Banach J. Math. Anal. 6 (2012), no. 2, 192–203

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

SQUARE ROOT FOR BACKWARD OPERATOR WEIGHTED SHIFTS WITH MULTIPLICITY 2

BINGZHE HOU AND GENG TIAN*

Communicated by J. A. Ball

Abstract. As is well-known, each positive operator T acting on a Hilbert space has a positive square root which is realized by means of functional calculus. However, it is not always true that an operator have a square root. In this paper, by means of Schauder basis theory we obtain that if a backward operator weighted shift T with multiplicity 2 is not strongly irreducible, then there exists a backward shift operator B (maybe unbounded) such that $T = B^2$. Furthermore, the backward operator weighted shifts in the sense of Cowen-Douglas are also considered.

1. Introduction and preliminaries

As is well-known, functional calculus is a fundamental tool in operator theory, and people obtain many beautiful results by means of it. A natural and simple question is "Given an operator, does it have a square root?" Although the answer is negative in general, by restricting the class of operators, we may obtain positive results. In this paper, we will consider strongly reducible backward operator weighted shifts, and give a square root for it by means of Schauder basis theory.

First, let us introduce some fundamental notations and results.

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$.

Date: Received: 9 July 2011; Revised: 5 May 2012; Accepted: 4 June 2012.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 47B37; Secondary 46A35.

Key words and phrases. backward operator weighted shifts with multiplicity 2, unconditional basis, strongly reducible operator.

Let S be a backward operator weighted shift with multiplicity 2. Its weights $\{W_k\}_{k=1}^{\infty}$ are a sequence of invertible linear operators on $\mathbb{C}^2$. We can denote S as a matrix

$$S = \begin{bmatrix} 0 & W_1 & 0 & \dots \\ 0 & 0 & W_2 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{array}{c} \mathbb{C}^2 \\ \mathbb{C}^2 \\ \mathbb{C}^2. \\ \vdots \end{array}$$

Furthermore, denote $M_0 = I, M_k = W_1 W_2 \cdots W_k$ and $\mathcal{A}(S) = \mathcal{A}'(S)|_{ker(S)}$.

Definition 1.1. Let $T \in \mathcal{L}(\mathcal{H})$. Then T is called strongly irreducible, if there does not exist a nontrivial idempotent operator commuting with T. The class of strongly irreducible operators is denoted by $S(I)$.

From C. L. Jiang and J. X. Li [4], one can see the following conclusion.

Lemma 1.2. *Let S be the backward operator weighted shift with weights $\{W_k\}_{k=1}^{\infty}$. Then the following are equivalent:*

(1) $S \notin (SI)$,

(2) there exists a nontrival idempotent operator P_0 such that $\sup\limits_k \{\|M_k^{-1} P_0 M_k\|\} < \infty$.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Then for each vector f in $\mathcal{H}$, one can write $f = \sum\limits_{n=1}^{\infty} \langle f, e_n \rangle e_n$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$.

Furthermore, for a sequence of vectors $\{f_m\}_{m=1}^{\infty}$ in $\mathcal{H}$, write

$$F_f = \begin{bmatrix} f_{11} & f_{12} & \cdots \\ f_{21} & f_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

where the entries f_{nm} is the n-th coordinate of vector f_m. We always use G_f^* to denote a left inverse of F_f whenever the left inverse exists. Notice that F_f and G_f^* may be unbounded operators.

Definition 1.3. A sequence of vectors $\{f_m\}_{m=1}^{\infty}$ in $\mathcal{H}$ is said to be quasinormed, if there exist constants K_1 and K_2 such that for all m, $0 < K_1 \leq \|f_m\| \leq K_2$.

Definition 1.4. A sequence of vectors $\{f_m\}_{m=1}^{\infty}$ in $\mathcal{H}$ is said to be a Schauder basis for $\mathcal{H}$ if every $x \in \mathcal{H}$ has an unique norm-convergent expansion

$$x = \sum_{m=1}^{\infty} c_m f_m.$$

The following results about unconditional basis will be used in this article. One can see [6], [2] and [3] for details respectively.

