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WEIGHTED COMPOSITION OPERATORS BETWEEN VECTOR-VALUED LIPSCHITZ FUNCTION SPACES

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ABSTRACT. We give necessary and sufficient conditions for the boundedness and compactness of weighted composition operators between spaces of vector-valued Lipschitz functions. We then show that a bounded separating linear operator between these spaces is indeed a weighted composition operator.

1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a compact metric space, $(E, \|\cdot\|)$ be a Banach space and $\alpha \in (0, 1]$. The space of all functions $f : X \to E$ for which

$$p_\alpha(f) = \sup\left\{\frac{\|f(x) - f(y)\|}{d^\alpha(x, y)} : x, y \in X, x \neq y\right\} < \infty,$$

is denoted by $\mathrm{Lip}_\alpha(X, E)$. The subspace of those functions f with

$$\lim_{d(x,y)\to 0} \frac{\|f(x) - f(y)\|}{d^\alpha(x, y)} = 0,$$

is denoted by $\mathrm{lip}_\alpha(X, E)$. The spaces $\mathrm{Lip}_\alpha(X, E)$ and $\mathrm{lip}_\alpha(X, E)$ are Banach spaces when equipped with the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup\{\|f(x)\| : x \in X\}$. These are called vector-valued Lipschitz function spaces. In the case where E is the scalar field of the complex numbers $\mathbb{C}$, to simplify the notation, we write $\mathrm{Lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(X)$ instead of $\mathrm{Lip}_\alpha(X, \mathbb{C})$ and $\mathrm{lip}_\alpha(X, \mathbb{C})$,

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respectively. In this case, $\mathrm{Lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(X)$ are Banach algebras which are also called Lipschitz algebras. The scalar-valued Lipschitz functions were first studied by de Leeuw [16] and Sherbert [17, 18]. The interested reader is also referred to [3, 6, 19] for further details on the subject. The spaces $\mathrm{Lip}_\alpha(X, E)$ and $\mathrm{lip}_\alpha(X, E)$ were first considered by Johnson [13]. Since then, there has been an extensive study on this subject.

In this paper we shall study a class of operators between vector-valued Lipschitz function spaces known as weighted composition operators. Given X and Y two compact Hausdorff spaces and given E and F two Banach spaces, let $\mathcal{S}(X, E)$ be any subspace of $C(X, E)$, the space of all continuous E-valued functions on X. A weighted composition operator between vector-valued function spaces is defined to be a linear operator $T : \mathcal{S}(X, E) \to \mathcal{S}(Y, F)$ of the form $Tf(y) = W_y(f(\varphi(y)))$ for every $f \in \mathcal{S}(X, E)$ and $y \in Y$ where W_y is a linear operator from E into F and $\varphi : Y \to X$ is a function. In the scalar case, a weighted composition operator is a composition operator followed by a multiplier. The compactness of the weighted composition operators on $C(X, \mathbb{C})$ has been characterized by Kamowitz [14]. Kamowitz and Scheinberg also determined necessary and sufficient conditions for composition operators on $\mathrm{Lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(X)$ to be compact [15]. Jamison and Rajagopalan provided a necessary and sufficient condition for weighted composition operators on $C(X, E)$ to be compact [7]. Chan has improved their results and characterized the compact weighted composition operators on the space of continuous functions defined on a locally compact Hausdorff space vanishing at infinity [4]. In section 2, we give necessary and sufficient conditions for weighted composition operators between spaces of vector-valued Lipschitz functions to be compact.

It is also interesting to determine which types of operators have the form of weighted composition operators. We say that a linear operator $T : \mathcal{S}(X, E) \to \mathcal{S}(Y, F)$ has the *disjoint support property* or is *separating*, if $\|Tf(y)\|\|Tg(y)\| = 0$ for all $y \in Y$ whenever $f, g \in \mathcal{S}(X, E)$ satisfy $\|f(x)\|\|g(x)\| = 0$ for all $x \in X$. We say that T is *biseparating* if it is bijective and both T and T^{-1} are separating. Note that a typical example of separating operators between function spaces are weighted composition operators and the standard problem is to determine whether these are the canonical examples. The notion of separating operator has been studied extensively. Separating linear maps between spaces of continuous scalar-valued functions were studied by Beckenstein, et al. [2], Font and Hernández [5], Jaroz [8] and Jeang and Wong [9]. Bounded separating linear operators on the space $C(X, E)$ were studied by Jamison and Rajagopalan [7] and Chan [4]. The study of separating linear maps between spaces of scalar-valued Lipschitz functions was initiated by Wu in [20]. Jiménez-Vargas in [10] obtained a representation of separating linear maps between scalar-valued Lipschitz algebras $\mathrm{lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(Y)$ when X and Y are compact and $\alpha \in (0, 1)$. Biseparating linear operators between spaces of vector-valued Lipschitz functions were studied by Araujo and Dubarbie in [1] and Jimenéz-Vargas, et al. in [11] and [12]. In section 3, we shall show that bounded separating linear operators between vector-valued Lipschitz function spaces are weighted composition operators.

Given Banach spaces E and F, we denote by $\mathcal{B}(E, F)$ and $\mathcal{K}(E, F)$ the space of all bounded linear operators and compact linear operators from E into F, respectively. The dual space of a Banach space E is denoted by E^*. In this note, the linear operators which we have considered are nonzero.

2. Compact weighted composition operators

In this section, we provide necessary and sufficient conditions for weighted composition operators between vector-valued Lipschitz spaces to be compact.

In what follows, we will assume that X and Y are compact metric spaces, E and F are Banach spaces and $\alpha \in (0, 1]$. It is interesting to note that $\mathrm{Lip}_1(X, E) \subseteq \mathrm{lip}_\alpha(X, E) \subseteq \mathrm{Lip}_\alpha(X, E)$ for $\alpha \in (0, 1)$. For each scalar-valued function h on X and E-valued function f on X, one can define the E-valued function hf on X by $(hf)(x) = h(x)f(x)$ for $x \in X$. If $h \in \mathrm{Lip}_1(X)$ and $f \in \mathrm{Lip}_\alpha(X, E)$, then $hf \in \mathrm{Lip}_\alpha(X, E)$. Similarly, it is true for $\mathrm{lip}_\alpha(X, E)$. That is, both spaces $\mathrm{Lip}_\alpha(X, E)$ and $\mathrm{lip}_\alpha(X, E)$ are $\mathrm{Lip}_1(X)$-modules. Moreover, $\mathrm{Lip}_\alpha(X, E)$ is a $\mathrm{Lip}_\alpha(X)$-module and $\mathrm{lip}_\alpha(X, E)$ is a $\mathrm{lip}_\alpha(X)$-module.

Identifying each $e \in E$ with the constant function $1_e(x) = e$ for $x \in X$, the Banach space E can be considered as a subspace of $\mathrm{Lip}_1(X, E)$. Hence, given $e \in E$ and $f \in \mathrm{Lip}_\alpha(X)$, the function f_e defined by $f_e(x) = f(x)e$ for $x \in X$ belongs to $\mathrm{Lip}_\alpha(X, E)$. Moreover, $\|f_e\|_X = \|f\|_X\|e\|$, $p_\alpha(f_e) = p_\alpha(f)\|e\|$ and hence $\|f_e\|_\alpha = \|e\|\|f\|_\alpha$.

