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OPERATOR INEQUALITIES AND NORMAL OPERATORS

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ABSTRACT. In the present paper, taking some advantages offered by the context of finite dimensional Hilbert spaces, we shall give a complete characterizations of certain distinguished classes of operators (self-adjoint, unitary reflection, normal) in terms of operator inequalities. These results extend previous characterizations obtained by the second author.

1. Introduction and preliminaries results

Let $\mathfrak{B}(H)$ be the C^*-algebra of all bounded linear operators acting on a complex Hilbert space H , and let $\mathfrak{N}(H)$,and $\mathscr{S}(H)$ denote the class of all normal operators and the class of all self-adjoint operators in $\mathfrak{B}(H)$, respectively.

We denote by

- $\mathfrak{I}(H)$, the set of all invertible elements in $\mathfrak{B}(H)$,
- $\mathscr{S}_0(H) = \mathscr{S}(H) \cap \mathfrak{I}(H)$, the set of all invertible self-adjoint operators in $\mathfrak{B}(H)$,
- $\mathfrak{N}_0(H)$, the set of all invertible normal operators in $\mathfrak{B}(H)$,
- $\mathfrak{U}_r(H)$, the set of all unitary reflection operators in $\mathfrak{B}(H)$,
- $\mathfrak{R}(H)$, the set of all operators with closed ranges in $\mathfrak{B}(H)$,
- $x \otimes y$ (where $x,\ y \in H$), the operator on H defined by $(x \otimes y)\, z = \langle z, y \rangle\, x$, for every $z \in H$.

For $S \in \mathfrak{B}(H)$, let $R(S)$ and $\ker(S)$ denote the range and the kernel of S, respectively. It is known that for every operator $S \in \mathfrak{R}(H)$, there exits an

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operator $X \in \mathfrak{B}(H)$ satisfying the following two equations $SXS = S$ and $XSX = X$, then X is called a generalized inverse of S and so SX and XS are idempotents. We recall that in general the generalized inverse is not unique. But there exits a unique generalized inverse X such that SX and XS are orthogonal projections. In this case, X is called the More–Penrose inverse S and it is denoted by S^+. In this case, SS^+ and S^+S are orthogonal projections onto $R(S)$ and $R(S^*)$, respectively, and hence $S^* = S^+SS^* = S^*SS^+$. We say that an operator $S \in \mathfrak{R}(H)$ is EP, if $R(S) = R(S^*)$ (or equivalently $S^+S = SS^+$). Note that any normal operator with closed range is EP, but the converse is not true even in a finite dimensional space.

The ascent and descent of $S \in \mathfrak{B}(H)$ are respectively defined by

$$\mathrm{asc}(S) = \min \left\{ p \in \mathbb{N} \cup \{0\} : \ker(S^p) = \ker(S^{p+1}) \right\}$$

and

$$\mathrm{dsc}(S) = \min \left\{ p \in \mathbb{N} \cup \{0\} : R(S^p) = R(S^{p+1}) \right\}$$

if they are finite, they are equal, and their common value is called the index of S and it is denoted by $ind(S)$.

For every S in $\mathfrak{R}(H)$, we associate the 2×2 matrix representation $\begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$ of S with respect to the orthogonal direct sum $H = R(S) \oplus KerS^*$. For $S \in \mathfrak{R}(H)$, it is easy to see that $ind(S) \leq 1$ if and only if S_1 is invertible, and S is an EP operator if and only if $S_2 = 0$ (and therefore S_1 is invertible).

In 1979, McIntosh [5] has proved the following operator inequality

$$\forall A, B, X \in \mathfrak{B}(H), \|A^*AX + XBB^*\| \geq 2\|AXB\| . \tag{1.1}$$

An operator inequality equivalent to (1.1) was proved by Corach–Porta–Recht [2] with a different motivation. It was given as follows

$$\forall S \in \mathscr{S}_0(H),\ \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\|X\| .$$

Based on this last inequality, the second author was interested to characterize some distinguished classes of operators (invertible self-adjoint operator, unitary reflection operator, invertible normal operator) in terms of operator inequalities. We cite here some of these characterizations:

($\cdot$) The following property ([6])

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\|X\| \quad (S \in \mathfrak{I}(H))$$

characterizes the class $\mathbb{C}^*\mathscr{S}_0(H)$, subclass of $\mathfrak{N}_0(H)$ characterized by the spectrum of each of its operator is included in some straight line passing through the origin.