Lemma 1.5 ([6]). *$\{f_m\}_{m=1}^{\infty}$ is an unconditional basis if and only if for any sequence of nonzero complex numbers $\{\lambda_m\}$, $\{\lambda_m f_m\}_{m=1}^{\infty}$ is an unconditional basis.*

Lemma 1.6 ([2]). *Let $\{f_m\}_{m=1}^{\infty}$ be an unconditional basis and let T be an invertible operator. Then TF_f generate an unconditional basis, i.e. $\{TF_f(e_n)\}_{n=1}^{\infty}$ is an unconditional basis.*

Lemma 1.7 ([3]). *$\{f_m\}_{m=1}^{\infty}$ is a quasinormed unconditional basis if and only if $\{f_m\}_{m=1}^{\infty}$ is a Riesz basis, if and only if F_f is bounded and invertible.*

2. Square root and Schauder basis

Proposition 2.1. *Let $\{f_m\}_{m=1}^{\infty}$ be a sequence of vectors in $\mathcal{H}$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$, and let F_f be the matrix defined as above. If*

$$F_f = \begin{bmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & 0 & \dots \\ 0 & 0 & A_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $A_k = \begin{bmatrix} a_k & b_k \\ 0 & c_k \end{bmatrix}$ is invertible, then the following conditions are equivalent

(1) $\{f_m\}_{m=1}^{\infty}$ is a Schauder basis.

(2) $\sup\limits_k |\dfrac{b_k}{c_k}| = K < \infty$.

(3) $\{f_m\}_{m=1}^{\infty}$ is an unconditional basis.

Proof. (1) $\Rightarrow$ (2). Suppose that $\{|\frac{b_k}{c_k}|\}_{k=1}^{\infty}$ is infinite. Then there is a subsequence k_i such that $\sum\limits_{i=1}^{\infty} |\frac{c_{k_i}}{b_{k_i}}|^2 < \infty$. Let $x = -\sum\limits_{i=1}^{\infty} \frac{c_{k_i}}{b_{k_i}} e_{2k_i}$. Then $x \in \mathcal{H}$. Notice that x has an unique expansion

$$x = \sum_{i=1}^{\infty} (\frac{1}{a_{k_i}} f_{2k_i-1} - \frac{1}{b_{k_i}} f_{2k_i}),$$

since $f_{2k-1} = a_k e_{2k-1}$ and $f_{2k} = b_k e_{2k-1} + c_k e_{2k}$. However, this expansion is not convergent in the norm topology since $\|\frac{1}{a_{k_i}} f_{2k_i-1}\| = 1$. It is a contradiction to $\{f_m\}_{m=1}^{\infty}$ being a Schauder basis.

(2) $\Rightarrow$ (3). If $\sup\limits_k |\frac{b_k}{c_k}| = K < \infty$, we can choose a sequence of nonzero complex numbers $\{\lambda_m\}$, such that $f'_m = \lambda_m f_m$,

$$F_{f'} == \begin{bmatrix} A'_1 & 0 & 0 & \dots \\ 0 & A'_2 & 0 & \dots \\ 0 & 0 & A'_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $A'_k = \begin{bmatrix} 1 & \frac{b_k}{c_k} \\ 0 & 1 \end{bmatrix}$. Since $F_{f'}$ is the matrix representation of an invertible bounded linear operator, $\{f'_m\}_{m=1}^{\infty}$ is an unconditional basis and hence $\{f_m\}_{m=1}^{\infty}$ is an unconditional basis by Lemma 1.5.

(3) $\Rightarrow$ (1). Follows immediately from the definitions of Schauder and unconditional bases. $\square$

Before introduce our main result, notice that there exists an unbounded operator densely defined whose square is bounded. For instance, let T be an unbounded operator defined by $T(e_1) = 0$, $T(e_{2n}) = \frac{1}{2n} e_{2n-1}$ and $T(e_{2n+1}) = 2n e_{2n}$ for $n \geq 1$.

Then $T^2(e_1) = T^2(e_2) = 0$ whereas $T^2(e_{2n}) = (1-\frac{1}{2n})e_{2n-2}$ and $T^2(e_{2n+1}) = e_{2n-1}$, that implies T^2 is bounded.

We always denote the backward shift on $\{e_n\}_{n=1}^{\infty}$ by B_s, i.e., $B_s(e_1) = 0$ and $B_s(e_n) = e_{n-1}$ for $n > 1$.