Recall that a weighted composition operator $T : \mathcal{S}(X, E) \to \mathcal{S}(Y, F)$ is of the form $Tf(y) = W_y(f(\varphi(y)))$ where $W_y : E \to F$ is a linear operator for each $y \in Y$ and $\varphi : Y \to X$ is a function. We denote by N the set of all y in Y for which W_y is the zero operator. A map $\varphi : Y \to X$ is said to be a *Lipschitz function* on a subset K of Y, if there exists a constant $c > 0$ such that $d(\varphi(y), \varphi(y')) \leq cd(y, y')$ for all y, y' in K.

We first characterize a bounded weighted composition operator between vector-valued Lipschitz function spaces.

Proposition 2.1. *Let $\mathcal{S}(X, E) \subseteq C(X, E)$ and $\mathcal{S}(Y, F) \subseteq C(Y, F)$ be two Banach spaces such that the topology of pointwise convergence is weaker than their norm topology. Suppose that $T : \mathcal{S}(X, E) \to \mathcal{S}(Y, F)$ is a weighted composition operator of the form $Tf(y) = W_y(f(\varphi(y)))$. If $W_y \in \mathcal{B}(E, F)$ for each $y \in Y$, then T is bounded.*

Proof. Let (f_n) be a sequence in $\mathcal{S}(X, E)$ that converges to zero and (Tf_n) converges to g in $\mathcal{S}(Y, F)$. Then by the hypothesis $f_n(\varphi(y)) \to 0$ and $Tf_n(y) \to g(y)$ for every $y \in Y$. The boundedness of each W_y implies that $W_y(f_n(\varphi(y))) \to 0$. Therefore $g = 0$ and by the closed graph theorem, the map T is bounded. □

Remark 2.2. The definition of Lipschitz norm $\|\cdot\|_\alpha$ asserts that $\|\cdot\|_\alpha$-convergence implies pointwise convergence. Therefore one can conclude Proposition 2.1 for $\mathrm{Lip}_\alpha(X, E)$ and $\mathrm{lip}_\alpha(X, E)$.

Theorem 2.3. *Let $T : \mathrm{Lip}_\alpha(X, E) \to \mathrm{Lip}_\alpha(Y, F)$ be a nonzero bounded weighted composition operator of the form $Tf(y) = W_y(f(\varphi(y)))$. Then $W \in$*

$\mathrm{Lip}_\alpha(Y,\mathcal{B}(E,F))$ *and* φ *is continuous on* $Y\setminus N$ *and Lipschitz on every compact subset of* $Y\setminus N$.

Proof. Let $y\in Y$. Then

$$\|W_y(e)\|=\|T1_e(y)\|\leq\|T1_e\|_Y\leq\|T1_e\|_\alpha\leq\|T\|\|e\|,$$

for all $e\in E$. Hence $W_y\in\mathcal{B}(E,F)$ and $\|W_y\|\leq\|T\|$. We now show that $W\in\mathrm{Lip}_\alpha(Y,\mathcal{B}(E,F))$. To do this, let $y_1,y_2\in Y$ with $y_1\neq y_2$. Then

$$\frac{\|W_{y_1}(e)-W_{y_2}(e)\|}{d^\alpha(y_1,y_2)}=\frac{\|T1_e(y_1)-T1_e(y_2)\|}{d^\alpha(y_1,y_2)}\leq p_\alpha(T1_e)\leq\|T1_e\|_\alpha\leq\|T\|\|e\|,$$

for all $e\in E$. It follows that

$$\frac{\|W_{y_1}-W_{y_2}\|}{d^\alpha(y_1,y_2)}=\sup_{\|e\|\leq1}\frac{\|W_{y_1}(e)-W_{y_2}(e)\|}{d^\alpha(y_1,y_2)}\leq\|T\|,$$

and $W\in\mathrm{Lip}_\alpha(Y,\mathcal{B}(E,F))$ with $p_\alpha(W)\leq\|T\|$.

To show the continuity of φ on $Y\setminus N$, we first prove for given $f\in\mathrm{Lip}_\alpha(X)$, the scalar-valued map $\Psi_f(y,e,u^*)=<Tf_e(y),u^*>$ is continuous on $Y\times E\times F^*$. Fix a nonzero function $f\in\mathrm{Lip}_\alpha(X)$ and $(y_0,e_0,u_0^*)\in Y\times E\times F^*$. Given $\varepsilon>0$, by the continuity of Tf_{e_0} at y_0, there exists a neighborhood U_1 of y_0 in Y such that $\|Tf_{e_0}(y)-Tf_{e_0}(y_0)\|<\frac{\varepsilon}{3(1+\|u_0^*\|)}$ for every $y\in U_1$. If $\delta<\frac{\varepsilon}{3(1+\|u_0^*\|)\|T\|}$, and $U_2:=\{e\in E:\|e-e_0\|<\frac{\delta}{\|f\|_\alpha}\}$, then

$$\|Tf_e-Tf_{e_0}\|_\alpha\leq\|T\|\|f_e-f_{e_0}\|_\alpha\leq\|T\|\|f\|_\alpha\|e-e_0\|<\frac{\varepsilon}{3(1+\|u_0^*\|)},$$

for every $e\in U_2$. If $U=U_1\times U_2\times U_3$ where $U_3=\{u^*\in F^*:\|u^*-u_0^*\|<r\}$ and $r=\frac{\varepsilon}{3}(\frac{\varepsilon}{3(1+\|u_0^*\|)}+\|Tf_{e_0}\|_\alpha)^{-1}$, then U is a neighborhood of (y_0,e_0,u_0^*) and

$$\begin{aligned}|\Psi_f(y,e,u^*)-\Psi_f(y_0,e_0,u_0^*)|=&|<Tf_e(y),u^*>-<Tf_{e_0}(y_0),u_0^*>|\\ \leq&\|Tf_e(y)\|\|u^*-u_0^*\|+\|u_0^*\|\|Tf_e(y)-Tf_{e_0}(y)\|\\ &+\|u_0^*\|\|Tf_{e_0}(y)-Tf_{e_0}(y_0)\|\\ <&\|u^*-u_0^*\|\|Tf_e\|_\alpha+\|u_0^*\|\|Tf_e-Tf_{e_0}\|_\alpha+\frac{\varepsilon}{3}\\ <&(\frac{\varepsilon}{3(1+\|u_0^*\|)}+\|Tf_{e_0}\|_\alpha)\|u^*-u_0^*\|+\frac{2\varepsilon}{3}<\varepsilon,\end{aligned}$$

for all $(y,e,u^*)\in U$.

Using the above result for the constant function $f=1$, we conclude that the map $\Psi_1(y,e,u^*)=<T1_e(y),u^*>$ is continuous on $Y\times E\times F^*$ and therefore,

$$\begin{aligned}\mathrm{coz}(\Psi_1)&=\{(y,e,u^*)\in Y\times E\times F^*:\Psi_1(y,e,u^*)\neq0\}\\ &=\{(y,e,u^*)\in Y\times E\times F^*:<T1_e(y),u^*>=<W_y(e),u^*>\neq0\},\end{aligned}$$

is an open set in $Y\times E\times F^*$. One can write,

$$f(\varphi(y))=\frac{<Tf_e(y),u^*>}{<W_y(e),u^*>}=\frac{<Tf_e(y),u^*>}{<T1_e(y),u^*>}=\frac{\Psi_f(y,e,u^*)}{\Psi_1(y,e,u^*)},$$

for every $(y, e, u^*) \in \mathrm{coz}(\Psi_1)$ and $f \in \mathrm{Lip}_\alpha(X)$. Therefore the continuity of Ψ_f and Ψ_1 implies that the map $(y, e, u^*) \mapsto f(\varphi(y))$ is continuous on $\mathrm{coz}(\Psi_1)$ for given $f \in \mathrm{Lip}_\alpha(X)$.