($\cdot\cdot$) The following property ([7])

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = 2\|X\| \quad (S \in \mathfrak{I}(H))$$

characterizes the class $\mathbb{C}^*\mathfrak{U}_r(H)$, subclass of $\mathfrak{N}_0(H)$ for which the spectrum of each of its operator is included in $\{-\lambda, \lambda\}$ for some nonzero complex number λ.

($\cdots$) The following property ([7])

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\|X\| \quad (S \in \mathfrak{I}(H))$$

characterizes the class $\mathfrak{N}_0(H)$, the class of all invertible normal operators. For more other characterizations of subclasses of normal operators in term of operator inequalities, we may see [7, 8, 9, 1].

In this paper, we are interested in the general form of each of the above three properties in two manners. Firstly, in the above three properties, we replace in each of left terms of the inequalities, we replace S^{-1} by S^{+} and we replace the domain $\mathfrak{I}(H)$ of each property by the new domain $\mathfrak{R}(H)$, so we obtain the first three following general forms:

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}+S^{+}XS\right\| \geq 2\left\|SS^{+}XS^{+}S\right\| \quad (S \in \mathfrak{R}(H)), \tag{1.2}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}+S^{+}XS\right\| = 2\left\|SS^{+}XS^{+}S\right\| \quad (S \in \mathfrak{R}(H)),$$

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}\right\|+\left\|S^{+}XS\right\| \geq 2\left\|SS^{+}XS^{+}S\right\| \quad (S \in \mathfrak{R}(H))\,. \tag{1.3}$$

Secondly, we replace X by SXS, and $\mathfrak{I}(H)$ by $\mathfrak{B}(H)$, so we obtain the second following three general forms::

$$\forall X \in \mathfrak{B}(H),\ \left\|S^{2}X+XS^{2}\right\| \geq 2\left\|SXS\right\| \quad (S \in \mathfrak{B}(H)), \tag{1.4}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^{2}X+XS^{2}\right\| = 2\left\|SXS\right\| \quad (S \in \mathfrak{B}(H)), \tag{1.5}$$

$$\forall X \in \mathfrak{B}(H),\ \left\|S^{2}X\right\|+\left\|XS^{2}\right\| \geq 2\left\|SXS\right\| \quad (S \in \mathfrak{B}(H))\,. \tag{1.6}$$

In this note, we shall show that:

(i) each of the two properties (1.2) and (1.4) characterize the class of all self-adjoint operators multiplying by scalars in each of the two conditions:

(a) "$S \in \mathfrak{R}(H)$ and $ind(S) < \infty$",

(b) $\dim H < \infty$ and the domain of each properties is all $\mathfrak{B}(H)$.

(ii) each of the two properties (1.3) and (1.6) characterize the class of all normal operators in each of the two conditions:

(a) "$S \in \mathfrak{R}(H)$ and $ind(S) < \infty$",

(b) $\dim H < \infty$ and the domain of each properties is all $\mathfrak{B}(H)$,

(iii) the property (1.5) characterize the class of all unitary reflections multiplying by scalars in the case of $\dim H < \infty$.

In this section, we present some preliminaries results. These results are needed in section 2.

It is easy to see that if S satisfies the property (1.3), then it satisfies the property (1.6). Indeed, assume that S satisfies the property (1.3). So we obtain

$$\forall X \in \mathfrak{B}(H),\ \left\|S^{2}XSS^{+}\right\|+\left\|S^{+}SXS^{2}\right\| \geq 2\left\|SS^{+}SXSS^{+}S\right\| = 2\left\|SXS\right\|\,.$$

Since $\|SS^{+}\| = \|S^{+}S\| = 1$ and using the triangular inequality, we deduce the following inequality

$$\forall X \in \mathfrak{B}(H),\ \left\|S^{2}X\right\|+\left\|XS^{2}\right\| \geq 2\left\|SXS\right\|\,.$$

Hence S satisfies the property (1.6).

