in [0,1]$ and $A, B \in \mathfrak{B}^{++}(\mathcal{H})$, are defined as follows:

$$A\sharp_\mu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\mu}A^{\frac{1}{2}},$$

$$A\nabla_\mu B = (1-\mu)A + \mu B.$$

If $\mu = 1/2$, we write $A\sharp B$, $A\nabla B$ for brevity, respectively.

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It is well-known that the arithmetic-geometric mean inequality, with respect to operator order, says that

$$A\sharp_\mu B \leq A\nabla_\mu B, \quad \mu \in [0,1].$$

Such mean inequalities for Hilbert space operators lie in the fields of interest of numerous mathematicians, and we refer here to some recent results.

Recently, Kittaneh et. al. [10], obtained the following refinement and reverse of the arithmetic-geometric operator mean inequality:

$$\begin{aligned} \frac{2\max\{p_1,p_2\}}{p_1+p_2}\left[A\nabla B - A\sharp B\right] &\geq A\nabla_{\frac{p_1}{p_1+p_2}} B - A\sharp_{\frac{p_1}{p_1+p_2}} B \\ &\geq \frac{2\min\{p_1,p_2\}}{p_1+p_2}\left[A\nabla B - A\sharp B\right], \end{aligned} \tag{1.1}$$

where $A,B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p}=(p_1,p_2)\in\mathbb{R}_+^2$.

Further, the paper [10] also deals with the Heinz operator mean, which is deduced from the geometric operator mean. Recall that the Heinz operator mean is defined by

$$H_\mu(A,B) = \frac{A\sharp_\mu B + A\sharp_{1-\mu} B}{2}, \tag{1.2}$$

where $A,B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mu\in[0,1]$. It is easy to see that the Heinz mean interpolates the arithmetic-geometric mean inequality (see [10]):

$$A\sharp B \leq H_\mu(A,B) \leq A\nabla B.$$

Even more, the paper [10] provides the following refinement of the arithmetic–Heinz operator mean inequality:

$$A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B) \geq \frac{2}{p_1+p_2}\min\{p_1,p_2\}\left[A\nabla B - A\sharp B\right]. \tag{1.3}$$

It should be mentioned here that the series of inequalities in (1.1) and inequality (1.3) were proved in [9] for positive definite matrices $A,B\in M_n(\mathbb{C})$. Here, $M_n(\mathbb{C})$ denotes the algebra of $n\times n$ complex matrices. The similar problem area is, for example, considered in [8] and [11]. In addition, for a comprehensive inspection of the recent results about inequalities for bounded self-adjoint operators on Hilbert space, the reader is referred to [4].

The main objective of this paper is an improvement of the series of inequalities (1.1) and inequality (1.3). In other words, we shall improve estimates for the lower and upper bounds for the difference between the arithmetic and geometric operator means considering an order in $\mathbb{R}_+^2$ and an operator order in $\mathfrak{B}^{++}(\mathcal{H})$.

The paper is organized in the following way: after this Introduction, in Section 2 we define a functional that measures the difference between the classical arithmetic and geometric means and also deduce some significant scalar inequalities which will help us in obtaining operator inequalities. Further, in Section 3 we obtain, under certain conditions, improvements of the series of inequalities in (1.1) as well as yet another lower bound for the difference between the arithmetic and geometric operator means. As an application, in Section 4 we give an improved variant of inequality (1.3), concerning the Heinz operator mean. In Section 5 we present some eigenvalue inequalities for differences of certain operator means.

2. Auxiliary results

In this section we give some auxiliary results which will help us to establish the announced improvements of the inequalities involving operator means on Hilbert space.

The starting point in obtaining the inequalities for bounded self-adjoint operators on Hilbert space is the following general monotonicity principle for operator functions: If $X \in \mathfrak{B}_h(\mathcal{H})$ with spectrum $\mathrm{Sp}(X)$ and f, g are continuous real-valued functions on $\mathrm{Sp}(X)$, then

$$f(t) \geq g(t),\ t \in \mathrm{Sp}(X), \text{ implies that } f(X) \geq g(X), \tag{2.1}$$

with equality if and only if $f(t) = g(t)$ for all $t \in \mathrm{Sp}(X)$. For more details about the described monotonicity principle and contributed consequences, the reader is referred to [4]. Clearly, the above principle allows us to raise some significant real inequalities on the level of self-adjoint operators on Hilbert space.

Accordingly, at the beginning, we consider an interesting inequality in one variable, dependent on a certain parameter.

Lemma 2.1. *If*

$$p \geq 1,\ x \geq 1 \quad \textit{or} \quad 0 < p \leq 1,\ 0 < x \leq 1, \tag{2.2}$$

then

$$1 + px^2 - (1+p)x^{\frac{2p}{1+p}} \geq \frac{2p}{1+p}(x-1)^2. \tag{2.3}$$

In addition, if

$$p \geq 1,\ 0 < x \leq 1 \quad \textit{or} \quad 0 < p \leq 1,\ x \geq 1,$$

then the sign of inequality (2.3) is reversed. Moreover, equality in (2.3) and in the reverse inequality holds if and only if $x = 1$ or $p = 1$.