Let $y \in Y \setminus N$. Then $< W_y(e), u^* > \neq 0$ for some $e \in E$ and $u^* \in F^*$. Hence $(y, e, u^*) \in \mathrm{coz}(\Psi_1)$. Suppose that (y_n) is a sequence in $Y \setminus N$ converging to y. Then, $(y_n, e, u^*) \in \mathrm{coz}(\Psi_1)$ for large enough n. If $\varphi(y_n)$ does not converge to $\varphi(y)$, there exists an open neighborhood V of $\varphi(y)$ in X and a subsequence (y_{n_k}) of (y_n) such that $\varphi(y_{n_k}) \notin V$ for each k. On the other hand, considering the function $f(x) = \mathrm{dist}(x, X \setminus V)$, we note that $f \in \mathrm{Lip}_\alpha(X)$, then by the above discussion the map $(y, e, u^*) \mapsto f(\varphi(y))$ is continuous on $\mathrm{coz}(\Psi_1)$. Hence $f(\varphi(y_{n_k})) \to f(\varphi(y))$ as $k \to \infty$. However, by the definition of f, $f(\varphi(y_{n_k})) = 0$ for each k and $f(\varphi(y)) \neq 0$ which is a contradiction. Therefore φ is continuous on $Y \setminus N$.

To finish the proof, it remains to be shown that the function φ is Lipschitz on every compact subset of $Y \setminus N$. Let K be a compact subset of $Y \setminus N$. For each $y \in Y$, define $f_y(x) = d^\alpha(x, \varphi(y))$ for all $x \in X$. Let $e \in E$ be with $\|e\| = 1$. Therefore, $(f_y)_e \in \mathrm{Lip}_\alpha(X, E)$ and $\|(f_y)_e\|_\alpha \leq 1 + (\mathrm{diam}(X))^\alpha$ for every $y \in Y$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Then

$$\begin{aligned}\frac{d^\alpha(\varphi(y_1), \varphi(y_2))}{d^\alpha(y_1, y_2)} \|W_{y_2}(e)\| &= \frac{\|T(f_{y_1})_e(y_1) - T(f_{y_1})_e(y_2)\|}{d^\alpha(y_1, y_2)} \\ &\leq p_\alpha(T(f_{y_1})_e) \leq \|T(f_{y_1})_e\|_\alpha \leq c\|T\|,\end{aligned}$$

for every $e \in E$ with $\|e\| = 1$ where $c = 1 + (\mathrm{diam}(X))^\alpha$. Therefore,

$$\frac{d^\alpha(\varphi(y_1), \varphi(y_2))}{d^\alpha(y_1, y_2)} \|W_{y_2}\| = \frac{d^\alpha(\varphi(y_1), \varphi(y_2))}{d^\alpha(y_1, y_2)} \sup_{\|e\|=1} \|W_{y_2}(e)\| \leq c\|T\|.$$

The continuity of $y \mapsto \|W_y\|$ implies that $\gamma = \inf\{\|W_y\| : y \in K\} > 0$. Then

$$\frac{d(\varphi(y_1), \varphi(y_2))}{d(y_1, y_2)} \leq \left(\frac{c\|T\|}{\gamma}\right)^{1/\alpha},$$

for every $y_1, y_2 \in K$ with $y_1 \neq y_2$ which implies that φ is Lipschitz on K and the proof of theorem is complete. ☐

Using the same argument as in the proof of Theorem 2.3, we can obtain the similar result for $\mathrm{lip}_\alpha(X, E)$.

Theorem 2.4. *Let $\alpha \in (0, 1)$ and let $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ be a nonzero bounded weighted composition operator of the form $Tf(y) = W_y(f(\varphi(y)))$. Then $W \in \mathrm{Lip}_\alpha(Y, \mathcal{B}(E, F))$ and φ is continuous on $Y \setminus N$ and Lipschitz on every compact subset of $Y \setminus N$.*

Proof. Exactly the same as the proof of Theorem 2.3, one can show that $W \in \mathrm{Lip}_\alpha(Y, \mathcal{B}(E, F))$ and φ is continuous on $Y \setminus N$. To show that φ is Lipschitz on every compact subset K of $Y \setminus N$, we employ the function

$$f_{y_1,y_2}(x) = (d(x, \varphi(y_2)) + d(\varphi(y_1), \varphi(y_2)))^\alpha - d^\alpha(\varphi(y_1), \varphi(y_2)) \quad (x \in X, y_1, y_2 \in Y).$$

In general, if $a > 0$, the function $g(t) = (t+a)^\alpha - a^\alpha$, $t \geq 0$ has bounded derivative. Then for any $b > 0$, $g \in \mathrm{Lip}_1([0, b]) \subseteq \mathrm{lip}_\alpha([0, b])$. Fix $y_1, y_2 \in Y$ with $y_1 \neq y_2$

and set $a = d(\varphi(y_1), \varphi(y_2))$ and $b = \operatorname{diam}(X)$. Then $f_{y_1,y_2}(x) = g(d(x, \varphi(y_2)))$ for $x \in X$. Therefore, $f_{y_1,y_2} \in \operatorname{Lip}_1(X) \subseteq \operatorname{lip}_\alpha(X)$, $\|f_{y_1,y_2}\|_\alpha \le 1 + (2\operatorname{diam}(X))^\alpha$, $f_{y_1,y_2}(\varphi(y_2)) = 0$ and $f_{y_1,y_2}(\varphi(y_1)) = (2^\alpha - 1)d^\alpha(\varphi(y_1), \varphi(y_2))$.

Let $e \in E$ with $\|e\| = 1$. Therefore, we have $(f_{y_1,y_2})_e \in \operatorname{lip}_\alpha(X, E)$ and

$$\begin{aligned}\frac{d^\alpha(\varphi(y_1), \varphi(y_2))}{d^\alpha(y_1, y_2)}\|W_{y_1}(e)\| &= \frac{1}{2^\alpha - 1}\frac{\|f_{y_1,y_2}(\varphi(y_1))W_{y_1}(e)\|}{d^\alpha(y_1, y_2)} \\ &= \frac{1}{2^\alpha - 1}\frac{\|T(f_{y_1,y_2})_e(y_1) - T(f_{y_1,y_2})_e(y_2)\|}{d^\alpha(y_1, y_2)} \\ &\le \frac{1}{2^\alpha - 1}p_\alpha(T(f_{y_1,y_2})_e) \le \frac{1}{2^\alpha - 1}\|T(f_{y_1,y_2})_e\|_\alpha \le c\|T\|,\end{aligned}$$

where $c = \frac{1}{2^\alpha - 1}(1 + (2\operatorname{diam}(X))^\alpha)$. Therefore,

$$\frac{d^\alpha(\varphi(y_1), \varphi(y_2))}{d^\alpha(y_1, y_2)}\|W_{y_1}\| = \frac{d^\alpha(\varphi(y_1), \varphi(y_2))}{d^\alpha(y_1, y_2)}\sup_{\|e\|=1}\|W_{y_1}(e)\| \le c\|T\|.$$

The continuity of $y \mapsto \|W_y\|$ implies that $\gamma = \inf\{\|W_y\| : y \in K\} > 0$. Then

$$\frac{d(\varphi(y_1), \varphi(y_2))}{d(y_1, y_2)} \le \left(\frac{c\|T\|}{\gamma}\right)^{1/\alpha},$$

for every $y_1, y_2 \in K$ with $y_1 \ne y_2$ which implies that φ is Lipschitz on K. □

The following example shows that in Theorems 2.3 and 2.4, φ is not necessarily Lipschitz on $Y \setminus N$.