Theorem 2.2. *Let S be the backward operator weighted shift of multiplicity 2 with weights $\{W_k\}_{k=1}^{\infty}$. Then the three following conditions are equivalent:*

(1) $S \notin (SI)$.

(2) There exists an unconditional basis $\{f_m\}_{m=1}^{\infty}$ such that $S = B^2$, where B is the backward shift on $\{f_m\}_{m=1}^{\infty}$. In other words, $S = F_f B_s^2 G_f^*$.

(3) There exists a backward weighted shift operator B_Λ on $\{e_n\}_{n=1}^{\infty}$, which may be unbounded, such that S is similar to B_Λ^2.

Proof. (1) $\Rightarrow$ (2).

Step 1. Suppose each W_k is upper triangular. By Lemma 1.2, there exists a nontrivial idempotent operator P_0 such that $\sup\limits_k\{\|M_k^{-1}P_0M_k\|\} < \infty$. Notice that nontrivial idempotent operators on $\mathbb{C}^2$ have matrices either of the form $\begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & \beta \\ 0 & 1 \end{bmatrix}$. Denote $M_k = \begin{bmatrix} m_{k1} & m_{k2} \\ 0 & m_{k3} \end{bmatrix}$.

Case 1. If $P_0 = \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix}$, then

$$M_k^{-1}P_0M_k = \begin{bmatrix} \frac{1}{m_{k1}} & -\frac{m_{k2}}{m_{k1}m_{k3}} \\ 0 & \frac{1}{m_{k3}} \end{bmatrix} \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_{k1} & m_{k2} \\ 0 & m_{k3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{m_{k2}+\beta m_{k3}}{m_{k1}} \\ 0 & 0 \end{bmatrix},$$

Thus $\sup\limits_k\{|\frac{m_{k2}+\beta m_{k3}}{m_{k1}}|\} < \infty$. Let $D = \begin{bmatrix} 1 & -\beta \\ 0 & 1 \end{bmatrix}$. Now for $k = 0, 1, 2, \ldots$, define $A_{k+1} = M_k^{-1}D = \begin{bmatrix} \frac{1}{m_{k1}} & -\frac{m_{k2}+\beta m_{k3}}{m_{k1}m_{k3}} \\ 0 & \frac{1}{m_{k3}} \end{bmatrix}$.

Let $\{f_m\}_{m=1}^{\infty}$ be a sequence of vectors in $\mathcal{H}$ such that

$$F_f = \begin{bmatrix} A_1 & 0 & 0 & \ldots \\ 0 & A_2 & 0 & \ldots \\ 0 & 0 & A_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

Since

$$\sup_k\{|(-\frac{m_{k2}+\beta m_{k3}}{m_{k1}m_{k3}})/(\frac{1}{m_{k3}})|\} = \sup_k\{|\frac{m_{k2}+\beta m_{k3}}{m_{k1}}|\} < \infty,$$

we have $\{f_m\}_{m=1}^{\infty}$ is an unconditional basis by Proposition 2.1.

In addition,

$$F_f B^2 G_f^* = \begin{bmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & 0 & \dots \\ 0 & 0 & A_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 & 0 & \dots \\ 0 & A_2^{-1} & 0 & \dots \\ 0 & 0 & A_3^{-1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & M_0^{-1}M_1 & 0 & 0 & \dots \\ 0 & 0 & M_1^{-1}M_2 & 0 & \dots \\ 0 & 0 & 0 & M_2^{-1}M_3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & W_1 & 0 & 0 & \dots \\ 0 & 0 & W_2 & 0 & \dots \\ 0 & 0 & 0 & W_3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, $S = B^2$, where B is the backward shift on $\{f_m\}_{m=1}^{\infty}$.

Case 2. If $P_0 = \begin{bmatrix} 0 & \beta \\ 0 & 1 \end{bmatrix}$, then

$$M_k^{-1} P_0 M_k = \begin{bmatrix} \frac{1}{m_{k1}} & -\frac{m_{k2}}{m_{k1}m_{k3}} \\ 0 & \frac{1}{m_{k3}} \end{bmatrix} \begin{bmatrix} 0 & \beta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{k1} & m_{k2} \\ 0 & m_{k3} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\beta m_{k3} - m_{k2}}{m_{k1}} \\ 0 & 1 \end{bmatrix},$$

Thus $\sup_k \{|\frac{\beta m_{k3} - m_{k2}}{m_{k1}}|\} < \infty$. Let $D = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$. Now for $k = 0, 1, 2, \dots$, define $A_{k+1} = M_k^{-1} D = \begin{bmatrix} \frac{1}{m_{k1}} & \frac{\beta m_{k3} - m_{k2}}{m_{k1}m_{k3}} \\ 0 & \frac{1}{m_{k3}} \end{bmatrix}$.