Proof. We consider the function $g : (0, \infty) \to \mathbb{R}$, defined by

$$g(x) = 1 + px^2 - (1+p)x^{\frac{2p}{1+p}} - \frac{2p}{1+p}(x-1)^2.$$

By taking the first and second derivatives of the function g, we get

$$g'(x) = \frac{2p(p-1)}{p+1}x - 2px^{\frac{p-1}{p+1}} + \frac{4p}{p+1},$$

$$g''(x) = \frac{2p(p-1)}{p+1}\left(1 - x^{-\frac{2}{p+1}}\right).$$

If $p \geq 1$, $x \geq 1$, then $g''(x) \geq 0$. Hence, g' is increasing on $[1, \infty)$, which implies the inequality $g'(x) \geq g'(1) = 0$ for $x \geq 1$. Since, $g'(x) \geq 0$ for $x \geq 1$, we conclude that g is increasing on interval $[1, \infty)$. This implies the inequality $g(x) \geq g(1) = 0$, which is valid for $x \geq 1$.

Similarly, if $0 < p \leq 1$, $0 < x \leq 1$, then $g''(x) \geq 0$ and g' is increasing on $(0, 1]$, which implies the inequality $g'(x) \leq g'(1) = 0$ for $0 < x \leq 1$. Since $g'(x) \leq 0$ for $0 < x \leq 1$, we conclude that g is decreasing on interval $(0, 1]$. This implies the inequality $g(x) \geq g(1) = 0$, which holds for $0 < x \leq 1$. Therefore, inequality (2.3) is valid under conditions (2.2).

On the other hand, the reverse inequality in (2.3) is deduced in the same way as inequality (2.3).

Finally, conditions for equality in (2.3) and in the reversed inequality follow immediately from the presented proof. □

Note that the sign of inequality in (2.3) depends on both, the variable x and the parameter p. In the sequel, we give another type of lower bounds for the left-hand side of inequality (2.3), which holds for all positive x and p.

Lemma 2.2. *If $p > 0$, $x > 0$, then*

$$1 + px - (1+p)x^{\frac{p}{1+p}} \geq \frac{3p(x-1)^2}{(4+2p)x + 2 + 4p}. \tag{2.4}$$

Equality in (2.4) holds if and only if $x = 1$.

Proof. We consider the function $g : (0, \infty) \to \mathbb{R}$, defined by

$$g(x) = [(4+2p)x + 2 + 4p]\left[1 + px - (1+p)x^{\frac{p}{1+p}}\right] - 3p(x-1)^2.$$

By taking its first three derivatives, we get

$$\begin{aligned} g'(x) &= 2(1+2p)\left[p\left(x - x^{-\frac{1}{1+p}}\right) - (2+p)\left(x^{\frac{p}{1+p}} - 1\right)\right], \\ g''(x) &= \frac{2p(1+2p)}{1+p}\left[1 + p + x^{-\frac{2+p}{1+p}} - (2+p)x^{-\frac{1}{1+p}}\right], \\ g'''(x) &= \frac{2p(1+2p)(2+p)}{(1+p)^2}(x-1)x^{-\frac{3+2p}{1+p}}. \end{aligned}$$

If $x \geq 1$, then $g'''(x) \geq 0$. Hence, g'' is increasing on $[1, \infty)$, which implies the inequality $g''(x) \geq g''(1) = 0$, for $x \geq 1$. Since $g''(x) \geq 0$, for $x \geq 1$, we conclude that g' is increasing on $[1, \infty)$, that is $g'(x) \geq g'(1) = 0$, for $x \geq 1$. Now, g is increasing on $[1, \infty)$, i.e. $g(x) \geq g(1) = 0$, for $x \geq 1$.

On the other hand, if $0 < x \leq 1$, then $g'''(x) \leq 0$, which means that g'' is decreasing on $(0, 1]$. Hence, $g''(x) \geq g''(1) = 0$, for $0 < x \leq 1$. The latter relations implies that g' is increasing on interval $(0, 1]$, that is $g'(x) \leq g'(1) = 0$, for $0 < x \leq 1$. Now, g is decreasing on $(0, 1]$, which implies that $g(x) \geq g(1) = 0$ for $0 < x \leq 1$. This completes the proof. □

Remark 2.3. It should be mentioned here that special cases of Lemmas 2.1 and 2.2, with similar proofs, can be found in [2]. In addition, some related results can also be found in [3].

If we take a closer look at inequalities (2.3) and (2.4), we see that the left-hand sides of these inequalities can be interpreted as the differences between classical weighted arithmetic and geometric means. For that sake, we need more convenient forms of inequalities (2.3) and (2.4). More precisely, we define the functional $J : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$J(\mathbf{x}, \mathbf{p}) = (p_1 + p_2)\left[A(\mathbf{x}, \mathbf{p}) - G(\mathbf{x}, \mathbf{p})\right], \tag{2.5}$$

where $\mathbf{x} = (x_1, x_2)$, $\mathbf{p} = (p_1, p_2)$, and

$$A(\mathbf{x}, \mathbf{p}) = \frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}, \quad G(\mathbf{x}, \mathbf{p}) = (x_1^{p_1} x_2^{p_2})^{\frac{1}{p_1+p_2}}.$$

Obviously, due to the relationship between the arithmetic and geometric means, functional (2.5) is non-negative.