Example 2.5. Suppose $X = [0, \sqrt{2}]$, $Y = [-1, 1]$ and E is an arbitrary Banach space. Define $W_y(e) = (1 + y)e$ and $\varphi(y) = \sqrt{1 + y}$ for every $y \in [-1, 1]$ and $e \in E$. Then $N = \{-1\}$. Let $T : \operatorname{Lip}_\alpha(X, E) \to \operatorname{Lip}_\alpha(Y, E)$ be the weighted composition operator induced by W and φ. Clearly, φ is continuous on $[-1, 1]$ and it is not Lipschitz on $(-1, 1]$. However, φ is Lipschitz on $[-1 + \delta, 1]$ for each $\delta \in (0, 2)$. Moreover, we have $Tf(y) = (1+y)f(\sqrt{1 + y})$ for every $f \in \operatorname{Lip}_\alpha(X, E)$ and $y \in [-1, 1]$ which is bounded.

To investigate the compactness of weighted composition operators between spaces of vector-valued Lipschitz functions, we need the following results. The first one is the generalized Arzela-Ascoli theorem for vector-valued continuous functions.

Theorem 2.6. [4, Theorem A] *A subset H of $C(X, E)$ is relatively compact if and only if the following conditions are satisfied:*

(i) *H is equicontinuous, and*
(ii) *$H(x) = \{f(x) : f \in H\}$ is relatively compact for every $x \in X$.*

The following definition and theorem are provided in [15].

Definition 2.7. A map $\varphi : Y \to X$ will be called a *supercontraction* on $K \subseteq Y$ if $\frac{d(\varphi(y),\varphi(y'))}{d(y,y')} \to 0$ whenever $y, y' \in K$ and $d(y, y') \to 0$

Theorem 2.8. [15, Theorem 1] *Let $T : \mathrm{Lip}_\alpha(X) \to \mathrm{Lip}_\alpha(Y)$ be a composition operator of the form $Tf(y) = f(\varphi(y))$ for every $f \in \mathrm{Lip}_\alpha(X)$ and $y \in Y$ where $\varphi : Y \to X$. Then T is compact if and only if φ is a supercontraction.*

It was established in [3] that $\mathrm{lip}_\alpha(X)^{**} = \mathrm{Lip}_\alpha(X)$ for $0 < \alpha < 1$. As mentioned in [15, page 260], using this fact the following corollary follows from the above theorem.

Corollary 2.9. *Let $\alpha \in (0, 1)$ and let $T : \mathrm{lip}_\alpha(X) \to \mathrm{lip}_\alpha(Y)$ be a composition operator of the form $Tf(y) = f(\varphi(y))$ for every $f \in \mathrm{lip}_\alpha(X)$ and $y \in Y$ where $\varphi : Y \to X$. Then T is compact if and only if φ is a supercontraction.*

Theorem 2.8 gives motivation to the following theorem which is our main result of this section.

Theorem 2.10. *Let $T : \mathrm{Lip}_\alpha(X, E) \to \mathrm{Lip}_\alpha(Y, F)$ be a nonzero weighted composition operator of the form $Tf(y) = W_y(f(\varphi(y)))$.*

(i) *If T is compact, then $W \in \mathrm{Lip}_\alpha(Y, \mathcal{K}(E, F))$, and φ is continuous on $Y \setminus N$ and a supercontraction on every compact subset of $Y \setminus N$.*

(ii) *If $W \in \mathrm{lip}_\alpha(Y, \mathcal{K}(E, F))$ and φ is a supercontraction on $Y \setminus N$, then T is compact.*

Proof. (i) By Theorem 2.3, $W \in \mathrm{Lip}_\alpha(Y, \mathcal{B}(E, F))$. Thus it is enough to show that $W_y \in \mathcal{K}(E, F)$ for each $y \in Y$. To do this, let (e_n) be a bounded sequence in E. The sequence (1_{e_n}) is bounded in $\mathrm{Lip}_\alpha(X, E)$. Since T is compact, there exists a subsequence (e_{n_k}) such that $(T1_{e_{n_k}})$ converges in $\mathrm{Lip}_\alpha(Y, F)$. In particular, $(W_y(e_{n_k}))$ converges in F that is $W_y \in \mathcal{K}(E, F)$ for each $y \in Y$.

By Theorem 2.3, φ is continuous on $Y \setminus N$. To show that φ is a supercontraction on every compact subset of $Y \setminus N$, we first fix a point $y_0 \in Y \setminus N$ and $e \in E$ with $\|e\| = 1$ such that $W_{y_0}(e) \neq 0$. The continuity of the function $y \mapsto W_y(e)$ from Y into F implies that, there exists $\delta > 0$ such that $\|W_y(e)\| > 0$ for every $y \in Y$ with $d(y, y_0) \leq \delta$. Consider the compact subset $\Delta = \overline{B(y_0, \delta)}$ of Y where $B(y_0, \delta) = \{y \in Y : d(y, y_0) < \delta\}$. Then $\gamma = \inf_{y \in \Delta} \|W_y(e)\| > 0$. Define $S : \mathrm{Lip}_\alpha(X) \to \mathrm{Lip}_\alpha(\Delta)$ by $Sf = f \circ \varphi$ on Δ. We show that S is compact. To do this, let (f_n) be a sequence in $\mathrm{Lip}_\alpha(X)$ such that $\|f_n\|_\alpha = 1$. Thus $(g_n) = ((f_n)_e)$ is a bounded sequence in $\mathrm{Lip}_\alpha(X, E)$. By the compactness of T, there exists a subsequence (g_{n_k}) of (g_n) such that (Tg_{n_k}) converges and hence it is a Cauchy sequence in $\mathrm{Lip}_\alpha(Y, F)$. By the definition of S, we have

$$\begin{aligned} |Sf_{n_k}(y) - Sf_{n_l}(y)| &= |f_{n_k}(\varphi(y)) - f_{n_l}(\varphi(y))| \\ &= \frac{\|f_{n_k}(\varphi(y))W_y(e) - f_{n_l}(\varphi(y))W_y(e)\|}{\|W_y(e)\|} \\ &\leq \frac{\|Tg_{n_k}(y) - Tg_{n_l}(y)\|}{\gamma} \leq \frac{\|Tg_{n_k} - Tg_{n_l}\|_Y}{\gamma}, \end{aligned}$$

for every $y \in \Delta$. Hence

$$\|Sf_{n_k} - Sf_{n_l}\|_\Delta \leq \frac{1}{\gamma}\|Tg_{n_k} - Tg_{n_l}\|_Y. \tag{2.1}$$