Let $\{f_m\}_{m=1}^{\infty}$ be a sequence of vectors in $\mathcal{H}$ such that

$$F_f = \begin{bmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & 0 & \dots \\ 0 & 0 & A_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

Since

$$\sup_k \{|(\frac{\beta m_{k3} - m_{k2}}{m_{k1}m_{k3}})/(\frac{1}{m_{k3}})|\} = \sup_k \{|\frac{\beta m_{k3} - m_{k2}}{m_{k1}}|\} < \infty,$$

we have $\{f_m\}_{m=1}^{\infty}$ is an unconditional basis by Proposition 2.1.

In addition,

$$F_f B^2 G_f^* = \begin{bmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & 0 & \dots \\ 0 & 0 & A_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A_1^{-1} & 0 & 0 & \dots \\ 0 & A_2^{-1} & 0 & \dots \\ 0 & 0 & A_3^{-1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & M_0^{-1}M_1 & 0 & 0 & \dots \\ 0 & 0 & M_1^{-1}M_2 & 0 & \dots \\ 0 & 0 & 0 & M_2^{-1}M_3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & W_1 & 0 & 0 & \dots \\ 0 & 0 & W_2 & 0 & \dots \\ 0 & 0 & 0 & W_3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore, $S = B^2$, where B is the backward shift on $\{f_m\}_{m=1}^{\infty}$.

Step 2. For S, there exists an unitary operator

$$U=\begin{bmatrix} U_1 & 0 & 0 & \dots \\ 0 & U_2 & 0 & \dots \\ 0 & 0 & U_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

such that $U_kW_kU_{k+1}^{-1}$ is upper triangular for each $k=1,2,\dots$. Let $T=USU^{-1}$. Then

$$T=\begin{bmatrix} 0 & U_1W_1U_2^{-1} & 0 & 0 & \dots \\ 0 & 0 & U_2W_2U_3^{-1} & 0 & \dots \\ 0 & 0 & 0 & U_3W_3U_4^{-1} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By the conclusion in step 1, there exists an unconditional basis $\{f'_m\}_{m=1}^{\infty}$ such that $T=B'^2$, where B' is the backward shift on $\{f'_m\}_{m=1}^{\infty}$. Now define $\{f_m\}_{m=1}^{\infty}$ such that $F_f=U^{-1}F_{f'}$. According to Lemma 1.6, $\{f_m\}_{m=1}^{\infty}$ is an unconditional basis. Since $T=USU^{-1}=F_{f'}B_s^2G_{f'}^*$, $S=U^{-1}F_{f'}B_s^2G_{f'}^*U=F_fB_s^2G_f^*$ and hence the unconditional basis $\{f_m\}_{m=1}^{\infty}$ is required.

(2) $\Rightarrow$ (3). Let

$$J=\begin{array}{c} \begin{bmatrix} \|f_1\| & 0 & 0 & \dots \\ 0 & \|f_2\| & 0 & \dots \\ 0 & 0 & \|f_3\| & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ \vdots \end{array},$$

$$J^*=\begin{bmatrix} \|f_1\|^{-1} & 0 & 0 & \dots \\ 0 & \|f_2\|^{-1} & 0 & \dots \\ 0 & 0 & \|f_3\|^{-1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ \vdots \end{array},$$

and let $\widetilde{F}=F_fJ^*$. There exists a backward weighted shift operator B_Λ such that

$$B_\Lambda^2=JB_s^2J^*=\begin{bmatrix} 0 & 0 & \frac{\|f_1\|}{\|f_3\|} & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{\|f_2\|}{\|f_4\|} & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{\|f_3\|}{\|f_5\|} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ \vdots \end{array}. \tag{2.1}$$