Now, Lemmas 2.1 and 2.2 enable us to establish appropriate bounds for functional (2.5), regarding the orders in $\mathbf{x}$ and $\mathbf{p}$.

More precisely, Lemma 2.1 yields lower and upper bounds for functional (2.5) with respect to the orders in $\mathbf{x} = (x_1, x_2)$ and $\mathbf{p} = (p_1, p_2)$.

Lemma 2.4. *Let $\mathbf{x} = (x_1, x_2), \mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$. If*

$$0 < p_1 \le p_2,\ 0 < x_1 \le x_2 \quad \text{or} \quad 0 < p_2 \le p_1,\ 0 < x_2 \le x_1, \tag{2.6}$$

then

$$J(\mathbf{x}, \mathbf{p}) \ge H(p_1, p_2)\left(\sqrt{x_1} - \sqrt{x_2}\right)^2, \tag{2.7}$$

where $H(p_1, p_2) = 2p_1p_2/(p_1 + p_2)$, that is, the harmonic mean of p_1 and p_2. In addition, if

$$0 < p_1 \le p_2,\ 0 < x_2 \le x_1 \quad \text{or} \quad 0 < p_2 \le p_1,\ 0 < x_1 \le x_2, \tag{2.8}$$

then the sign of inequality (2.7) is reversed. Further, equality in (2.7) and in the reverse inequality holds if and only if $x_1 = x_2$ or $p_1 = p_2$.

Proof. Suppose that conditions (2.6) are fulfilled and rewrite the functional (2.5) in the form:

$$J(\mathbf{x}, \mathbf{p}) = p_1 x_1 \left[1 + \frac{p_2}{p_1} \cdot \frac{x_2}{x_1} - \left(1 + \frac{p_2}{p_1}\right)\left(\frac{x_2}{x_1}\right)^{\frac{p_2}{p_1+p_2}}\right]. \tag{2.9}$$

If we denote $p = p_2/p_1$ and $x = \sqrt{x_2/x_1}$, we see that conditions (2.6) are equivalent to those in (2.2). Besides, under the above notations, the expression in the square brackets represents the left-hand side of inequality (2.3). Thus, by applying (2.3) on (2.9), we have

$$\begin{aligned} J(\mathbf{x}, \mathbf{p}) &\ge p_1 x_1 \frac{\frac{2p_2}{p_1}}{1 + \frac{p_2}{p_1}} \left(\sqrt{\frac{x_2}{x_1}} - 1\right)^2 \\ &= \frac{2p_1p_2}{p_1 + p_2}\left(\sqrt{x_1} - \sqrt{x_2}\right)^2, \end{aligned}$$

as required.

On the other hand, if conditions (2.8) are fulfilled, then, again by using Lemma 2.1, the sign of inequality (2.7) is reversed. □

Lemma 2.2 yields another lower bound for functional (2.5) regardless of an order in $\mathbf{x} = (x_1, x_2)$ and $\mathbf{p} = (p_1, p_2)$.

Lemma 2.5. *If $\mathbf{x} = (x_1, x_2), \mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$, then*

$$J(\mathbf{x}, \mathbf{p}) \ge \frac{3p_1p_2(x_1 - x_2)^2}{(2p_1 + 4p_2)x_1 + (4p_1 + 2p_2)x_2}. \tag{2.10}$$

In addition, equality in (2.10) holds if and only if $x_1 = x_2$.

Proof. Note that functional (2.5) can be written in the form:

$$J(\mathbf{x},\mathbf{p}) = p_1x_1\left[1+\frac{p_2}{p_1}\cdot\frac{x_2}{x_1}-\left(1+\frac{p_2}{p_1}\right)\left(\frac{x_2}{x_1}\right)^{\frac{p_2}{p_1+p_2}}\right].$$

Applying inequality (2.4) on the above expression in the square brackets for $p=p_2/p_1$ and $x=x_2/x_1$, we have

$$J(\mathbf{x},\mathbf{p}) \geq p_1x_1\frac{\frac{3p_2}{p_1}\cdot\left(\frac{x_2}{x_1}-1\right)^2}{\left(4+\frac{2p_2}{p_1}\right)\cdot\frac{x_2}{x_1}+\left(2+\frac{4p_2}{p_1}\right)} = \frac{3p_1p_2(x_1-x_2)^2}{(2p_1+4p_2)x_1+(4p_1+2p_2)x_2},$$

which completes the proof. □

3. Main results

In this section, with the help of Lemma 2.4, we are ready to deduce refinements and reverses of the arithmetic-geometric mean inequality for Hilbert space operators. More precisely, by following the ideas developed in Lemma 2.4, we get refinements and reverses of the arithmetic-geometric operator mean inequality which yield better results than the original result (1.1), presented in the Introduction.