Proposition 1.1. *Let $S, T \in \mathfrak{R}(H)$. Then the following inequality holds:*

$$\forall X \in \mathfrak{B}(H),\ \left\|S^*XT^+ + T^+XS^*\right\| \geq 2\left\|SS^+XT^+T\right\| .$$

Proof. First we prove that the inequality holds for $S = T$. Let $X \in \mathfrak{B}(H)$. From inequality (1.1), we obtain

$$\left\|S^*XS^+ + S^+XS^*\right\| = \left\|S^*SS^+XS^+ + S^+XS^+SS^*\right\| \geq 2\left\|SS^+XS^+S\right\| .$$

So the inequality for S, T follows immediately from the first step and by using the known Berberian method. □

Proposition 1.2. (i) *The property* (1.2) *is satisfied for every self-adjoint operator in* $\mathfrak{R}(H)$*,*

(ii) *The property* (1.4) *is satisfied for every self-adjoint operator in* $\mathfrak{B}(H)$*,*

(iii) *The property* (1.3) *is satisfied for every normal operator in* $\mathfrak{R}(H)$*,*

(iv) *The property* (1.6) *is satisfied for every normal operator in* $\mathfrak{B}(H)$*.*

Proof. (i) follows immediately from Proposition 1.1.

(ii) follows immediately from inequality (1.1).

(iii) Let S be a normal operator in $\mathfrak{R}(H)$ and let $X \in \mathfrak{B}(H)$.

Since S is normal, then $\|SXS^+\| = \|S^*XS^+\|$ and $\|S^+XS\| = \|S^+XS^*\|$. so it follows that :

$$\left\|SXS^+\right\| + \left\|S^+XS\right\| = \left\|S^*XS^+\right\| + \left\|S^+XS^*\right\| \geq \left\|S^*XS^+ + S^+XS^*\right\| .$$

Hence, by Proposition 1.1, we obtain

$$\left\|SXS^+\right\| + \left\|S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\| .$$

(iv) Let S be a normal operator in $\mathfrak{B}(H)$ and let $X \in \mathfrak{B}(H)$. Since S is normal then $\|S^2X\| = \|S^*SX\|$ and $\|XS^2\| = \|XSS^*\|$, so we obtain

$$\left\|S^2X\right\| + \left\|XS^2\right\| = \|SS^*SX\| + \|XSS^*\| \geq \|SS^*SX + XSS^*\| .$$

Thus, the result follows immediately from inequality (1.1). □

2. Characterizations of classes of operators by operator inequalities

In this section, we consider $S \in \mathfrak{R}(H)$ and $\begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$ be the 2×2 matrix representation of S with respect to the orthogonal direct sum $H = R(S) \oplus KerS^*$.

Theorem 2.1. *Assume that $ind(S) < \infty$. Then the following properties are equivalent:*

(i) *S is normal,*

(ii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+\| + \|S^+XS\| \geq 2\|SS^+XS^+S\|$,

(iii) $\forall X \in \mathfrak{B}(H),\ \|S^2X\| + \|XS^2\| \geq 2\|SXS\|$.

Proof. The implication (i) ⇒ (ii) holds (see Proposition 1.2).

The implication (ii) ⇒ (iii) holds (see introduction).

(iii) ⇒ (i): Assume that (iii) holds.

All 2×2 matrices given below are given with respect to the decomposition $H = R(S) \oplus KerS^*$. We prove that $ind(S) \leq 1$.

Assume that $ind(S) > 1$. By choosing $X = x \otimes y$, for $x,\ y \in (H)_1$, then using (iii) we obtain

$$\forall x, y \in (H)_1 : \left\| S^2 x \right\| + \left\| (S^*)^2 y \right\| \geq 2 \left\| Sx \right\| \left\| S^* y \right\| . \tag{2.1}$$

Since $ind(S) > 1$, then $KerS^2 \neq KerS$. Hence there exists $x \in (H)_1$ such that $S^2 x = 0$ and $Sx \neq 0$. Using (2.1), we obtain $\|(S^*)^2 y\| \geq k \|S^* y\|$, for every $y \in H$ (where $k = 2\|Sx\| > 0$). Thus, $S^2(S^*)^2 \geq k^2 SS^*$, and so that $R(S^2) = R(S)$ (see [4]). Contradiction with $ind(S) > 1$. So we obtain $ind(S) \leq 1$ and hence S_1 is invertible.

