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# WEIGHTED CLASSES OF QUATERNION-VALUED FUNCTIONS

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Abstract. In this paper, we define the classes $F(p,q,s)$ of quaternion-valued functions, then we characterize quaternion Bloch functions by quaternion $F(p,q,s)$ functions in the unit ball of $\mathbb{R}^3$. Further, some important basic properties of these functions are also considered.

## 1. Introduction and preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disk. Let $0 < p < \infty$. An analytic function $f$ in $\mathbb{D}$ belongs to the Hardy space $H^p$ (see [11, 18]), if

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}|^p d\theta < \infty;$$

$f$ is in $H^\infty$, if

$$\sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

It is well known that $f \in H^2$ if and only if

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)\, dA(z) < \infty,$$

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where $dA(z)$ is the Euclidean area element $dx\,dy$. For $0 < p < \infty$, an analytic function $f$ in $\mathbb{D}$ belongs to the Bergman space $L^p_a$ (see [12]), if

$$\int\limits_{\mathbb{D}} |f(z)|^2 dA(z) < \infty.$$

The well known $\alpha$-Bloch space (see [29]) is defined by:

$$\mathcal{B}^\alpha = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \mathcal{B}^\alpha(f) = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty\},$$

where $0 < \alpha < \infty$. The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$. The little $\alpha$-Bloch space $\mathcal{B}^\alpha_0$ is a subspace of $\mathcal{B}^\alpha$ consisting of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |f'(z)| = 0.$$

The Dirichlet space is given by:

$$\mathcal{D} = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \int\limits_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty\}.$$

Let $0 < p < \infty$. Then the Besov-type spaces

$$B^p = \big\{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \big|f'(z)\big|^p \big(1-|z|^2\big)^{p-2} \big(1-|\varphi_a(z)|^2\big)^2 dA(z) < \infty\big\}$$

are introduced and studied intensively (see [24]). Here, $\varphi_a$ always stands for the Möbius transformation $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. From [24] it is known that the $B^p$ spaces can be used to describe the Bloch space $\mathcal{B}$ equivalently by the integral norms of $B^p$. Composing the Möbius transform $\varphi_a(z)$, which maps the unit disk $\mathbb{D}$ onto itself, and the fundamental solution of the two-dimensional real Laplacian on $\mathbb{D}$, we obtain the Green's function $g(z,a) = \ln|\frac{1-\bar{a}z}{a-z}|$ with logarithmic singularity at $a \in \mathbb{D}$. Then the spaces

$$Q_p = \{f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}} \int\limits_{\mathbb{D}} |f'(z)|^2 g^p(z,a) dA(z) < \infty\}$$

are defined in [6]. The idea of these $Q_p$-spaces is to find a scale of spaces with $\mathcal{D}$ and $\mathcal{B}$, respectively, "at the both end points" of the scale. In [28] Zhao gave the following definition:

**Definition 1.1.** Let $f$ be an analytic function in $\mathbb{D}$ and let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) = 0,$$

then $f \in F_0(p,q,s)$.

The spaces $F(p, q, s)$ were intensively studied by Zhao in [28] and Rättyä in [21]. It is known from ([28], Theorem 2.10) that, for $p \geq 1$, the spaces $F(p, q, s)$ are Banach spaces under the norm

$$\|f\| = \|f\|_{F(p,q,s)} + |f(0)|.$$

Moreover, it is known that in (Definition 1.1) the Green's function $g(z, a)$ can be replaced by the weight function $1 - |\varphi_a(z)|^2$ and that for $q + s \leq -1$ the spaces $F(p, q, s)$ and $F_0(p, q, s)$ both reduce to the space of constant functions (see [28], theorem 2.4 and proposition 2.12 ). It is sometimes convenient to replace the parameter $q$ by $p - 2$ and consider the spaces $F(p, p - 2, s)$ and $F_0(p, p - 2, s)$ instead of the spaces $F(p, q, s)$ and $F_0(p, q, s)$ (see [21]).
If $q = p - 2$ and $s = 0$, we denote $F(p, p - 2, 0) = F_0(p, p - 2, 0) = B^p$.