Also, using the inequality $p_\alpha(W) \leq \|T\|$, we obtain

$$\begin{aligned}
&\frac{|S(f_{n_k}-f_{n_l})(y)-S(f_{n_k}-f_{n_l})(y')|}{d^\alpha(y,y')} \\
&\quad= \frac{\|(f_{n_k}-f_{n_l})(\varphi(y))W_y(e)-(f_{n_k}-f_{n_l})(\varphi(y'))W_y(e)\|}{\|W_y(e)\|d^\alpha(y,y')} \\
&\quad\leq \frac{\|T(g_{n_k}-g_{n_l})(y)-T(g_{n_k}-g_{n_l})(y')\|}{\|W_y(e)\|d^\alpha(y,y')} + \frac{\|(W_y-W_{y'})(e)\|}{\|W_y(e)\|d^\alpha(y,y')}|(f_{n_k}-f_{n_l})(\varphi(y'))| \\
&\quad\leq \frac{1}{\gamma}p_\alpha(T(g_{n_k}-g_{n_l})) + \frac{1}{\gamma}p_\alpha(W)\|Sf_{n_k}-Sf_{n_l}\|_\Delta \\
&\quad\leq \frac{1}{\gamma}p_\alpha(T(g_{n_k}-g_{n_l})) + \frac{1}{\gamma^2}\|T\|\|Tg_{n_k}-Tg_{n_l}\|_Y,
\end{aligned}$$

for every $y, y' \in \Delta$ with $y \neq y'$. Therefore,

$$p_\alpha(Sf_{n_k}-Sf_{n_l}) \leq \frac{1}{\gamma}p_\alpha(T(g_{n_k}-g_{n_l})) + \frac{1}{\gamma^2}\|T\|\|Tg_{n_k}-Tg_{n_l}\|_Y. \tag{2.2}$$

From (2.1) and (2.2), we conclude that (Sf_{n_k}) is a Cauchy and then a convergent sequence in $\mathrm{Lip}_\alpha(\Delta)$. It implies that S is compact. By Theorem 2.8, $\varphi_{|\Delta}$ is a supercontraction.

We have shown that for every $y \in Y \setminus N$ there exists some $\delta > 0$ such that φ is a supercontraction on $\overline{B(y,\delta)}$. We now assume that K is a compact subset of $Y \setminus N$. Then there exist $y_1, \ldots, y_n$ in K and positive numbers $\delta_1, \ldots, \delta_n$ such that $\{B(y_i, \frac{\delta_i}{2})\}_{i=1}^n$ covers K and φ is a supercontraction on each $\overline{B(y_i,\delta_i)}$. Then for any $\varepsilon > 0$, one can choose a positive number $\delta < \min\{\frac{\delta_1}{2}, \ldots, \frac{\delta_n}{2}\}$ such that $\frac{d(\varphi(y),\varphi(y'))}{d(y,y')} < \varepsilon$ for every $y, y' \in K$ with $0 < d(y,y') < \delta$. This completes the proof of (i).

(ii) Using Proposition 2.1 and Remark 2.2, one can say that T is bounded. For compactness of T, we assume that (f_n) is a bounded sequence in $\mathrm{Lip}_\alpha(X,E)$ with $\|f_n\|_\alpha \leq 1$. Then boundedness of T and the fact that $Tf_n \in \mathrm{Lip}_\alpha(Y,F)$ for all $n \in \mathbb{N}$ imply that (Tf_n) is an equicontinuous subset of $C(Y,F)$, and compactness of each W_y $(y \in Y)$ implies that $(Tf_n(y))$ is relatively compact for each $y \in Y$. Therefore, by Theorem 2.6, (Tf_n) is relatively compact in $C(Y,F)$. Hence there exists a subsequence (f_{n_k}) of (f_n) such that (Tf_{n_k}) converges in $C(Y,F)$ and then it is a Cauchy sequence in $C(Y,F)$. We will show that (Tf_{n_k}) is a Cauchy sequence in $\mathrm{Lip}_\alpha(Y,F)$. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\frac{d(\varphi(y),\varphi(y'))}{d(y,y')} < \left(\frac{\varepsilon}{3\|W\|_\alpha}\right)^{1/\alpha}, \tag{2.3}$$

for every $y, y' \in Y \setminus N$ with $0 < d(y,y') < \delta$,

$$\frac{\|W_y - W_{y'}\|}{d^\alpha(y,y')} < \frac{\varepsilon}{6}, \tag{2.4}$$

for every $y, y' \in Y$ with $0 < d(y, y') < \delta$, and

$$\sup_{y\in Y} \|Tf_{n_k}(y) - Tf_{n_l}(y)\| < \frac{1}{2}\delta^\alpha \varepsilon, \tag{2.5}$$

for large enough k, l, since φ is a supercontraction on $Y \setminus N$, the weight W is in $\mathrm{lip}_\alpha(Y, \mathcal{K}(E, F))$ and (Tf_{n_k}) is a Cauchy sequence in $C(Y, F)$, respectively.

Let $y, y' \in Y$ and k, l be large enough. We consider three cases.

Case 1. If $d(y, y') \geq \delta$, using (2.5) we obtain

$$\frac{\|Tf_{n_k}(y) - Tf_{n_l}(y) - Tf_{n_k}(y') + Tf_{n_l}(y')\|}{d^\alpha(y, y')} \leq \frac{2}{\delta^\alpha} \sup_{y\in Y} \|Tf_{n_k}(y) - Tf_{n_l}(y)\| < \varepsilon.$$

Case 2. If $y, y' \in Y \setminus N$ with $0 < d(y, y') < \delta$ and $\varphi(y) \neq \varphi(y')$, then applying (2.3) and (2.4) we get

$$\begin{aligned}
&\frac{\|Tf_{n_k}(y) - Tf_{n_l}(y) - Tf_{n_k}(y') + Tf_{n_l}(y')\|}{d^\alpha(y, y')} \\
=&\frac{\|W_y(f_{n_k}(\varphi(y))) - W_y(f_{n_l}(\varphi(y))) - W_{y'}(f_{n_k}(\varphi(y'))) + W_{y'}(f_{n_l}(\varphi(y')))\|}{d^\alpha(y, y')} \\
\leq&\frac{\|W_y\|\|f_{n_k}(\varphi(y)) - f_{n_k}(\varphi(y'))\|}{d^\alpha(y, y')} + \frac{\|W_y - W_{y'}\|\|f_{n_k}(\varphi(y')) - f_{n_l}(\varphi(y'))\|}{d^\alpha(y, y')} \\
&+ \frac{\|W_y\|\|f_{n_l}(\varphi(y')) - f_{n_l}(\varphi(y))\|}{d^\alpha(y, y')} \\
\leq&\|W\|_Y p_\alpha(f_{n_k}) \frac{d^\alpha(\varphi(y), \varphi(y'))}{d^\alpha(y, y')} + \frac{\varepsilon}{6}(\|f_{n_k}\|_X + \|f_{n_l}\|_X) \\
&+ \|W\|_Y p_\alpha(f_{n_l}) \frac{d^\alpha(\varphi(y), \varphi(y'))}{d^\alpha(y, y')} \\
<&2\|W\|_\alpha \cdot \frac{\varepsilon}{3\|W\|_\alpha} + 2\frac{\varepsilon}{6} = \varepsilon.
\end{aligned}$$

Case 3. If $y \in Y \setminus N$ and $y' \in N$ with $0 < d(y, y') < \delta$, applying (2.4), we obtain

$$\begin{aligned}
\frac{\|Tf_{n_k}(y) - Tf_{n_l}(y) - Tf_{n_k}(y') + Tf_{n_l}(y')\|}{d^\alpha(y, y')} =&\frac{\|Tf_{n_k}(y) - Tf_{n_l}(y)\|}{d^\alpha(y, y')} \\
\leq&\frac{\|W_y\|\|f_{n_k}(\varphi(y)) - f_{n_l}(\varphi(y))\|}{d^\alpha(y, y')} \\
\leq&2\frac{\|W_y - W_{y'}\|}{d^\alpha(y, y')} < \frac{\varepsilon}{3} < \varepsilon.
\end{aligned}$$

Therefore, (Tf_{n_k}) is a Cauchy sequence in $\mathrm{Lip}_\alpha(Y, F)$ from which we have T is compact. □

Using Corollary 2.9, similar to the proof of Theorem 2.10, one can conclude the following results for vector-valued little Lipschitz function spaces $\mathrm{lip}_\alpha(X, E)$.

