Banach J. Math. Anal. 6 (2012), no. 2, 86–97

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

UPPER BEURLING DENSITY OF SYSTEMS FORMED BY TRANSLATES OF FINITE SETS OF ELEMENTS IN $L^p(\mathbb{R}^d)$

BEI LIU[1] AND RUI LIU[2]*

Communicated by D. R. Larson

ABSTRACT. In this paper, we prove that if a finite disjoint union of translates $\bigcup_{k=1}^n \{f_k(x-\gamma)\}_{\gamma\in\Gamma_k}$ in $L^p(\mathbb{R}^d)$ $(1<p<\infty)$ is a p'-Bessel sequence for some $1<p'<\infty$, then the disjoint union $\Gamma=\bigcup_{k=1}^n \Gamma_k$ has finite upper Beurling density, and that if $\bigcup_{k=1}^n \{f_k(x-\gamma)\}_{\gamma\in\Gamma_k}$ is a (C_q)-system with $1/p+1/q=1$, then Γ has infinite upper Beurling density. Thus, no finite disjoint union of translates in $L^p(\mathbb{R}^d)$ can form a p'-Bessel (C_q)-system for any $1<p'<\infty$. Furthermore, by using techniques from the geometry of Banach spaces, we obtain that, for $1<p\le 2$, no finite disjoint union of translates in $L^p(\mathbb{R}^d)$ can form an unconditional basis.

1. INTRODUCTION

Given $1<p<\infty$ and $\gamma\in\mathbb{R}^d$, we define the translation operator T_γ on $L^p(\mathbb{R}^d)$ by $(T_\gamma f)(x)=f(x-\gamma)$ for all $x\in\mathbb{R}^d$. If $\Gamma\subset\mathbb{R}^d$, then the collection of translations of $f\in L^p(\mathbb{R}^d)$ along Γ is defined to be $T_p(f,\Gamma)=\{T_\gamma f\}_{\gamma\in\Gamma}$. Our main focus shall be on the upper Beurling density of such Γ, the disjoint union $\bigcup_{k=1}^n \Gamma_k$, given that $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ has some additional structure in $L^p(\mathbb{R}^d)$. The "additional structure" takes two forms: $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ is a p'-Bessel sequence or is a (C_q)-system.

The nature of $T_p(f,\Gamma)$ has been studied in a number of papers [21, 2, 10, 19], mainly using techniques of harmonic analysis. Our techniques will come partially from the geometry of Banach spaces. Recall that, in 1992, Olson and Zalik [20]

Date: Received: 29 November 2011; Accepted: 25 March 2012.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 42C30; Secondary 46E30, 46B15.

Key words and phrases. (C_q)-system, L^p-space, p'-Bessel sequence, translate, upper Beurling density.

proved that there do not exist any Riesz bases for $L^2(\mathbb{R})$ generated by $T_2(f,\Gamma)$. Then Christensen [6] conjectured that there are no frames for $L^2(\mathbb{R})$ of the form $\bigcup_{k=1}^n T_2(f_k,\Gamma_k)$. In 1999, Christensen, Deng and Heil [11] proved this conjecture by studying density of frames. For more density theorems, please see the research survey [12]. Recently, Odell, Sari, Schlumprecht and Zheng [18] used techniques largely from the geometry of Banach spaces to consider the closed subspace of $L^p(\mathbb{R})$ generated by translates of one element in $L^p(\mathbb{R})$.

In Section 2, we extend the concept of (C_q)-system from Hilbert spaces to reflexive Banach spaces and give our basic Lemma 3.3 and examples in $L^p(\mathbb{R}^d)$.

In section 3, by using techniques in [11, 18], we prove that if $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ in $L^p(\mathbb{R}^d)$ $(1<p<\infty)$ is a p'-Bessel sequence for some $1<p'<\infty$, then the disjoint union $\Gamma=\bigcup_{k=1}^n \Gamma_k$ has finite upper Beurling density, and that if $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ is a (C_q)-system with $1/p+1/q=1$, then Γ has infinite upper Beurling density. Thus, no collection $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ of pure translates can form a p'-Bessel (C_q)-system in $L^p(\mathbb{R}^d)$ for any $1<p'<\infty$. This extends the Christensen/Deng/Heil density result in [11] from classical (Hilbert) frames in $L^2(\mathbb{R}^d)$ to more general p'-Bessel (C_q)-systems in $L^p(\mathbb{R}^d)$.

In the last section. by using techniques from the geometry of Banach spaces, we obtain that there is no unconditional basis of $L^p(\mathbb{R}^d)$ of the form $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ for $1<p\le 2$. It partially extends the latest work [18] on uniformly separated translates of one element in $L^p(\mathbb{R})$. The extension is to higher dimensions, to multiple generating functions, and to completely arbitrary sets of translates.

2. Preliminaries and notation

In 2001, Aldroubi, Sun and Tang [3] introduced the concept of p-frame in $L^p(\mathbb{R})$, which is a generalization of classical (Hilbert) frames [9, 4, 7] and can be naturally extended to Banach spaces [8, 5].