*Remark* 1.2. The interest of the spaces $F(p, q, s)$ come from that these spaces cover a lot of known spaces. Zhao in [28] collected the following immediate relations of $F(p, q, s)$ and $F_0(p, q, s)$ :

(1) $F(p, q, s) = \mathcal{B}^{\frac{q+2}{p}}$ and $F_0(p, q, s) = \mathcal{B}_0^{\frac{q+2}{p}}$, for $s > 1$.
(2) $F(2, 0, s) = Q_s$, $F_0(2, 0, s) = Q_{s,0}$.
(3) $F(2, 1, 0) = H^2$.
(4) $F(p, p, 0) = L^p_a$, for $1 \leq p < \infty$.
(5) $F(p, p - 2, 0) = B^p$ , for $1 < p < \infty$.

For more studies on the spaces $F(p, q, s)$ in the unit disk or in the unit ball of $\mathbb{C}^n$, we refer to [4, 17, 19, 20, 21, 26, 27, 28, 30].

## 2. Quaternion function spaces

Let $\mathbb{H}$ be the skew field of quaternions. This means we can write each element $z \in \mathbb{H}$ in the form

$$z = z_0 + z_1 i + z_2 j + z_3 k, \quad z_0, z_1, z_2, z_3 \in \mathbb{R},$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, kj = -jk = i, ki = -ik = j.$$

The conjugate element $\bar{z}$ is given by $\bar{z} = z_0 - z_1 i - z_2 j - z_3 k$ and we have the property

$$z\bar{z} = \bar{z}z = \|z\|^2 = z_0^2 + z_1^2 + z_2^2 + z_3^2.$$

Moreover, we can identify each vector $\vec{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion $x$ of the form

$$x = x_0 + x_1 i + x_2 j.$$

In what follows we will work in $\mathbb{B} \subset \mathbb{R}^3$, the unit ball in the real three-dimensional space. $\mathbb{B}$ is a bounded, simply connected domain with a $C^\infty$-boundary $S_1(0)$. Moreover, we will consider functions $f$ defined on $\mathbb{B}$ with values in $\mathbb{H}$. Let $\Omega$ be a domain in $\mathbb{R}^3$, then we will consider $\mathbb{H}$-valued functions defined in $\Omega$ (depending on $x = (x_0, x_1, x_2)$):

$$f : \Omega \longrightarrow \mathbb{H}.$$

The notation $C^p(\Omega;\mathbb{H}), p \in \mathbf{N}\cup\{0\}$, has the usual component-wise meaning. On $C^1(\Omega;\mathbb{H})$ we define a generalized Cauchy-Riemann operator $D$ by

$$Df = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2}$$

and it's conjugate operator by

$$\overline{D}f = \frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2}.$$

The solutions of $Df = 0,\ x \in \Omega$, are called (left) hyperholomorphic (or monogenic) functions and generalize the class of holomorphic functions from the one-dimensional complex function theory. For more details about quaternionic analysis and general Clifford analysis, we refer to [9], [14], [16] and [25] and others. For $|a| < 1$, we will denote by

$$\varphi_a(x) = (a-x)(1-\bar{a}x)^{-1}$$

the Möbius transform, which maps the unit ball onto itself. Furthermore, let

$$g(x,a) = \frac{1}{4\pi}\left(\frac{1}{|\varphi_a(x)|} - 1\right)$$

be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$ composed with the Möbius transform $\varphi_a(x)$. Especially, we denote for all $p \geq 0$

$$g^p(x,a) = \frac{1}{4^p\pi^p}\left(\frac{1}{|\varphi_a(x)|} - 1\right)^p.$$

Let $f : \mathbb{B} \mapsto \mathbb{H}$ be a hyperholomorphic function. Then from [13], we have the seminorms

- $\mathcal{B}(f) = \sup\limits_{x\in\mathbb{B}}(1-|x|^2)^{3/2}|\overline{D}f(x)|$,
- $Q_p(f) = \sup\limits_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^2 g^p(x,a)d\mathbb{B}_x$,

which lead to the following definitions:

**Definition 2.1.** (see [13]) The spatial (or three-dimensional) Bloch space $\mathcal{B}$ is the right $\mathbb{H}$-module of all hyperholomorphic functions $f : \mathbb{B} \mapsto \mathbb{H}$ with $\mathcal{B}(f) < \infty$.

**Definition 2.2.** (see [13]) The right $\mathbb{H}$-module of all quaternion-valued functions $f$ defined on the unit ball, which are hyperholomorphic and satisfy $Q_p(f) < \infty$, is called $Q_p$-space.