Theorem 2.11. *Let $\alpha \in (0, 1)$ and let $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ be a nonzero weighted composition operator of the form $Tf(y) = W_y(f(\varphi(y)))$.*

(i) *If T is compact, then $W \in \mathrm{Lip}_\alpha(Y, \mathcal{K}(E,F))$ and φ is a supercontraction on every compact subset of $Y \setminus N$.*
(ii) *If $W \in \mathrm{lip}_\alpha(Y, \mathcal{K}(E,F))$ and φ is a supercontraction on $Y \setminus N$, then T is compact.*

3. The separating operators from $\mathrm{lip}_\alpha(X,E)$ into $\mathrm{lip}_\alpha(Y,F)$

In this section, we characterize the representation of bounded separating linear operators between spaces of vector-valued little Lipschitz functions on compact metric spaces. For this, we need the following result which is a modified version of Theorem 2.2 in [10]. In fact, by imposing an extra assumption on T to be bounded, we provide a simpler statement of the mentioned theorem that serves our purpose in the present paper.

In what follows, we will assume that X and Y are compact metric spaces, E and F are Banach spaces and $\alpha \in (0,1)$. For each $y \in Y$, let δ_y be the linear functional on $\mathrm{lip}_\alpha(Y)$ defined by $\delta_y(f) = f(y)$.

Theorem 3.1. [10, Theorem 2.2] *Let $\alpha \in (0,1)$ and let T be a bounded separating linear operator from $\mathrm{lip}_\alpha(X)$ into $\mathrm{lip}_\alpha(Y)$. If $Y_0 = \{y \in Y : \delta_y \circ T = 0\}$ and Y_0^c is the complement of Y_0, then there exist a continuous map $\varphi : Y_0^c \to X$, and a non-vanishing function $h \in \mathrm{lip}_\alpha(Y_0^c)$ such that $Tf(y) = h(y)f(\varphi(y))$ for every $y \in Y_0^c$ and $f \in \mathrm{lip}_\alpha(X)$, and $Tf(y) = 0$ for every $y \in Y_0$ and $f \in \mathrm{lip}_\alpha(X)$.*

First, using Theorem 3.1 and a similar argument as in [7, Theorem 1], we show that a bounded separating linear operator $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ is a weighted composition operator on the $\mathrm{span}\{f_e : f \in \mathrm{lip}_\alpha(X), e \in E\}$. Then we generalized this fact.

Theorem 3.2. *Let $\alpha \in (0,1)$ and let $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ be a nonzero bounded separating linear operator. Then there exist a function $W : Y \to \mathcal{B}(E,F)$ and a map $\varphi : Y \to X$ continuous on $Y \setminus N$ such that*

$$Tf_e(y) = W_y(f_e(\varphi(y)))$$

for every $f \in \mathrm{lip}_\alpha(X)$, $e \in E$ and $y \in Y$ where $N = \{y \in Y : W_y = 0\}$ is the kernel of W. Moreover, $W \in \mathrm{Lip}_\alpha(Y, \mathcal{B}(E,F))$ and φ is Lipschitz on every compact subset of $Y \setminus N$.

Proof. Given a fixed $e \in E$ and $u^* \in F^*$, consider the map $T_{e,u^*} : \mathrm{lip}_\alpha(X) \to \mathrm{lip}_\alpha(Y)$ by $T_{e,u^*}f(y) =< Tf_e(y), u^* >$. Then T_{e,u^*} is a bounded linear map and $\|T_{e,u^*}\| \le 2\|u^*\|\|T\|\|e\|$. We show that T_{e,u^*} is a separating map. To do this, let $f, g \in \mathrm{lip}_\alpha(X)$ with $fg = 0$ on X. Then $\|f_e(x)\|\|g_e(x)\| = |f(x)||g(x)|\|e\|^2 = 0$ for every $x \in X$. Hence, due to the separating property of T, $\|Tf_e(y)\|\|Tg_e(y)\| = 0$ for every $y \in Y$. Therefore,

$$\begin{aligned} |T_{e,u^*}f(y)T_{e,u^*}g(y)| &= | < Tf_e(y), u^* > || < Tg_e(y), u^* > | \\ &\le \|u^*\|^2\|Tf_e(y)\|\|Tg_e(y)\| = 0, \end{aligned}$$

for every $y \in Y$. That is, T_{e,u^*} is a bounded separating linear map from $\mathrm{lip}_\alpha(X)$ into $\mathrm{lip}_\alpha(Y)$. Thus, Theorem 3.1 ensures the existence of a map $\varphi_{e,u^*} : Y_0^c \to X$ and a non-vanishing function $h_{e,u^*} : Y_0^c \to \mathbb{C}$ with $T_{e,u^*}f(y) = h_{e,u^*}(y)f(\varphi_{e,u^*}(y))$

for every $y \in Y_0^c$ and $f \in \mathrm{lip}_\alpha(X)$, and $T_{e,u^*}f = 0$ on Y_0 for every $f \in \mathrm{lip}_\alpha(X)$. Note that Y_0 depends on e and u^*. In fact, $Y_0 = \{y \in Y : \delta_y \circ T_{e,u^*} = 0\}$. Extending h_{e,u^*} to Y by defining zero on Y_0, one can say $T_{e,u^*}f(y) = h_{e,u^*}(y)f(\varphi_{e,u^*}(y))$ for every $f \in \mathrm{lip}_\alpha(X)$, $y \in Y$ and any extension of φ_{e,u^*} to Y. Take $f = 1$, the constant function in $\mathrm{lip}_\alpha(X)$. Then $h_{e,u^*}(y) = T_{e,u^*}1(y) =< T1_e(y), u^* >$ for every $y \in Y$. We now define $W : Y \to \mathcal{B}(E, F)$, by $W_y(e) = T1_e(y)$ for every $y \in Y$ and $e \in E$. It is easy to see that W is well-defined, $\|W_y\| \leq \|T\|$ for every $y \in Y$. Similar to the proof of Theorem 2.3, $W \in \mathrm{Lip}_\alpha(Y, \mathcal{B}(E, F))$. By the definition of W, one can write $h_{e,u^*}(y) =< W_y(e), u^* >$ and then

$$\begin{aligned} < Tf_e(y), u^* > &=< W_y(e), u^* > f(\varphi_{e,u^*}(y)) \\ &=< f(\varphi_{e,u^*}(y))W_y(e), u^* >=< W_y(f_e(\varphi_{e,u^*}(y))), u^* >, \end{aligned}$$

for every $(e, u^*, y) \in E \times F^* \times Y$ and $f \in \mathrm{lip}_\alpha(X)$.