For instance, we can choose

$$B_\Lambda=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{\|f_1\|}{\|f_3\|} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{\|f_2\|\|f_3\|}{\|f_1\|\|f_4\|} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{\|f_1\|\|f_4\|}{\|f_2\|\|f_5\|} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \frac{\|f_2\|\|f_5\|}{\|f_1\|\|f_6\|} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ \vdots \end{array}. \tag{2.2}$$

Then

$$S = F_f B_s^2 G_f^* = F_f J^* B_\Lambda^2 J G_f^*.$$

By Lemma 1.7, $\widetilde{F}$ is an invertible operator and consequently $S = \widetilde{F} B_\Lambda^2 \widetilde{F}^{-1}$, i.e. S is similar to B_Λ^2.

(3) $\Rightarrow$ (1). Let

$$P = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \vdots \end{array},$$

then $PB_\Lambda^2 = B_\Lambda^2 P$ and hence B_Λ^2 is strongly reducible. Since strong reducibility is similar invariant, $S \notin (SI)$. □

Proposition 2.3. *Let S be the backward operator weighted shift of multiplicity 2 with weights*

$$W_k = \begin{bmatrix} 1 & w_k \\ 0 & 1 \end{bmatrix}, \quad for\ k = 1, 2, \cdots,$$

and let $\{\sum_{k=1}^n w_k\}_{n=1}^\infty$ be bounded. Then S is similar to B_s^2.

Proof. Let I_2 be the identity 2×2 matrix acting on $\mathbb{C}^2$. Thus

$$S = \begin{bmatrix} 0 & W_1 & 0 & \dots \\ 0 & 0 & W_2 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$B_s^2 = \begin{bmatrix} 0 & I_2 & 0 & \dots \\ 0 & 0 & I_2 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Now choose invertible 2×2 matrices

$$A_1 = I_2 \ and \ A_n = W_1 \dots W_{n-1} = \begin{bmatrix} 1 & \sum_{k=1}^{n-1} w_k \\ 0 & 1 \end{bmatrix} \ for\ n \geq 2.$$

Let

$$A = \begin{bmatrix} A_1 & 0 & 0 & \dots \\ 0 & A_2 & 0 & \dots \\ 0 & 0 & A_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Since $\{\sum_{k=1}^n w_k\}_{n=1}^\infty$ is bounded, then A is an invertible bounded linear operator and it is easy to see $ASA^{-1} = B_s^2$. □

3. Square root in the sense of Cowen-Douglas

Now let us consider the operator weighted shift Cowen-Douglas operators. The definition given by Cowen and Douglas [1] is well known as follows.

Definition 3.1. For Ω a connected open subset of $\mathbb{C}$ and n a positive integer, let $\mathcal{B}_n(\Omega)$ denotes the operators T in $\mathcal{L}(\mathcal{H})$ which satisfy:
(a) $\Omega \subseteq \sigma(T) = \{\omega \in \mathbb{C} : T - \omega \textit{ not invertible}\}$;
(b) $ran(T-\omega) = \mathcal{H}$ *for* ω *in* Ω;
(c) $\bigvee ker_{\omega\in\Omega}(T-\omega) = \mathcal{H}$; and
(d) $dim\ ker(T-\omega) = n$ for ω in Ω.

JueXian Li et. al. gave an sufficient and necessary condition for operator weighted shift Cowen-Douglas operators in [5].

Lemma 3.2. *Let $n \geq 1$ and let S be a backward operator weighted shift of multiplicity n with weights $\{W_k\}_{k=1}^{\infty}$. Then $S \in \mathcal{B}_n(\Omega)$ if and only if*

$$\sup_k \{\|W_k\|, \|W_k^{-1}\|\} < \infty.$$

Proposition 3.3. *Let $S \in B_2(\Omega)$ be a backward operator weighted shift of multiplicity 2 with weights $\{W_k\}_{k=1}^{\infty}$. Then the two following conditions are equivalent:*
(1) $S \notin (SI)$.
(2) There exists a backward weighted shift operator $B_\Lambda \in B_1(\Omega)$ such that S is similar to $B_s B_\Lambda$.