Let $X \in \mathfrak{B}(H)$ given by $X = S_1^{-2} \oplus 0$. By a simple computation, we obtain $S^2 X = I_1 \oplus 0$, and $XS^2 = SXS = \begin{bmatrix} I_1 & S_1^{-1} S_2 \\ 0 & 0 \end{bmatrix}$, where I_1 is the identity operator on $R(S)$. So that $\|S^2 X\| = 1$ and $\|XS^2\|^2 = \|SXS\|^2 = \left\| I_1 + \left(S_1^{-1} S_2\right) \left(S_1^{-1} S_2\right)^* \right\|$. Applying (iii) for S and X, then we obtain $1 \geq \left\| I_1 + \left(S_1^{-1} S_2\right) \left(S_1^{-1} S_2\right)^* \right\|$. Hence $\left(S_1^{-1} S_2\right) \left(S_1^{-1} S_2\right)^* = 0$, since $\left(S_1^{-1} S_2\right) \left(S_1^{-1} S_2\right)^*$ is a positive operator and so $\left\| I_1 + \left(S_1^{-1} S_2\right) \left(S_1^{-1} S_2\right)^* \right\| > 1$ if $\left(S_1^{-1} S_2\right) \left(S_1^{-1} S_2\right)^* \neq 0$. Thus $S_2 = 0$, so that $S = S_1 \oplus 0$. Applying (iii), for S and $X = X_1 \oplus 0$ (where X_1 is an arbitrary operator on $R(S)$), so we obtain $\|S_1^2 X_1\| + \|X_1 S_1^2\| \geq 2\|S_1 X_1 S_1\|$, for every bounded operator X_1 on $R(S)$, and where S_1 is invertible. Hence the inequality $\left\|S_1 X_1 S_1^{-1}\right\| + \left\|S_1^{-1} X_1 S_1\right\| \geq 2\|X_1\|$ holds, for every bounded operator X_1 on $R(S)$. So we obtain that S_1 is an invertible normal operator on $R(S)$. Hence, S is normal. □

Theorem 2.2. *Assume that* $\dim H < \infty$. *Then the following properties are equivalent:*

(i) *S is normal,*
(ii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+\| + \|S^+ XS\| \geq 2\|SS^+ XS^+ S\|$,
(iii) $\forall X \in \mathfrak{B}(H),\ \|S^2 X\| + \|XS^2\| \geq 2\|SXS\|$.

Proof. The two implications (i) ⇒ (ii) and (ii) ⇒ (iii) hold (see the above proof).

(iii) ⇒ (i): Assume (iii) holds. We shall prove that $Ker\,(S^*)^2 = KerS^*$. Assume that $Ker\,(S^*)^2 \neq KerS^*$. Using the same argument as used in the above proof, there exists $y \in (H)_1$ such that $(S^*)^2 y = 0$ and $S^* y \neq 0$. Using (2.1), we obtain $\|S^2 x\| \geq k\|Sx\|$, for every $x \in H$ (where $k = 2\|S^* y\| > 0$). Hence $\|S_1 u\| \geq k\|u\|$, for every $u \in R(S)$. Thus S_1 is injective. So that S_1 is invertible (or equivalently $ind(S) \leq 1$). Using Theorem 2.1, we deduce that S is normal. Then S^* is also normal, and so $Ker\,(S^*)^2 = KerS^*$. Contradiction. Therefore, we obtain that $Ker\,(S^*)^2 = KerS^*$.

Using the same argument as used above and since S^* satisfies (iii), we find that $KerS^2 = KerS$.

So, we obtain that $KerS^2 = KerS$ and $R(S^2) = R(S)$. Then $ind(S) \leq 1$. So, using Theorem 2.1, S is normal. □

Theorem 2.3. *Assume that $ind(S) < \infty$. Then the following properties are equivalent:*

(i) $S \in \mathbb{C}\mathscr{S}(H)$,
(ii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+ XS\| \geq 2\|SS^+ XS^+ S\|$,

(iii) $\forall X \in \mathfrak{B}(H),\ \|S^2X + XS^2\| \geq 2\|SXS\|$.

Proof. The two implications (i) ⇒ (ii) and (i) ⇒ (iii) hold from Proposition 1.2.

Assume now that (ii) or (iii) holds.