Theorem 3.1. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p}=(p_1,p_2)\in\mathbb{R}_+^2$. If*

$$A \geq B,\ 0<p_1\leq p_2 \quad \text{or} \quad A\leq B,\ 0<p_2\leq p_1, \tag{3.1}$$

then

$$A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B \geq \frac{2H(p_1,p_2)}{p_1+p_2}\left[A\nabla B - A\sharp B\right]. \tag{3.2}$$

In addition, if

$$A \geq B,\ 0<p_2\leq p_1 \quad \text{or} \quad A\leq B,\ 0<p_1\leq p_2, \tag{3.3}$$

then the sign of inequality (3.2) is reversed. Equality in (3.2) and in the reverse inequality holds if and only if $A=B$ or $p_1=p_2$.

Proof. If we put $x_1=x$ and $x_2=1$ in (2.7), we get the inequality

$$p_1x+p_2-(p_1+p_2)x^{\frac{p_1}{p_1+p_2}} \geq 2H(p_1,p_2)\left(\frac{x+1}{2}-\sqrt{x}\right), \tag{3.4}$$

which is valid if

$$0<p_1\leq p_2,\ 0<x\leq 1 \quad \text{or} \quad 0<p_2\leq p_1,\ x\geq 1.$$

Now, considering the operator $A^{-\frac12}BA^{-\frac12}\in\mathfrak{B}^{++}(\mathcal{H})$ and the monotonicity principle (2.1), applied on inequality (3.4), we get the inequality

$$p_1A^{-\frac12}BA^{-\frac12}+p_21_{\mathcal{H}}-(p_1+p_2)\left(A^{-\frac12}BA^{-\frac12}\right)^{\frac{p_1}{p_1+p_2}}$$
$$\geq 2H(p_1,p_2)\left[\frac{A^{-\frac12}BA^{-\frac12}+1_{\mathcal{H}}}{2}-\left(A^{-\frac12}BA^{-\frac12}\right)^{\frac12}\right], \tag{3.5}$$

which is valid if

$$0 < p_1 \le p_2,\ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le 1_{\mathcal{H}} \ \text{ or } \ 0 < p_2 \le p_1,\ A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \ge 1_{\mathcal{H}}. \tag{3.6}$$

Here, $1_{\mathcal{H}}$ denotes the identity operator on the Hilbert space $\mathcal{H}$. Clearly, conditions (3.6) are equivalent to those in (3.1). Now, multiplying inequality (3.5) by $A^{\frac{1}{2}}$ from both sides, we get inequality (3.2).

On the other hand, by considering the reverse inequality in (3.4), which is valid under the conditions

$$0 < p_1 \le p_2,\ x \ge 1 \quad \textit{or} \quad 0 < p_2 \le p_1,\ 0 < x \le 1,$$

we get the reverse inequality in (3.2), following the same lines as in the first part of the proof. Clearly, the reverse inequality in (3.2) is fulfilled under conditions (3.3).

Finally, equality conditions follow immediately from Lemma 2.4 and the monotonicity principle (2.1) for operator functions. □

Remark 3.2. Since $\min\{p_1, p_2\} \le H(p_1, p_2) \le \max\{p_1, p_2\}$, inequality (3.2), which holds under conditions (3.1), represents an improvement of the refined inequality in (1.1). On the other hand, the reverse inequality in (3.2), which is valid under conditions (3.3), also represents an improvement of the reverse inequality in (1.1).

Inequality (2.10) from Lemma 2.5 yields yet another lower bound for functional (2.5), regardless of an order in $\mathbf{x} = (x_1, x_2)$ and $\mathbf{p} = (p_1, p_2)$. By raising to the level of operators in the Hilbert space, we get another lower bound for the difference between the arithmetic and geometric operator means, without any conditions on the positive operators A and B, and the positive parameters p_1 and p_2.

Theorem 3.3. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$. Then,*

$$A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B \ge \frac{3p_1p_2}{p_1+p_2}A^{\frac{1}{2}}D^{-\frac{1}{2}}\left[A^{-\frac{1}{2}}(A-B)A^{-\frac{1}{2}}\right]^2 D^{-\frac{1}{2}}A^{\frac{1}{2}}, \tag{3.7}$$

where $D = A^{-\frac{1}{2}}\left[(4p_1+2p_2)A + (2p_1+4p_2)B\right]A^{-\frac{1}{2}}$. Moreover, equality in (3.7) holds if and only if $A = B$.

Proof. If we put $x_1 = x$ and $x_2 = 1$ in (2.10), we get the inequality

$$p_1x + p_2 - (p_1+p_2)x^{\frac{p_1}{p_1+p_2}} \ge \frac{3p_1p_2(x-1)^2}{(2p_1+4p_2)x + 4p_1 + 2p_2},$$

which can be rewritten in the form

$$p_1x + p_2 - (p_1+p_2)x^{\frac{p_1}{p_1+p_2}} \ge 3p_1p_2k^{-\frac{1}{2}}(x)(x-1)^2k^{-\frac{1}{2}}(x), \tag{3.8}$$

where $k(x) = (2p_1+4p_2)x + 4p_1 + 2p_2$.