Proof. (2) $\Rightarrow$ (1) is similar to the last part in the proof of Theorem 2.2.
(1) $\Rightarrow$ (2). Let $S \notin (SI)$, then S is similar to B_Λ^2, where B_Λ is defined by (2.2). Set

$$\widetilde{B_\Lambda} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{\|f_1\|}{\|f_3\|} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{\|f_2\|}{\|f_4\|} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{\|f_3\|}{\|f_5\|} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \vdots \end{matrix},$$

then $B_\Lambda^2 = B_s \widetilde{B_\Lambda}$. It suffices to prove that $\widetilde{B_\Lambda}$ belongs to $B_1(\Omega)$. Since $JB_s^2J^*$ in (2.1) belongs to $B_2(\Omega)$, then according to Lemma 3.2, there exist constants K_1 and K_2 such that $0 < K_1 < \frac{\|f_n\|}{\|f_{n+2}\|} < K_2$ for each $n \geq 1$. Then $\widetilde{B_\Lambda}$ belongs to $B_1(\Omega)$ by Lemma 3.2. $\square$

Let $S_d \in B_2(\Omega)$ be a backward operator weighted shift of multiplicity 2 with weights $\{W_k\}_{k=1}^{\infty}$, where $W_k = \begin{bmatrix} w_{k1} & 0 \\ 0 & w_{k2} \end{bmatrix}$. Denote $M_k = \prod_{i=1}^{k} W_i = \begin{bmatrix} m_{k1} & 0 \\ 0 & m_{k2} \end{bmatrix}$.
One can write

$$S_d = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{matrix} H_1 \\ H_2 \end{matrix},$$

where H_1 and H_2 are subspaces with orthonormal basis $\{e_{2k-1}\}_{k=1}^{\infty}$ and $\{e_{2k}\}_{k=1}^{\infty}$ respectively, and

$$T_1 = \begin{bmatrix} 0 & w_{11} & 0 & 0 & \dots \\ 0 & 0 & w_{21} & 0 & \dots \\ 0 & 0 & 0 & w_{31} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & w_{12} & 0 & 0 & \dots \\ 0 & 0 & w_{22} & 0 & \dots \\ 0 & 0 & 0 & w_{32} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We shall use the following theorem and give the proof in appendix.

Theorem 3.4. *Let S and S' be two backward operator weighted shifts of multiplicity r with weights $\{W_k\}_{k=1}^{\infty}$ and $\{W'_k\}_{k=1}^{\infty}$ respectively. Then S and S' are similar if and only if there exist an invertible $r \times r$ matrix D and a constant C such that*

$$\sup_k\{\|M_k^{-1}DM'_k\|, \|M_k'^{-1}D^{-1}M_k\|\} \leq C.$$

The following two lemmas are special cases of the above theorem.

Lemma 3.5. *Let B and B' be two backward weighted shift operators with weighted sequences $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\lambda'_k\}_{k=1}^{\infty}$ respectively. Denote $\beta_k = \prod_{i=1}^{k} \lambda_i$ and $\beta'_k = \prod_{i=1}^{k} \lambda'_i$. Then B and B' are similar if and only if there exist constants C_1 and C_2 such that*

$$C_1 < \inf_k\{|\frac{\beta_k}{\beta'_k}|\} \leq \sup_k\{|\frac{\beta_k}{\beta'_k}|\} < C_2.$$

Lemma 3.6. *Let S and S' be two backward operator weighted shifts of multiplicity 2 with weights $\{W_k\}_{k=1}^{\infty}$ and $\{W'_k\}_{k=1}^{\infty}$ respectively, where $W_k = \begin{bmatrix} w_{k1} & 0 \\ 0 & w_{k2} \end{bmatrix}$ and $W'_k = \begin{bmatrix} w'_{k1} & 0 \\ 0 & w'_{k2} \end{bmatrix}$. Denote $M_k = \prod_{i=1}^{k} W_i = \begin{bmatrix} m_{k1} & 0 \\ 0 & m_{k2} \end{bmatrix}$ and $M'_k = \prod_{i=1}^{k} W'_i = \begin{bmatrix} m'_{k1} & 0 \\ 0 & m'_{k2} \end{bmatrix}$. Then S and S' are similar if and only if there exist constants C_1 and C_2 such that*

$$C_1 < \inf_k\{|\frac{m_{k1}}{m'_{k1}}|, |\frac{m_{k2}}{m'_{k2}}|\} \leq \sup_k\{|\frac{m_{k1}}{m'_{k1}}|, |\frac{m_{k2}}{m'_{k2}}|\} < C_2.$$

Theorem 3.7. *Let S_d be defined above. Then the following three conditions are equivalent:*

(1) There exists a backward weighted shift operator $B_\Lambda \in B_1(\Omega)$ such that S_d is similar to B_Λ^2.