*Remark* 2.3. Because of the special structure of $g(x,a)$ the seminorms $Q_p(f)$ make sense for $p < 3$ only. Consequently, we will consider in this paper $Q_p$-spaces for $p < 3$ only.

With the generalized Cauchy-Riemann operator $D$, its adjoint $\overline{D}$, the hypercomplex Möbius transformation $\varphi_a(x) = (a-x)(1-\bar{a}x)^{-1}$, and a modified fundamental solution $g$ of the real Laplacian Gürlebeck et al. [13] considered generalized

$Q_p$-spaces defined by

$$Q_p = \{f \in \ker D : \sup_{a\in\mathbb{B}} \int\limits_{\mathbb{B}} |\overline{D}f(x)|^2 \big(g(\varphi_a(x))\big)^p \, d\mathbb{B}_x < \infty\}.$$

where $\mathbb{B}$ stands for the unit ball in $\mathbb{R}^3$.

**Definition 2.4.** The right $\mathbb{H}$-module of all quaternion-valued functions $f$ defined on the unit ball, which are hyperholomorphic and satisfy the condition

$$\int_{\mathbb{B}} |\overline{D}f(x)|^2 d\mathbb{B}_x < \infty,$$

is called spatial (or three-dimensional) Dirichlet space $\mathcal{D}$.

The quaternion $\alpha$-Bloch space (see [2]) is defined by:

$$\mathcal{B}^\alpha = \{f : f \in KerD \ \text{ and } \ \mathcal{B}^\alpha(f) = \sup_{x\in\mathbb{B}} (1-|x|^2)^{\frac{3}{2}\alpha} |\overline{D}f(x)| < \infty\},$$

where $0<\alpha<\infty$. The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$. The little quaternion $\alpha$-Bloch space $\mathcal{B}^\alpha_0$ is a subspace of $\mathcal{B}^\alpha$ consisting of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|x|\to 1} (1-|x|^2)^{\frac{3}{2}\alpha} |\overline{D}f(x)| = 0.$$

Now, we give the following definition:

**Definition 2.5.** Let $f$ be quaternion-valued function in $\mathbb{B}$. For $0<p<\infty$, $-2<q<\infty$ and $0<s<\infty$. If

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^p (1-|x|^2)^{\frac{3q}{2}} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{B}} |\overline{D}f(x)|^p (1-|x|^2)^{\frac{3q}{2}} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x = 0,$$

then $f \in F_0(p,q,s)$.

*Remark* 2.6. Obviously, these spaces are not Banach spaces. Nevertheless, if we consider a small neighborhood of the origin $N_\epsilon$, with an arbitrary but fixed $\epsilon>0$, then we can add the $L_1$-norm of the function $f$ over $N_\epsilon$ to the seminorms, so $F(p,q,s)$ spaces will become Banach spaces. Also, $F(p,q,s)$ spaces are not linear spaces.

*Remark* 2.7. It should be remarked that if we put $q=0$ and $p=2$, then $F(2,0,s)=Q_s$. Also, if $p=2$ and $s=q=0$, then $F(2,0,0)=\mathcal{D}$, the quaternion Dirichlit space.

The main aim of this paper is to study these $F(p,q,s)$ spaces and their relations to the above mentioned quaternionic Bloch space. It will be shown that these exponents $p$ and $q$ generate a new scale of spaces, equivalent to the Bloch space for all $p$ and $q$. The concept may be generalized in the context of Clifford analysis to arbitrary real dimensions. We will restrict us for simplicity to $\mathbb{R}^3$ and quaternion-valued functions as (the lowest non-commutative case) a model case.

For more studies on quaternion function spaces, we refer to [1, 2, 3, 5, 7, 8, 10, 13, 15, 22] and others.
Let $U(a, R) = \{x : |\varphi_a(x)| < R\}$ be the pseudo-hyperbolic ball with radius $R$, where $0 < R < 1$. Analogously to the complex case (see [24]), for a point $a \in \mathbb{B}$ and $0 < R < 1$, we can get that $U(a, R)$ with pseudo-hyperbolic center $a$ and pseudo hyperbolic radius $R$ is a Euclidean disc: its Euclidean center and Euclidean radius are $\frac{(1-R^2)a}{1-R^2|a|^2}$ and $\frac{(1-|a|^2)R}{1-R^2|a|^2}$, respectively.
We will need the following lemma in the sequel:

**Lemma 2.8.** [22] *Let $f : \mathbb{B} \longrightarrow \mathbb{H}$ be a hyperholomorphic function. Suppose that $0 < R < 1$ and $1 < q < \infty$. Then for every $a \in \mathbb{B}$, we have*

$$|\overline{D}f(a)|^q \leq \frac{3(4)^{2+q}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^3} \int_{U(a,R)} \left|\overline{D}f(x)\right|^q d\mathbb{B}_x.$$

## 3. $F(p, q, s)$-spaces in Clifford Analysis

In this section, relations between $F(p, q, s)$ and Bloch spaces, which have been attracted considerable attention are given in quaternion sense. Our results are extensions of the results due to Zhao (see [28]) in quaternion sense. We consider some essential properties of $F(p, q, s)$ spaces of quaternion-valued functions as basic scale properties.

**Proposition 3.1.** *Let $f$ be a hyperholomorphic function in $\mathbb{B}$ and $f \in \mathcal{B}^{\frac{3(q+2)}{2p}}$. Then for $0 < p < \infty$, $-2 < q < \infty$ and $2 < s < \infty$, we have that*

$$\int_{\mathbb{B}} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x \leq \lambda\left(\mathcal{B}(f)\right)^{\frac{3}{2}(q+2)}.$$

*Proof.* For $\alpha > 0$, we have

$$\left(1-|x|^2\right)^{\frac{3}{2}\alpha}\left|\overline{D}f(x)\right| \leq \mathcal{B}^\alpha(f).$$

Then, for $\alpha = \frac{3(q+2)}{2p}$, we deduce that

$$\begin{aligned}
& \int_{\mathbb{B}} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x \\
\leq\ & \left(\mathcal{B}(f)\right)^{\frac{3}{2}(q+2)} \int_{\mathbb{B}} \left(1-|x|^2\right)^{-3} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x \\
=\ & \left(\mathcal{B}(f)\right)^{\frac{3}{2}(q+2)} \int_{\mathbb{B}} \left(1-|\varphi_a(x)|^2\right)^{-3} \left(1-|x|^2\right)^s \frac{(1-|a|^2)^3}{|1-\bar{a}x|^6} d\mathbb{B}_x\,.
\end{aligned}$$

Here, we used that the Jacobian determinant is $\frac{(1-|a|^2)^3}{|1-\bar{a}x|^6}$. Now, using the equality

$$\left(1-|\varphi_a(x)|^2\right) = \frac{\left(1-|a|^2\right)\left(1-|x|^2\right)}{|1-\bar{a}x|^2}$$

we obtain that,

$$\begin{aligned}&\int_{\mathbb{B}} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x\\ \leq\ & \lambda \left(\mathcal{B}(f)\right)^{\frac{3}{2}(q+2)} \int_0^1 \left(1-r^2\right)^{s-3} r^2 dr,\end{aligned}$$

where $\lambda$ is a positive constant. The integral

$$\int_0^1 \left(1-r^2\right)^{s-3} r^2 dr < \infty$$

for $2 < s < \infty$. This completes the proof.

**Corollary 3.2.** *From proposition 3.1, for $0 < p < \infty$, $-2 < q < \infty$ and $2 < s < \infty$, then we have that*

$$\mathcal{B}^{\frac{3(q+2)}{2p}} \subset F(p,q,s).$$

**Proposition 3.3.** *Let $f$ be a hyperholomorphic function in the unit ball $\mathbb{B}$. Then for $1 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$, we have*

$$\left(1-|a|^2\right)^{\frac{3}{2}(q+2)} \left|\overline{D}f(a)\right|^p \leq \frac{48(2)^{2p}}{\pi R^3 (1-R^2)^{s+2p}} \int_{\mathbb{B}} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x,$$

*where $0 < R < 1$.*

*Proof.* For a fixed $R \in (0,1)$, let

$$E(a,R) = \left\{x \in \mathbb{B} : |x-a| < R|1-a|\right\}$$

Then, we have

$$\begin{aligned}&\int_{\mathbb{B}} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}p} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x\\ \geq\ & \int_{U(a,R)} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x\\ \geq\ & (1-R^2)^s \int_{U(a,R)} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} d\mathbb{B}_x\\ \geq\ & (1-R^2)^s \int_{E(a,R)} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} d\mathbb{B}_x\\ \geq\ & (1-R^2)^s \left(1-|a|^2\right)^{\frac{3}{2}q} \int_{E(a,R)} \left|\overline{D}f(x)\right|^p d\mathbb{B}_x\,.\end{aligned}$$