We now show that φ_{e,u^*} is independent of e and u^* on $Y \setminus N$ where $N = \{y \in Y : W_y = 0\}$. Given $y_0 \in Y \setminus N$, choose $e \in E$ whit $W_{y_0}(e) = T1_e(y_0) \neq 0$. Then there exists $u^* \in F^*$ such that $< T1_e(y_0), u^* >\neq 0$. We claim that for every t^* in F^* with $< T1_e(y_0), t^* >\neq 0$, we have $\varphi_{e,u^*}(y_0) = \varphi_{e,t^*}(y_0)$. To prove this claim, let $x_1 = \varphi_{e,u^*}(y_0)$ and $x_2 = \varphi_{e,t^*}(y_0)$. If $x_1 \neq x_2$, then there exist open neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$. Define $f_i(x) = \mathrm{dist}(x, X \setminus U_i)$ and note that $f_i \in \mathrm{lip}_\alpha(X)$ and $f_i(x_i) \neq 0$ for each $i = 1, 2$. Moreover, $f_1(x)f_2(x) = 0$ for every $x \in X$. Hence,

$$\|(f_1)_e(x)\|\|(f_2)_e(x)\| = |f_1(x)||f_2(x)|\|e\|^2 = 0,$$

for all $x \in X$. Thus, by the separating property of T,

$$\|T(f_1)_e(y)\|\|T(f_2)_e(y)\| = 0,$$

for all $y \in Y$. On the other hand,

$$< T(f_1)_e(y_0), u^* >=< T1_e(y_0), u^* > f_1(\varphi_{e,u^*}(y_0)) =< T1_e(y_0), u^* > f_1(x_1) \neq 0,$$

and

$$< T(f_2)_e(y_0), t^* >=< T1_e(y_0), t^* > f_2(\varphi_{e,t^*}(y_0)) =< T1_e(y_0), t^* > f_2(x_2) \neq 0.$$

Therefore,

$$\begin{aligned} 0 &< | < T(f_1)_e(y_0), u^* > || < T(f_2)_e(y_0), t^* > | \\ &\leq \|u^*\|\|t^*\|\|T(f_1)_e(y_0)\|\|T(f_2)_e(y_0)\| = 0, \end{aligned}$$

which is a contradiction. Hence if $W_y(e) = (T1_e)(y) \neq 0$, then $< T1_e(y), u^* >\neq 0$ for some $u^* \in F^*$ and $\varphi_{e,u^*}(y) = \varphi_{e,t^*}(y)$ for every t^* in $\{t^* \in F^* :< T1_e(y), t^* >\neq 0\}$. Thus, for a fixed $e \in E$, one can define φ_e on $\{y \in Y : W_y(e) = T1_e(y) \neq 0\}$ by $\varphi_e(y) = \varphi_{e,u^*}(y)$ where $< T1_e(y), u^* >\neq 0$. Similarly, one can show that $\varphi_e(y)$ does not depend on e for every $y \in Y \setminus N$, that is, if $y \in Y$ and $e_1, e_2 \in E$ such that $W_y(e_1) \neq 0$ and $W_y(e_2) \neq 0$, then $\varphi_{e_1}(y) = \varphi_{e_2}(y)$. We are then able to define the function $\varphi : Y \setminus N \to X$ by $\varphi(y) = \varphi_e(y)$ where $W_y(e) = T1_e(y) \neq 0$ for some $e \in E$. Hence, $Tf_e(y) = W_y(f_e(\varphi(y)))$ if $y \in Y \setminus N$ and $Tf_e(y) = 0$ if $y \in N$, for every $f \in \mathrm{lip}_\alpha(X)$ and $e \in E$. Therefore, one can write $Tf_e(y) = W_y(f_e(\varphi(y)))$ for every $y \in Y$, $f \in \mathrm{lip}_\alpha(X)$, $e \in E$ and any extension of φ to Y.

Similar to the proof of Theorem 2.3, one can show that φ is continuous on $Y \setminus N$ and Lipschitz on every compact subset of $Y \setminus N$. □

As an immediate application of Theorem 3.2, we provide the following result which characterizes the general form of a bounded separating linear operator $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ under certain conditions on X and E.

Corollary 3.3. *Let X be a compact metric space, E be a Banach space and $\alpha \in (0,1)$. Suppose that the linear span of $\{f_e : f \in \mathrm{lip}_\alpha(X), e \in E\}$ is dense in $\mathrm{lip}_\alpha(X,E)$. Then every bounded separating linear operator $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ is a weighted composition operator $Tf(y) = W_y(f(\varphi(y)))$ for every $f \in \mathrm{lip}_\alpha(X,E)$ and $y \in Y$, where φ and W are the same as found in Theorem 3.2.*

Note that the density of the linear span of $\{f_e : f \in \mathrm{lip}_\alpha(X), e \in E\}$ in $\mathrm{lip}_\alpha(X,E)$ is not very restrictive and there are many cases with such property. For instance, as follows from [13, page 167 and Cor. 5.17], if X is an infinite compact set in $\mathbb{R}^n$ and E is a dual space of a Banach space, then the linear span of $\{f_e : f \in \mathrm{lip}_\alpha(X), e \in E\}$ is dense in $\mathrm{lip}_\alpha(X,E)$.

To characterize a bounded separating linear operator $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ in the general case, we need the following lemma.

Lemma 3.4. *Let $z \in X$ and set*

$$J_z = \{f \in \mathrm{lip}_\alpha(X,E) : z \notin \overline{\mathrm{coz}(f)} = \mathrm{supp}(f)\},$$

$$M_z = \{f \in \mathrm{lip}_\alpha(X,E) : f(z) = 0\}.$$

Then J_z is a dense subspace of M_z.

Proof. Let $f \in M_z$ and $\varepsilon > 0$. By the definition of $\mathrm{lip}_\alpha(X,E)$, there exists $\delta \in (0,1)$ such that $\|f(x_1)-f(x_2)\| < \varepsilon d^\alpha(x_1,x_2)$ for every $x_1, x_2 \in X$ with $d(x_1,x_2) < \delta$. Let $U = B(z, \frac{\delta}{4})$ and $V = B(z, \frac{\delta}{2})$. Define $h(x) = \min\{\frac{4}{\delta}\,\mathrm{dist}(x,U), 1\}$ and $g = hf$. Note that $h \in \mathrm{Lip}_1(X)$, $0 \le h \le 1$, $h = 0$ on U, $h = 1$ on $X \setminus V$ and $\|1-h\|_X = 1$. Then $g \in \mathrm{lip}_\alpha(X,E)$ since $\mathrm{lip}_\alpha(X,E)$ is a $\mathrm{Lip}_1(X)$-module, and $g \in J_z$ since $g = 0$ on U. Also $f = g$ on $X \setminus V$ and

$$\|f(x)\| = \|f(x) - f(z)\| < \varepsilon d^\alpha(x,z) < \varepsilon(\frac{\delta}{2})^\alpha,$$

for every $x \in V$. Therefore,

$$\|f-g\|_X = \sup_{x\in V} \|f(x) - g(x)\| = \sup_{x \in V} \|f(x)\||1-h(x)| < \varepsilon(\frac{\delta}{2})^\alpha < \varepsilon.$$

Next we show that $p_\alpha(f-g) \le 5\varepsilon$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$.