(2) There exist constants C_1 and C_2 such that

$$0 < C_1 < \inf_k\{|\frac{m_{k1}}{m_{k2}}|\} \leq \sup_k\{|\frac{m_{k1}}{m_{k2}}|\} < C_2.$$

(3) T_1 is similar to T_2.

Proof. (1) $\Rightarrow$ (2). Let $B_\Lambda \in B_1(\Omega)$ be the backward weighted shift operator with weighted sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that S_d is similar to B_Λ^2, which is a backward operator weighted shift of multiplicity 2 with weights $\{W'_k\}_{k=1}^{\infty}$, where $W'_k =$

$\begin{bmatrix}\lambda_{2k-1}\lambda_{2k}, & 0\\ 0, & \lambda_{2k}\lambda_{2k+1}\end{bmatrix}$. Then $m'_{k1}=\beta_{2k}$ and $m'_{k2}=\frac{\lambda_{2k+1}}{\lambda_1}\beta_{2k}$. Consequently, by Lemma 3.6, there exist constants C'_1 and C'_2 such that

$$C'_1<\inf_k\{|\frac{m_{k1}}{m'_{k1}}|,|\frac{m_{k2}}{m'_{k2}}|\}\leq\sup_k\{|\frac{m_{k1}}{m'_{k1}}|,|\frac{m_{k2}}{m'_{k2}}|\}<C'_2,$$

then there exist constants C_1 and C_2 such that

$$C_1<\inf_k\{|\frac{m_{k1}}{\beta_{2k}}|,|\frac{m_{k2}}{\beta_{2k}}|\}\leq\sup_k\{|\frac{m_{k1}}{\beta_{2k}}|,|\frac{m_{k2}}{\beta_{2k}}|\}<C_2.$$

Therefore,

$$C_1{C_2}^{-1}<\inf_k\{|\frac{m_{k1}}{m_{k2}}|\}\leq\sup_k\{|\frac{m_{k1}}{m_{k2}}|\}<C_2{C_1}^{-1}.$$

(2) $\Rightarrow$ (3). It is obtained immediately by Lemma 3.5.

(3) $\Rightarrow$ (1). Since T_1 is similar to T_2, then S_d is similar to $S'_d=T_1\oplus T_1$. Notice

$$S'_d=\begin{bmatrix}0&0&w_{11}&0&0&0&\dots\\0&0&0&w_{11}&0&0&\dots\\0&0&0&0&w_{21}&0&\dots\\0&0&0&0&0&w_{21}&\dots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}.$$

Let

$$B_\Lambda=\begin{bmatrix}0&1&0&0&0&\dots\\0&0&w_{11}&0&0&\dots\\0&0&0&1&0&\dots\\0&0&0&0&w_{21}&\dots\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{bmatrix}.$$

Then $B_\Lambda\in B_1(\Omega)$ and $B_\Lambda^2=S'_d$, i.e., S_d is similar to B_Λ^2. □

Remark 3.8. If $S_d\in B_2(\Omega)$ is a backward operator weighted shift of multiplicity 2 with weights $W_k=\begin{bmatrix}1&0\\0&2\end{bmatrix}$, $k=1,2,\cdots$, then by Theorem 2.2, there exists a backward weighted shift operator B_Λ such that $S_d\sim B_\Lambda^2$. But B_Λ is not a Cowen-Douglas operator from Theorem 3.7.

4. Appendix: the proof of Theorem 3.4

The proof of Theorem 3.4. " $\Leftarrow$ ". Let

$$A=\begin{bmatrix}M_0^{-1}DM'_0&0&0&\dots\\0&M_1^{-1}DM'_1&0&\dots\\0&0&M_2^{-1}DM'_2&\dots\\\vdots&\vdots&\vdots&\ddots\end{bmatrix}.$$

Then A is an invertible bounded linear operator and

$$SA = \begin{bmatrix} 0 & M_0^{-1}DM_1' & 0 & 0 & \dots \\ 0 & 0 & M_1^{-1}DM_2' & 0 & \dots \\ 0 & 0 & 0 & M_2^{-1}DM_3' & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = AS'.$$

Thus S and S' are similar.