Then, applying Lemma 2.8, we obtain

$$\begin{aligned}&\int_{\mathbb{B}} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x\\ \geq\ & \frac{4^{-(p+2)}\pi R^3}{3} \left(1-R^2\right)^{s+2p} \left(1-|a|^2\right)^{\frac{3}{2}q+3} |\overline{D}f(a)|^p,\end{aligned}$$

which implies that,

$$\left(1-|a|^2\right)^{\frac{3}{2}(q+2)} \left|\overline{D}f(a)\right|^p \leq \frac{48(2)^{2p}}{\pi R^3 (1-R^2)^{s+2p}} \int_{\mathbb{B}} \left|\overline{D}f(x)\right|^p \left(1-|x|^2\right)^{\frac{3}{2}q} \left(1-|\varphi_a(x)|^2\right)^s d\mathbb{B}_x,$$

This completes the proof.

**Corollary 3.4.** *From proposition 3.2, we get for* $1 < p < \infty$, $-2 < q < \infty$ *and* $0 < s < \infty$ *that*

$$F(p, q, s) \subset \mathcal{B}^{\frac{3(q+2)}{2p}}.$$

The following result gives a characterization of the quaternion Bloch space by quaternion $F(p, q, s)$ spaces.

**Theorem 3.5.** *Let f be hyperholomorphic in the unit ball* $\mathbb{B}$. *Then for* $1 < p < \infty$, $-2 < q < \infty$ *and* $2 < s < \infty$, *we have that*

$$F(p, q, s) = \mathcal{B}^{\frac{3(q+2)}{2p}}.$$

*Proof.* The proof follows from Corollaries 3.2 and 3.4.
The importance of the above theorem is to give us a characterization for the hyperholomorphic Bloch space by the help of integral norms of $F(p, q, s)$ spaces of hyperholomorphic functions.

Also, with the same arguments used to prove the previous theorem, we can prove the following theorem for characterization of little hyperholomorphic Bloch space.

**Theorem 3.6.** *Let f be hyperholomorphic in the unit ball* $\mathbb{B}$. *Then, for* $1 < p < \infty$, $-2 < q < \infty$ *and* $2 < s < \infty$, *we have that*

$$F_0(p, q, s) = \mathcal{B}_0^{\frac{3(q+2)}{2p}}.$$

## 4. Weights in quaternion $F(p, q, s)$-spaces

In this section, we give a characterization for the quaternion $F(p, q, s)$ spaces in terms of some different weighted functions in the unit ball of $\mathbb{R}^3$.

**Theorem 4.1.** *Let f be a hyperholomorphic function in* $\mathbb{B}$. *Then, for* $1 < q < 4$ *and* $1 \leq p \leq 2 + \frac{q}{4}$, *we have that*

$$f \in F(p, q, s) \Leftrightarrow \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |\overline{D} f(x)|^p (1 - |x|^2)^{\frac{3}{2}q} \big(g(x, a)\big)^s d\mathbb{B}_x < \infty.$$

*Proof.* First, we consider the equivalence

$$\int_{\mathbb{B}} |\overline{D} f(x)|^p (1 - |x|^2)^{\frac{3}{2}q} (1 - |\varphi_a(x)|^2)^s d\mathbb{B}_x \simeq \int_{\mathbb{B}} |\overline{D} f(x)|^p (1 - |x|^2)^{\frac{3}{2}q} \big(g(x, a)\big)^s d\mathbb{B}_x,$$

with $g(x, a) = \frac{1}{4\pi} \left( \frac{1}{|\varphi_a(x)|} - 1 \right)$ and $\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}$ the Möbius-transform, which maps the unit ball onto itself. After a change of variables $w = \varphi_a(x)$ (the Jacobian determinant $\left( \frac{1-|a|^2}{|1-\bar{a}w|^2} \right)^3$ has no singularities) we get

$$\begin{aligned} &\int_{\mathbb{B}} |\overline{D}_x f(\varphi_a(w))|^p (1 - |\varphi_a(w)|^2)^{\frac{3}{2}q} (1 - |w|^2)^s \left( \frac{1 - |a|^2}{|1 - \bar{a}w|^2} \right)^3 d\mathbb{B}_w \\ \simeq \quad &\int_{\mathbb{B}} |\overline{D}_x f(\varphi_a(w))|^p (1 - |\varphi_a(w)|^2)^{\frac{3}{2}q} g^s(w, 0) \left( \frac{1 - |a|^2}{|1 - \bar{a}w|^2} \right)^3 d\mathbb{B}_w, \end{aligned}$$