Then applying the triangular inequality, we obtain from Theorem 2.2 that S is normal. Then $S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}$, where S_1 is invertible. Applying (ii) or (iii) for S and $X = X_1 \oplus 0$ (where X_1 is an arbitrary operator on $R(S)$), so we obtain $\|S_1^2X_1 + X_1S_1^2\| \geq 2\|S_1X_1S_1\|$, for every bounded operator X_1 on $R(S)$, and where S_1 is invertible. Hence the inequality $\left\|S_1X_1S_1^{-1} + S_1^{-1}X_1S_1\right\| \geq 2\|X_1\|$ holds, for every bounded operator X_1 on $R(S)$. So we obtain that S_1 is an invertible self-adjoint operator on $R(S)$ multiplying by a non-zero scalar. Hence $S \in \mathbb{C}\mathscr{S}(H)$. □

Theorem 2.4. *Assume that* $\dim H < \infty$. *Then the following properties are equivalent:*

(i) $S \in \mathbb{C}\mathscr{S}(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| \geq 2\|SS^+XS^+S\|$,

(iii) $\forall X \in \mathfrak{B}(H),\ \|S^2X + XS^2\| \geq 2\|SXS\|$.

Proof. The proof is similar to the previous proof. □

Theorem 2.5. *Assume that* $\dim H < \infty$. *Then the following two properties are equivalent:*

(i) $S \in \mathbb{C}\mathfrak{U}_r(H)$,

(ii) $\forall X \in \mathfrak{B}(H),\ \|S^2X + XS^2\| = 2\|SXS\|$.

Proof. (i) ⇒ (ii) is obvious.

(ii) ⇒ (i): Assume (ii) holds. From Theorem 2.4, S is of the form $S = \lambda_0 T$, where $\lambda_0 \in \mathbb{C}$ and T is a self-adjoint operator in $\mathfrak{B}(H)$.

We may assume without loss of the generality that $S \neq O$. Thus the following equality holds

$$\forall X \in \mathfrak{B}(H),\ \left\|T^2X + XT^2\right\| = 2\|TXT\|\,. \tag{2.2}$$

Since T is self-adjoint, then there exists an eigenvalue λ_1 of T such that $|\lambda_1| = \|T\|$. Let λ be an arbitrary eigenvalue of T. Then there exist two unit vectors $x,\ y \in H$ such that $Tx = \lambda_1 x$ and $Ty = \lambda y$. By taking $X = x \otimes y$ in (2.2), we obtain that $\lambda^2 + \lambda_1^2 = 2|\lambda||\lambda_1|$. Hence $|\lambda| = |\lambda_1|$. So we have $\sigma(\frac{T}{\|T\|}) \subset \{-1, 1\}$ and where $\frac{T}{\|T\|}$ is self-adjoint. Thus $\frac{T}{\|T\|}$ is a unitary reflection. Therefore $S = (\lambda_0\|T\|)\frac{T}{\|T\|}$. □

Corollary 2.6. *Assume that* $\dim H < \infty$. *Then the following two properties are equivalent:*

(i) S is an EP operator with its nonzero part is a unitary reflection on $R(S)$ multiplying by a nonzero scalar,

(ii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| = 2\|SS^+XS^+S\|$.

Proof. (i) ⇒ (ii). This implication is obvious.

(ii) ⇒ (i).

Assume that (ii) holds.

Then from Theorem 2.4, $S = \lambda T$ for some scalar λ and some self-adjoint operator $T \in \mathfrak{B}(H)$. Then $T = T_1 \oplus 0$ with respect to the orthogonal direct sum $H = R(T) \oplus KerT$ and T_1 is invertible. So from (ii), we obtain the following inequality

$$\forall X \in \mathfrak{B}(R(T)), \ \left\|T_1 X T_1^{-1} + T_1^{-1} X T_1\right\| = 2\left\|X\right\| .$$

Then T_1 is a unitary reflection operator on $R(T)$ multiplying by a nonzero scalar. Hence S satisfies (i). □

Remark 2.7. 1. The results presented in this paper are extensions of the results of Khosravi [3].

2. Proposition 1.1, which is showed with an easy and direct proof, was given in [3].

3. Theorem 2.3 was given in [3] with the restricted condition "S is an EP operator", and without inequality (iii).

4. Does the characterizations given in Theorem 2.1 and Theorem 2.3 remain true without the assumption "$ind(S) < \infty$".

5. Does the characterizations given in Theorem 2.1 (without the condition (ii)) and Theorem 2.3 (without the condition (ii)) remain true without the assumption "$S \in \mathfrak{R}(H)$ and $ind(S) < \infty$".

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