" $\Rightarrow$ " Suppose S and S' are similar. Then there exists an invertible bounded linear operator A such that $SA = AS'$. Moreover, $A^{-1}S = S'A^{-1}$. Write

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} & \dots \\ A_{21} & A_{22} & \dots & A_{2n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad B = A^{-1} = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} & \dots \\ B_{21} & B_{22} & \dots & B_{2n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mn} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where A_{ij} and B_{ij} are $r \times r$ matrices for every $i, j \in \mathbb{N}$. Since

$$\begin{aligned} SA &= \begin{bmatrix} W_1A_{21} & W_1A_{22} & \dots & W_1A_{2n} & \dots \\ W_2A_{31} & W_2A_{32} & \dots & W_2A_{3n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W_{m-1}A_{m1} & W_{m-1}A_{m2} & \dots & W_{m-1}A_{mn} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\ = AS' &= \begin{bmatrix} 0 & A_{11}W_1' & \dots & A_{1n-1}W_{n-1}' & \dots \\ 0 & A_{21}W_1' & \dots & A_{2n-1}W_{n-1}' & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & A_{m1}W_1' & \dots & A_{mn-1}W_{n-1}' & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \end{aligned}$$

then

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & \dots \\ 0 & M_1^{-1}A_{11}M_1' & M_1^{-1}A_{12}M_1'^{-1}M_2' & M_1^{-1}A_{13}M_2'^{-1}M_3' & \dots \\ 0 & 0 & M_2^{-1}A_{11}M_2' & M_2^{-1}A_{12}M_1'^{-1}M_3' & \dots \\ 0 & 0 & 0 & M_3^{-1}A_{11}M_3' & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Similarly,

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & \dots \\ 0 & M_1'^{-1}B_{11}M_1 & M_1'^{-1}B_{12}M_1{}^{-1}M_2 & M_1'^{-1}B_{13}M_2{}^{-1}M_3 & \dots \\ 0 & 0 & M_2'^{-1}B_{11}M_2 & M_2'^{-1}B_{12}M_1{}^{-1}M_3 & \dots \\ 0 & 0 & 0 & M_3'^{-1}B_{11}M_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Notice A and B are bounded and $A_{11}^{-1} = B_{11}$. Consequently, we can take $C = \max\{\|A\|, \|A^{-1}\|\}$ and $D = A_{11}$ which is an invertible $r \times r$ matrix, then

$$\sup_k\{\|M_k^{-1}DM_k'\|, \|M_k'^{-1}D^{-1}M_k\|\} \leq C.$$

Acknowledgement. A large part of this article was developed during the seminar on operator theory and dynamical systems held at the University of Jilin in China. The authors thank Professor Cao Yang for useful discussions and suggestions. We want to also thank the referee whose comments produced an improvement of the presentation of the paper. This work was supported by the National Nature Science Foundation of China (Grant No. 11001099).

References

1. M.J. Cowen and R.G. Douglas, *Complex geometry and operator theory*, Acta Mathematica **141** (1978), no. 1, 187–261.
2. I.C. Gohberg and M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Trans. Math. Monogr. AMS, Providence, 1969.
3. S. Jaffard and R.M. Young, *A representation theorem for Schauder bases in Hilbert space*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 553–560.
4. C.L. Jiang and J.X. Li, *The irreducible decomposition of Cowen-Douglas operators and operator weighted shifts*, Acta Sci. Math. (Szeged) **66** (2000), no. 3-4, 679–695.
5. J.X. Li, Y.Q. Ji and S.L. Sun, *The essential spectrum and Banach reducibility of operator weighted shifts*, Acta Math. Sin. (Engl. Ser.) **17** (2001), no. 3, 413–424.
6. R.E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics 183, Springer-Verlag New York, Inc., 1998.

Department of Mathematics , Jilin university, 130012, Changchun, P.R. China.
E-mail address: houbz@jlu.edu.cn
E-mail address: tiangeng09@mails.jlu.edu.cn