where $D_x$ means the Cauchy-Riemann-operator with respect to $x$.
The problem here is, that $\overline{D}_x f(x)$ is hyperholomorphic, but after the change of variables $\overline{D}_x f(\varphi_a(w))$ is not hyperholomorphic. But we know from [23] that $\frac{1-\bar{w}a}{|1-\bar{a}w|^3}\overline{D}_x f(\varphi_a(w))$ is again hyperholomorphic. We also refer to [25] who studied this problem for the four-dimensional case already in 1979. Therefore, we get

$$\begin{aligned}
&\int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}\, d\mathbb{B}_w \\
\simeq\ & \frac{1}{(4\pi)^s}\int_{\mathbb{B}} |\psi(w,a)|^p \left(\frac{1}{|w|}-1\right)^s \frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}\, d\mathbb{B}_w,
\end{aligned}$$

with $\psi(w,a)=\frac{1-\bar{w}a}{|1-\bar{a}w|^3}\overline{D}_x f(\varphi_a(w))$. This means we have to find constants $C_1(s)$ and $C_2(s)$ with

$$\begin{aligned}
&\frac{1}{(4\pi)^s} C_1(s) \int_{\mathbb{B}} |\psi(w,a)|^p \left(\frac{1}{|w|}-1\right)^s \frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}\, d\mathbb{B}_w \\
\leq\ & \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}\, d\mathbb{B}_w \\
\leq\ & \frac{1}{(4\pi)^s} C_2(s) \int_{\mathbb{B}} |\psi(w,a)|^p \left(\frac{1}{|w|}-1\right)^s \frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}\, d\mathbb{B}_w.
\end{aligned}$$

**Part 1**

Let $C_2(s)=2^s(4\pi)^s$. Then, using the inequalities

$$1-|a| \leq |1-\bar{a}w| \leq 1+|a| \quad \text{and} \quad 1-|w| \leq |1-\bar{a}w| \leq 1+|w|,$$

we obtain that

$$\begin{aligned}
I_1 =\ & \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} d\mathbb{B}_w \\
-\ & 2^s \int_{\mathbb{B}} |\psi(w,a)|^p \left(\frac{1}{|w|}-1\right)^s \frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} d\mathbb{B}_w \\
=\ & \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \left\{1-\frac{2^s(1-|w|)^s}{|w|^s(1-|w|^2)^s}\right\} d\mathbb{B}_w \\
=\ & \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|^2)^{\frac{3}{2}q+s} \frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}} \left\{1-\frac{2^s}{|w|^s(1+|w|)^s}\right\} d\mathbb{B}_w \\
\leq\ & (2)^{3q+s+3} \int_{\mathbb{B}} |\psi(w,a)|^p (1-|w|)^{s-2p-3} \left\{1-\frac{2^s}{|w|^s(1+|w|)^s}\right\} d\mathbb{B}_w \\
=\ & (2)^{3q+s+3} \int_0^1 \left(M_p(\overline{D}f,r)\right)^p (1-r)^{s-2p-3} \left(1-\frac{2^s}{r^s(1+r)^s}\right) r^2 dr \leq 0
\end{aligned}$$

with

$$\left(M_p(\overline{D}f,r)\right)^p = \int_0^{\pi}\int_0^{2\pi} \left|h(r)\overline{D}f(r,\theta_1,\theta_2)\right|^p \sin\theta_1 d\theta_2 d\theta_1,$$

where, $h(r)$ stands for $\frac{1}{|1-\bar{a}w|^2}$ in spherical coordinates.