Now, considering the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \in \mathfrak{B}^{++}(\mathcal{H})$ and the monotonicity principle (2.1), applied on inequality (3.8), we get the inequality

$$\begin{aligned} &p_1A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + p_21_{\mathcal{H}} - (p_1+p_2)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{p_1}{p_1+p_2}} \\ &\quad \ge 3p_1p_2k^{-\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}}\right)^2 k^{-\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right). \end{aligned} \tag{3.9}$$

Now,

$$\begin{aligned} k\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) &= (2p_1+4p_2)A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (4p_1+2p_2)1_{\mathcal{H}} \\ &= (2p_1+4p_2)A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (4p_1+2p_2)A^{-\frac{1}{2}}AA^{-\frac{1}{2}} \\ &= A^{-\frac{1}{2}}\left[(4p_1+2p_2)A+(2p_1+4p_2)B\right]A^{-\frac{1}{2}} = D, \end{aligned}$$

and

$$\begin{aligned} \left[A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}}\right]^2 &= \left[A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - A^{-\frac{1}{2}}AA^{-\frac{1}{2}}\right]^2 \\ &= \left[A^{-\frac{1}{2}}(B-A)A^{-\frac{1}{2}}\right]^2 = \left[A^{-\frac{1}{2}}(A-B)A^{-\frac{1}{2}}\right]^2, \end{aligned}$$

so inequality (3.9) takes the form

$$\begin{aligned} p_1A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + p_2 1_{\mathcal{H}} - (p_1+p_2)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{p_1}{p_1+p_2}} & \\ \geq 3p_1p_2D^{-\frac{1}{2}}\left[A^{-\frac{1}{2}}(A-B)A^{-\frac{1}{2}}\right]^2 D^{-\frac{1}{2}}. & \quad (3.10) \end{aligned}$$

Finally, if we multiply inequality (3.10) by $A^{\frac{1}{2}}$ from both sides, we get (3.7). This completes the proof. □

4. Applications to Heinz means

In this section we apply our Theorems 3.1 and 3.3 to the operator Heinz mean. Recall that the Heinz mean (1.2) is closely connected with the arithmetic and geometric means and it interpolates the two mentioned means. As an application of Theorem 3.1, we obtain a better lower bound, for the difference between the arithmetic and Heinz means, than the original refinement (1.3) presented in the Introduction.

Theorem 4.1. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$. If $A \geq B$ or $A \leq B$, then*

$$A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B) \geq \frac{H(p_1,p_2)+\min\{p_1,p_2\}}{p_1+p_2}\left[A\nabla B - A\sharp B\right]. \quad (4.1)$$

Moreover, equality in (4.1) holds if and only if $A = B$ or $p_1 = p_2$.

Proof. Consider the right inequality in (1.1) with reversed roles of the parameters p_1 and p_2:

$$A\nabla_{\frac{p_2}{p_1+p_2}}B - A\sharp_{\frac{p_2}{p_1+p_2}}B \geq \frac{2\min\{p_1,p_2\}}{p_1+p_2}\left[A\nabla B - A\sharp B\right]. \quad (4.2)$$

Clearly, inequality (4.2) is valid for any choice of positive parameters p_1 and p_2, where $A, B \in \mathfrak{B}^{++}(\mathcal{H})$. On the other hand, inequality (3.2) is valid under conditions (3.1). Now, adding inequalities (3.2) and (4.2), it follows that inequality (4.1) holds under conditions (3.1).

Now, considering (3.2) with reversed roles of the parameters p_1 and p_2, we get the inequality

$$A\nabla_{\frac{p_2}{p_1+p_2}} B - A\sharp_{\frac{p_2}{p_1+p_2}} B \geq \frac{2H(p_1,p_2)}{p_1+p_2}\left[A\nabla B - A\sharp B\right], \tag{4.3}$$

which is valid under conditions (3.3). Same as before, if we add the right inequality in (1.1) and (4.3), we obtain again inequality (4.1) which holds under conditions (3.3).

From this we conclude that (4.1) holds if one of the conditions in (3.1) or (3.3) is fulfilled. This means that we can drop the conditions concerning the parameters p_1 and p_2 in (3.1) and (3.3), that is, (4.1) holds if $A \leq B$ or $A \geq B$. □

Remark 4.2. Since $H(p_1,p_2)+\min\{p_1,p_2\} \geq 2\min\{p_1,p_2\}$, inequality (4.1) represents an improved form of inequality (1.3) from the Introduction. Note also that inequality (4.1) is established by successive application of the improved inequality (3.2) and the weaker refined inequality in (1.1). Therefore, in inequality (4.1) we do not get so good constant factor $2H(p_1,p_2)$, as in (3.2), but we can omit the conditions concerning the parameters p_1 and p_2.