If $x_1, x_2 \in V$, then by [19, Proposition 1.5.5], we have

$$\frac{|h(x_1)-h(x_2)|}{d^\alpha(x_1,x_2)} \le p_1(h)(\mathrm{diam}(V))^{1-\alpha} \le \frac{4}{\delta}\delta^{1-\alpha} = \frac{4}{\delta^\alpha},$$

which implies

$$\begin{aligned}\frac{\|(f-g)(x_1)-(f-g)(x_2)\|}{d^\alpha(x_1,x_2)} \leq & \frac{\|f(x_1)\||h(x_1)-h(x_2)|}{d^\alpha(x_1,x_2)} \\ & + \frac{|1-h(x_2)|\|f(x_1)-f(x_2)\|}{d^\alpha(x_1,x_2)} \\ < & \varepsilon(\frac{\delta}{2})^\alpha \cdot \frac{4}{\delta^\alpha} + \varepsilon < 5\varepsilon.\end{aligned}$$

If $x_1 \in X \setminus V$ and $x_2 \in V$, then $d(x_1, x_2) \geq \frac{\delta}{2} - d(x_2, z)$ and

$$\begin{aligned}\frac{\|(f-g)(x_1)-(f-g)(x_2)\|}{d^\alpha(x_1,x_2)} = & \frac{\|f(x_2)\|(1-\frac{4}{\delta}\operatorname{dist}(x_2,U))}{d^\alpha(x_1,x_2)} \\ \leq & \varepsilon(\frac{\delta}{2})^\alpha \frac{2-\frac{4}{\delta}d(x_2,z)}{(\frac{\delta}{2}-d(x_2,z))^\alpha} \\ = & 2\varepsilon(\frac{\delta-2d(x_2,z)}{\delta})^{1-\alpha} \leq 2\varepsilon.\end{aligned}$$

Therefore, $p_\alpha(f-g) \leq 5\varepsilon$. Thus $\|f-g\|_\alpha = \|f-g\|_X + p_\alpha(f-g) \leq 6\varepsilon$. □

Theorem 3.5. *Let $\alpha \in (0,1)$ and let $T : \operatorname{lip}_\alpha(X,E) \to \operatorname{lip}_\alpha(Y,F)$ be a nonzero bounded separating linear operator. Then T is a weighted composition operator $Tf(y) = W_y(f(\varphi(y)))$ for every $f \in \operatorname{lip}_\alpha(X,E)$ and $y \in Y \setminus N$, where φ and W are defined as in Theorem 3.2.*

Proof. In the proof of Theorem 3.2, we have shown that $Tf_e(y) = W_y(f_e(\varphi(y)))$ for every $f \in \operatorname{lip}_\alpha(X)$, $e \in E$ and $y \in Y$. We now show that this still holds for every $f \in \operatorname{lip}_\alpha(X,E)$ and $y \in Y \setminus N$.

Fix $y \in Y \setminus N$. Let $f \in J_{\varphi(y)}$. Then there exists $\delta > 0$ with $B(\varphi(y),\delta) \cap \operatorname{coz}(f) = \emptyset$. We choose $e \in E$ with $\|e\| = 1$ such that $W_y(e) \neq 0$. Define $g(x) = \frac{1}{\delta}\operatorname{dist}(x, X \setminus B(\varphi(y),\delta))$ for $x \in X$. Then $g_e \in \operatorname{lip}_\alpha(X,E)$ and $\|f(x)\|\|g_e(x)\| = 0$ for every $x \in X$. The separating property of T implies that, $\|Tf(z)\|\|Tg_e(z)\| = 0$ for every $z \in Y$. In particular, $\|Tf(y)\|\|Tg_e(y)\| = 0$. On the other hand, by what we have proved in Theorem 3.2, $Tg_e(y) = g(\varphi(y))W_y(e) = W_y(e) \neq 0$. Therefore, $\|Tf(y)\| = 0$ and $Tf(y) = 0$.

We now assume that $f \in M_{\varphi(y)}$. By Lemma 3.4, there exists a sequence (f_n) in $J_{\varphi(y)}$ converging to f in $\operatorname{lip}_\alpha(X,E)$. Then the sequence (Tf_n) converges to Tf in $\operatorname{lip}_\alpha(Y,F)$. In particular, $Tf(y) = \lim Tf_n(y)$. Therefore, $Tf(y) = 0$.

Finally, let $f \in \operatorname{lip}_\alpha(X,E)$ and $e = f(\varphi(y))$. Then $g = f - 1_e$ is in $M_{\varphi(y)}$ and hence $Tf(y) - T1_e(y) = Tg(y) = 0$. Therefore,

$$Tf(y) = T1_e(y) = W_y(e) = W_y(f(\varphi(y))).$$

This completes the proof of the theorem. □

Remark 3.6. By imposing certain conditions on T, the kernel N of W will be empty. In this case, the operator T in Theorem 3.5, will be weighted composition of the form $Tf(y) = W_y(f(\varphi(y)))$ for every $y \in Y$, and φ will be continuous on Y. For instance, $N = \emptyset$, if either $T1_e = 1_u$ for some nonzero elements $e \in E, u \in F$ or if the family $\{T1_e : e \in E\}$ vanishes at no point of Y.

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References

1. J. Araujo and L. Dubarbie *Biseparating maps between Lipschitz function spaces*, J. Math. Anal. Appl. **357** (2009), no. 1, 191–200.
2. E. Beckenstein, L. Narici and A. Todd, *Automatic continuity of linear maps on spaces of continuous functions*, Manuscripta Math. **62** (1988), no. 3, 257–275.
3. W.G. Bade, P.C. Curtis, Jr. and H.G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. (3)**55** (1987), no. 2, 359–377.
4. J.T. Chan, *Operators with the disjoint support property*, J. Operator Theory **24** (1990), no. 2, 383–391.
5. J.J. Font and S. Hernández, *On separating maps between locally compact spaces*, Arch. Math. (Basel) **63** (1994), 158–165.
6. T.G. Honary and H. Mahyar, *Approximation in Lipschitz algebras*, Quaest. Math. **23** (2000), no. 1, 13–19.
7. J.E. Jamison and M. Rajagopalan, *Weighted composition operator on* $C(X, E)$, J. Operator Theory **19** (1988), no. 2, 307–317.
8. K. Jarosz, *Automatic continuity of separating linear isomorphisms*, Canad. Math. Bull. **33** (1990), 139–144.
9. J.S. Jeang and N.C. Wong, *Weighted composition operators of* $C_0(X)$*'s*, J. Math. Anal. Appl. **201** (1996), 981–993.
10. A. Jiménez-Vargas, *Disjointness preserving operators between little Lipschitz algebras*, J. Math. Anal. Appl. **337** (2008), 984–993.
11. A. Jiménez-Vargas, M.V. Vallecillos and Y.S. Wang, *Banach-Stone theorems for vector-valued little Lipschitz functions*, Publ. Math. Debrecen **74** (2009), no. 1-2, 81–100.
12. A. Jiménez-Vargas and Y.S. Wang, *Linear biseparating maps between vector-valued little Lipschitz function spaces*, Acta Math. Sin. **26** (2010), no. 6, 1005–1018.
13. J.A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, Trans. Amer. Math. Soc. **148** (1970), 147–169.
14. H. Kamowitz, *Compact weighted endomorphisms of* $C(X)$, Proc. Amer. Math. Soc. **83** (1981), no. 3, 517–521.
15. H. Kamowitz and S. Scheinberg, *Some properties of endomorphisms of Lipschitz algebras*, Studia Math. **24** (1990), no. 3, 383–391.
16. K. de Leeuw, *Banach spaces of Lipschitz functions*, Studia Math. **21** (1961/1962), 55–66.
17. D.R. Sherbert, *Banach algebras of Lipschitz functions*, Pacific J. Math. **13** (1963), 1387–1399.
18. D.R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. **111** (1964), 240–272.
19. N. Weaver, *Lipschitz algebras*, World Scientific, Singapore, 1999.
20. T.C. Wu, *Disjointness preserving operators between Lipschitz spaces*, Master thesis, Kaohsiung, NSYSU, Taiwan, 2006.

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