Because $\left(M_p(\overline{D}f,r)\right)^p \geq 0 \ \forall\ r \in\ [0,1]$ and $(1-r)^{s-2p-3}\left(1-\frac{2^s}{r^s(1+r)^s}\right)r^2 \leq 0$ $\forall\ r \in\ [0,1]$, $0<p<\frac{s}{2}-1;\ \ 2<s<\infty$ and $0<q<\infty$. Hence, we deduce that

$$\begin{aligned}
&\int_{\mathbb{B}} |\psi(w,a)|^p(1-|w|^2)^{\frac{3}{2}q+s}\frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}\,d\mathbb{B}_w \\
\leq\ & \frac{1}{(4\pi)^s}C_2(s)\int_{\mathbb{B}}|\psi(w,a)|^p\left(\frac{1}{|w|}-1\right)^s\frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}\,d\mathbb{B}_w.
\end{aligned}$$

**Part 2**

Let $C_1(s)=\left(\frac{11}{100}\right)^s(4\pi)^s$. Then,

$$\begin{aligned}
I_2 =& \int_{\mathbb{B}} |\psi(w,a)|^p(1-|w|^2)^{\frac{3}{2}q+s}\frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}d\,\mathbb{B}_w \\
-\ & \frac{C_1(s)}{(4\pi)^s}\int_{\mathbb{B}}|\psi(w,a)|^p\left(\frac{1}{|w|}-1\right)^s\frac{(1-|w|^2)^{\frac{3}{2}q}(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}d\mathbb{B}_w \\
=\ & \int_{\mathbb{B}} |\psi(w,a)|^p(1-|w|^2)^{\frac{3}{2}q+s}\frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}qp+3)}}d\mathbb{B}_w
\end{aligned}$$

where $G(|w|)=1-\left(\frac{11}{100}\right)^s\left(\frac{1}{|w|(1+|w|)}\right)^s$ To get our estimates, the integral $I_2$ must be greater than or equal to zero. Now, we have

$$\begin{aligned}
I_2 \ =\ & -\int_{\mathbb{B}_{\frac{1}{10}}}|\psi(w,a)|^p(1-|w|^2)^{\frac{3}{2}q+s}\frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}G(|w|)d\mathbb{B}_w \\
+\ & \int_{\mathbb{B}_{\frac{5}{10}}\setminus\mathbb{B}_{\frac{1}{10}}}|\psi(w,a)|^p(1-|w|^2)^{\frac{3}{2}q+s}\frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}G(|w|)d\mathbb{B}_w \\
+\ & \int_{\mathbb{B}_{\frac{6}{10}}\setminus\mathbb{B}_{\frac{5}{10}}}|\psi(w,a)|^p(1-|w|^2)^{\frac{3}{2}q+s}\frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}G(|w|)d\mathbb{B}_w \\
+\ & \int_{\mathbb{B}\setminus\mathbb{B}_{\frac{6}{10}}}|\psi(w,a)|^p(1-|w|^2)^{\frac{3}{2}q+s}\frac{(1-|a|^2)^{\frac{3}{2}q+3}}{|1-\bar{a}w|^{2(\frac{3}{2}q+p+3)}}G(|w|)d\mathbb{B}_w, \qquad (4.1)
\end{aligned}$$

where $\mathbb{B}_r$ is the ball centered at zero with radius $r$. It is clear that the second and the fourth integrals in (4.1) are greater than zero. Therefore, it is sufficient to compare the first and the third integrals in (4.1). Now, since in $\mathbb{B}_{\frac{1}{10}}$, we have that $\frac{9}{10}\leq 1-|w|\leq|1-\bar{a}w|$ and in $\mathbb{B}_{\frac{6}{10}}\setminus\mathbb{B}_{\frac{5}{10}}$, we have

$$1-|w|\leq|1-\bar{a}w|\leq\frac{16}{10}.$$

Then,

$$-\left(\frac{10}{9}\right)^{2(\frac{3}{2}q+p+3)}\int\limits_{0}^{\frac{1}{10}}\left(M_p(\overline{D}f,r)\right)^p\left(1-r^2\right)^{\frac{3}{2}q+s}\left(1-\left(\frac{11}{100}\right)^s\frac{1}{r^s(1+r)^s}\right)r^2dr$$

$$\leq \left(\frac{10}{16}\right)^{2(\frac{3}{2}q+p+3)}\int\limits_{\frac{5}{10}}^{\frac{6}{10}}\left(M_p(\overline{D}f,r)\right)^p\left(1-r^2\right)^{\frac{3}{2}q+s}\left(1-\left(\frac{11}{100}\right)^s\frac{1}{r^s(1+r)^s}\right)r^2dr.$$

In particular we have that $M_p(\overline{D}f,r)$ is a nondecreasing function, this because $\overline{D}f$ is harmonic in $\mathbb{B}$ and belongs to $L_p(\mathbb{B})$; $\forall\ 0\leq r<1$.
Thus, $I_2\geq 0$, and our theorem is therefore established.

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