Remark 4.3. In a similar way as in Theorem 4.1, we get a reverse inequality related to (4.1), that is

$$A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B) \leq r\left[A\nabla B - A\sharp B\right], \tag{4.4}$$

where

$$r = \frac{H(p_1,p_2)+\max\{p_1,p_2\}}{p_1+p_2} \geq \frac{\min\{p_1,p_2\}+\max\{p_1,p_2\}}{p_1+p_2} = 1.$$

Reverse relation (4.4) can be rewritten in the following form:

$$H_{\frac{p_1}{p_1+p_2}}(A,B) \geq A\sharp B - (r-1)\left[A\nabla B - A\sharp B\right]. \tag{4.5}$$

Now, since $r \geq 1$ and $A\nabla B \geq A\sharp B$, inequality (4.5) is weaker than the original relationship between the Heinz and geometric means, i.e. $H_{p_1/(p_1+p_2)}(A,B) \geq A\sharp B$. It is reasonable since the inequality between the geometric and Heinz means is established throughout the scalar inequality $x^{p_1/(p_1+p_2)} + x^{p_2/(p_1+p_2)} \geq 2\sqrt{x}$. Namely, the last inequality is non-weighted, and that is the best possible which gives the method developed in this paper.

We conclude this section with another type of refinements of the arithmetic–Heinz mean inequality, arising from Theorem 3.3.

Theorem 4.4. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p} = (p_1,p_2) \in \mathbb{R}_+^2$. Then,*

$$\begin{aligned} &A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B) \\ &\geq \frac{3p_1p_2}{2\left(p_1+p_2\right)} A^{\frac{1}{2}}\left(D^{-\frac{1}{2}} + D'^{-\frac{1}{2}}\right)\left[A^{-\frac{1}{2}}(A-B)A^{-\frac{1}{2}}\right]^2\left(D^{-\frac{1}{2}} + D'^{-\frac{1}{2}}\right)A^{\frac{1}{2}}, \end{aligned} \tag{4.6}$$

where $D = A^{-\frac{1}{2}}\left[(4p_1+2p_2)A + (2p_1+4p_2)B\right]A^{-\frac{1}{2}}$ and D' is obtained from D by changing the roles of the parameters p_1 and p_2. In addition, equality in (4.6) holds if and only if $A = B$.

Proof. If we rewrite inequality (3.7) with reversed roles of the parameters p_1 and p_2, we get the inequality

$$A\nabla_{\frac{p_2}{p_1+p_2}}B - A\sharp_{\frac{p_2}{p_1+p_2}}B \geq \frac{3p_1p_2}{p_1+p_2}A^{\frac{1}{2}}D'^{-\frac{1}{2}}\left[A^{-\frac{1}{2}}(A-B)A^{-\frac{1}{2}}\right]^2 D'^{-\frac{1}{2}}A^{\frac{1}{2}}. \quad (4.7)$$

Finally, by adding inequalities (3.7) and (4.7), we obtain (4.6). Now, equality in (4.6) holds if and only if it holds in inequalities (3.7) and (4.7), that is, if $A = B$. □

5. Eigenvalues of differences of certain operator means

This section is devoted to eigenvalue inequalities for differences of means of positive invertible operators, under certain compactness assumption. Related results have been recently obtained in [7].

For a compact operator $A \in \mathfrak{B}^+(\mathcal{H})$, let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq 0$ denote the eigenvalues of A arranged in decreasing order and repeated according to multiplicity. In order to present our promised eigenvalue inequalities, we need the following three lemmas. The first lemma follows from the well-known characterization of compact operators (see e.g., [6] or [12, p. 59]):

$$A \in \mathfrak{B}(\mathcal{H}) \text{ is compact if and only if } \langle Ae_n, e_n\rangle \to 0 \text{ as } n \to \infty$$

for every orthonormal set $\{e_n\}$ in $\mathcal{H}$, where $\langle\cdot,\cdot\rangle$ denotes the inner product defined on $\mathcal{H}$. The second lemma is known as the Weyl's monotonicity principle for compact positive operators (see e.g., [1, p. 63] or [5, p. 26]). While, the third lemma is a basic property of the eigenvalues of compact operators.

Lemma 5.1. *Let $A, B \in \mathfrak{B}^+(\mathcal{H})$ such that A is compact and $A \geq B$. Then B is compact.*

Lemma 5.2. *Let $A, B \in \mathfrak{B}^+(\mathcal{H})$ be compact such that $A \geq B$. Then $\lambda_j(A) \geq \lambda_j(B)$ for $j = 1, 2, \ldots$.*

Lemma 5.3. *Let $X \in \mathfrak{B}(\mathcal{H})$ be compact. Then $\lambda_j(X^*X) = \lambda_j(XX^*)$ for $j = 1, 2, \ldots$.*

Our first eigenvalue inequality in this section, which is closely related to inequality (3.7), gives a lower bound for the difference between the arithmetic and geometric means.

Theorem 5.4. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$. If $A - B$ is compact and $D = A^{-\frac{1}{2}}\left[(4p_1 + 2p_2)A + (2p_1 + 4p_2)B\right]A^{-\frac{1}{2}}$, then*

$$\lambda_j\left(A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\right) \geq \frac{3p_1p_2}{p_1+p_2}\lambda_j\left(D^{-\frac{1}{2}}A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}D^{-\frac{1}{2}}\right) \quad (5.1)$$

for $j = 1, 2, \ldots$.

Proof. Inequality (3.8) can be written as

$$\frac{p_1}{p_1+p_2}x + \frac{p_2}{p_1+p_2} - x^{\frac{p_1}{p_1+p_2}} \geq \frac{3p_1p_2}{p_1+p_2}(x-1)\,k^{-1}(x)(x-1) \quad (5.2)$$

for $x > 0$, where $k(x) = (2p_1 + 4p_2)x + 4p_1 + 2p_2$. Since $Sp(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \subseteq (0, \infty)$, the monotonicity principle (2.1), applied on inequality (5.2), implies that

$$\begin{aligned}
&\frac{p_1}{p_1+p_2}A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+\frac{p_2}{p_1+p_2}-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{p_1}{p_1+p_2}}\\
&\qquad \geq \frac{3p_1p_2}{p_1+p_2}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}}\right)k^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})\\
&\qquad = \frac{3p_1p_2}{p_1+p_2}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})
\end{aligned}$$

and so

$$\begin{aligned}
&A\nabla_{\frac{p_1}{p_1+p_2}}B-A\sharp_{\frac{p_1}{p_1+p_2}}B\\
&\qquad = \frac{p_1}{p_1+p_2}B+\frac{p_2}{p_1+p_2}A-A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{p_1}{p_1+p_2}}A^{\frac{1}{2}}\\
&\qquad \geq \frac{3p_1p_2}{p_1+p_2}A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})A^{\frac{1}{2}}. \qquad (5.3)
\end{aligned}$$

Since $A - B$ is compact, it follows from the spectral theorem, applied in the Calkin algebra setting, that the operator $A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B$ is also compact, and since the operator $A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})A^{\frac{1}{2}}$ is positive, it follows from inequality (5.3), together with Lemma 5.1, that the operator

$$A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})A^{\frac{1}{2}}$$

is compact. Lemma 5.2, together with the inequality (5.3), implies that

$$\begin{aligned}
&\lambda_j\left(A\nabla_{\frac{p_1}{p_1+p_2}}B-A\sharp_{\frac{p_1}{p_1+p_2}}B\right)\\
&\qquad \geq \frac{3p_1p_2}{p_1+p_2}\lambda_j\left(A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})A^{\frac{1}{2}}\right) \qquad (5.4)
\end{aligned}$$

for $j = 1, 2, \ldots$. Let $X = D^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}})A^{\frac{1}{2}}$. It follows from Lemma 5.3 that

$$\begin{aligned}
&\lambda_j\left(A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})A^{\frac{1}{2}}\right)\\
&\qquad = \lambda_j\left(X^*X\right)\\
&\qquad = \lambda_j\left(XX^*\right)\\
&\qquad = \lambda_j\left(D^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})A(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-1_{\mathcal{H}})D^{-\frac{1}{2}}\right)\\
&\qquad = \lambda_j\left(D^{-\frac{1}{2}}A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}D^{-\frac{1}{2}}\right) \qquad (5.5)
\end{aligned}$$

for $j = 1, 2, \ldots$. Now, the result follows from inequality (5.4) and identity (5.5). □

In order to investigate the equality condition of inequality (5.1) in Theorem 5.4, we need the following lemma (see, e.g., [5, p. 26]).

Lemma 5.5. *Let $A, B \in \mathfrak{B}^+(\mathcal{H})$ be compact such that $A \geq B$. Then $A = B$ if and only if $\lambda_j(A) = \lambda_j(B)$ for $j = 1, 2, ...$.*

Theorem 5.6. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$. If $A - B$ is compact and $D = A^{-\frac{1}{2}}\left[(4p_1 + 2p_2)A + (2p_1 + 4p_2)B\right]A^{-\frac{1}{2}}$, then*

$$\lambda_j\left(A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\right) = \frac{3p_1p_2}{p_1+p_2}\lambda_j\left(D^{-\frac{1}{2}}A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}D^{-\frac{1}{2}}\right)$$

for $j = 1, 2, ...$ if and only if $A = B$.

Proof. Suppose that

$$\lambda_j\left(A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\right) = \frac{3p_1p_2}{p_1+p_2}\lambda_j\left(D^{-\frac{1}{2}}A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}D^{-\frac{1}{2}}\right)$$

for $j = 1, 2, ...$. Then, it follows from identity (5.5) that

$$\begin{aligned}&\lambda_j\left(A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\right)\\ &\qquad = \frac{3p_1p_2}{p_1+p_2}\lambda_j\left(A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}})A^{\frac{1}{2}}\right) \qquad (5.6)\end{aligned}$$

for $j = 1, 2, ...$. It follows from relation (5.6), together with inequality (5.3) and Lemma 5.5, that

$$\begin{aligned}&A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\\ &\qquad = \frac{3p_1p_2}{p_1+p_2}A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}})A^{\frac{1}{2}}. \qquad (5.7)\end{aligned}$$

The equality in (5.7) implies that

$$\begin{aligned}\frac{J(\mathbf{x},\mathbf{p})}{p_1+p_2} &= \frac{p_1}{p_1+p_2}x + \frac{p_2}{p_1+p_2} - x^{\frac{p_1}{p_1+p_2}}\\ &= \frac{3p_1p_2}{p_1+p_2}(x-1)\,k^{-1}(x)(x-1)\\ &= \frac{3p_1p_2}{p_1+p_2}k^{-1}(x)(x-1)^2 \qquad (5.8)\end{aligned}$$

for all $x \in Sp\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)$, where $\mathbf{x} = (x, 1)$. Now, it follows from (5.8) and the equality conditions of inequality (2.10) that $x = 1$ for all $x \in Sp\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)$. Using this, together with the equality condition of inequality (2.1), we have $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} = 1_{\mathcal{H}}$, that is, $B = A$.

The converse is trivial, and the proof is complete. ☐

Our second eigenvalue inequality in this section gives a lower bound for the difference between the arithmetic and Heinz means.

Theorem 5.7. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$. If $A - B$ is compact and $D = A^{-\frac{1}{2}}\left[(4p_1 + 2p_2)A + (2p_1 + 4p_2)B\right]A^{-\frac{1}{2}}$, then*

$$\lambda_j\left(A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B)\right) \geq \frac{3p_1p_2}{p_1+p_2}\lambda_j\left(D^{-\frac{1}{2}}A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}D^{-\frac{1}{2}}\right) \qquad (5.9)$$

for $j = 1, 2, \ldots$.

Proof. In inequality (5.3) interchanging p_1 and p_2, we have

$$\begin{aligned} &A\nabla_{\frac{p_2}{p_1+p_2}}B - A\sharp_{\frac{p_2}{p_1+p_2}}B \\ &\geq \frac{3p_1p_2}{p_1+p_2}A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}})A^{\frac{1}{2}}. \quad (5.10) \end{aligned}$$

Adding inequalities (5.3) and (5.10), we have

$$\begin{aligned} &A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B) \\ &= \frac{A + B - \left(A\sharp_{\frac{p_1}{p_1+p_2}}B + A\sharp_{\frac{p_2}{p_1+p_2}}B\right)}{2} \\ &= \frac{\left(A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B\right) + \left(A\nabla_{\frac{p_2}{p_1+p_2}}B - A\sharp_{\frac{p_2}{p_1+p_2}}B\right)}{2} \\ &\geq \frac{3p_1p_2}{p_1+p_2}A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}})A^{\frac{1}{2}}. \quad (5.11) \end{aligned}$$

Since $A - B$ is compact, the operators

$$A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B)$$

and

$$\frac{3p_1p_2}{p_1+p_2}A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}}\right)D^{-1}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - 1_{\mathcal{H}})A^{\frac{1}{2}}$$

are compact. It follows from inequality (5.11), together with identity (5.5) and Lemma 5.2, that

$$\lambda_j\left(A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B)\right) \geq \frac{3p_1p_2}{p_1+p_2}\lambda_j\left(D^{-\frac{1}{2}}A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}D^{-\frac{1}{2}}\right)$$

for $j = 1, 2, \ldots$, as required. □

The equality condition of inequality (5.9) in Theorem 5.7 can be stated as follows. The proof is similar to that given for Theorem 5.6.

Theorem 5.8. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$. If $A - B$ is compact and $D = A^{-\frac{1}{2}}\left[(4p_1 + 2p_2)A + (2p_1 + 4p_2)B\right]A^{-\frac{1}{2}}$, then*

$$\lambda_j\left(A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A,B)\right) = \frac{3p_1p_2}{p_1+p_2}\lambda_j\left(D^{-\frac{1}{2}}A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}D^{-\frac{1}{2}}\right)$$

for $j = 1, 2, \ldots$ if and only if $A = B$.

Finally, we remark that, using Lemma 5.2 and the spectral theorem for compact positive operators, Theorems 5.4 and 5.7 can be formulated in the following forms.

Theorem 5.9. *Let $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ and $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$. If $A - B$ is compact and $D = A^{-\frac{1}{2}}\left[(4p_1 + 2p_2)A + (2p_1 + 4p_2)B\right]A^{-\frac{1}{2}}$, then*

$$A\nabla_{\frac{p_1}{p_1+p_2}}B - A\sharp_{\frac{p_1}{p_1+p_2}}B \geq \frac{3p_1p_2}{p_1+p_2}U^*D^{-\frac{1}{2}}A^{-\frac{1}{2}}(A-B)^2A^{-\frac{1}{2}}D^{-\frac{1}{2}}U \quad (5.12)$$

for some unitary operator $U \in \mathfrak{B}(\mathcal{H})$. *Moreover, equality holds in* (5.12) *if and only if* $A = B$.

Theorem 5.10. *Let* $A, B \in \mathfrak{B}^{++}(\mathcal{H})$ *and* $\mathbf{p} = (p_1, p_2) \in \mathbb{R}_+^2$. *If* $A - B$ *is compact and* $D = A^{-\frac{1}{2}}\left[(4p_1 + 2p_2)A + (2p_1 + 4p_2)B\right]A^{-\frac{1}{2}}$, *then*

$$A\nabla B - H_{\frac{p_1}{p_1+p_2}}(A, B) \geq \frac{3p_1p_2}{p_1 + p_2} U^* D^{-\frac{1}{2}} A^{-\frac{1}{2}} (A - B)^2 A^{-\frac{1}{2}} D^{-\frac{1}{2}} U \tag{5.13}$$

for some unitary operator $U \in \mathfrak{B}(\mathcal{H})$. *Moreover, equality holds in* (5.13) *if and only if* $A = B$.

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