Definition 2.1. Let X be a separable Banach space and $1<p<\infty$. A family $\{f_k\}_{k=1}^\infty \subset X$ is a *p-frame* for X^* if there exist constants $A,B>0$ such that

$$A\|h\|^p \le \sum_{k=1}^\infty |\langle h, f_k\rangle|^p \le B\|h\|^p \quad \text{for all } h\in X^*.$$

The number A and B are called the lower and upper p-frame bounds. The sentence $\{f_k\}_{k=1}^\infty$ is a *p-Bessel sequence* if the right-hand side inequality holds. We say that $\{f_k\}_{k=1}^\infty$ is a *Bessel sequence* if it is a 2-Bessel sequence.

In 2007, S. Nitzan and A. Olevskii introduced the concept of (C_q)-system in Hilbert spaces [15, 16, 17]. It is a weaker form of the frame-type condition, which is a relaxed version of this inequality:

$$A\|f\|^2 \le \sum_{k=1}^\infty |\langle f, f_k\rangle|^2 \quad \text{for all } f\in \mathcal{H}.$$

Now we extend this useful definition of (C_q)-system to reflexive Banach spaces.

Definition 2.2. Let X be a separable reflexive Banach space and $1<q<\infty$ be a fixed number. We say that a sequence of $\{f_k\}_{k=1}^\infty \subset X$ is a (C_q)-*system in X* *with constant $C > 0$ (complete with ℓ_q control over the coefficients)* if for every $f \in X$ and $\varepsilon > 0$, there exists a linear combination $g = \sum a_k f_k$ such that

$$\|f - g\|_X < \varepsilon \quad \text{and} \quad \Big(\sum |a_n|^q\Big)^{1/q} \leq C\|f\|_X, \tag{2.1}$$

where $C = C(q)$ is a positive constant not depending on f.

Remark 2.3. By Proposition 4.2 in Section 4, given $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, we have that: if $1 < p \leq 2$, then every seminormalized unconditional basis of $L^p(\mathbb{R}^d)$ is a q-Bessel (C_2)-system; if $2 \leq p < \infty$, then every seminormalized unconditional basis of $L^p(\mathbb{R}^d)$ is a Bessel (C_p)-system.

We define some types of sequences in $\mathbb{R}^d$ and upper Beurling density [11, 7].

Definition 2.4. Let $\Gamma = \{\gamma_i\}_{i \in I} \subset \mathbb{R}^d$.

(*i*) A point $\gamma \in \mathbb{R}^d$ is an *accumulation point* for Γ if every open ball in $\mathbb{R}^d$ centered at γ contains infinitely many γ_k.

(*ii*) Γ is *δ-uniformly separated* if $\delta = \inf_{i \neq j} |\gamma_i - \gamma_j| > 0$. The number δ is the *separation constant.*

(*iii*) Γ is *relatively uniformly separated* if it is a finite union of uniformly separated sequences Γ_k. That is to say that I can be partitioned into finite disjoint sets $I_1, ..., I_n$ such that each sequence $\Gamma_k = \{\gamma_i\}_{i \in I_k}$ is δ_k-uniformly separated for some $\delta_k > 0$.

For $h > 0$ and $x \in \mathbb{R}^d$, we define cube $Q_h(x)$ by

$$Q_h(x) = \prod_{i=1}^{d} [x_i - h/2, x_i + h/2), \quad \text{where } x = (x_1, ..., x_d).$$

When $x = 0$, we use Q_h instead of $Q_h(0)$ for simplicity. Let $\Gamma = \{\gamma_i\}_{i \in I} \subset \mathbb{R}^d$. For each $h > 0$, let $\nu_\Gamma^+(h)$ denote the largest number of points from Γ that lie in any cube $Q_h(x)$, i.e.,

$$\nu_\Gamma^+(h) = \sup_{x \in \mathbb{R}^d} \#(\Gamma \cap Q_h(x)).$$

The *upper Beurling density* of Γ is defined by

$$D^+(\Gamma) = \limsup_{h \to \infty} \frac{\nu_\Gamma^+(h)}{h^d}.$$

Lemma 2.5. *Let $\Gamma = \{\gamma_i\}_{i \in I}$ be a sequence in $\mathbb{R}^d$. Then the following statements are equivalent.*

(*i*) $D^+(\Gamma) < \infty$.

(*ii*) *Γ is relatively uniformly separated.*

(*iii*) *For some (and therefore every) $h > 0$, there is a natural number N_h such that each cube $Q_h(hn)$, $n \in \mathbb{Z}^d$, contains at most N_h points from Λ. That is,*

$$N_h = \sup_{n \in \mathbb{Z}^d} \#(\Lambda \cap Q_h(hn)) < \infty.$$

3. Main results

First we need the following basic lemma.

Lemma 3.1. *Let Γ be a sequence in $\mathbb{R}^d$, and $1 < p, q < \infty$ with $1/p + 1/q = 1$. Assume that $f \in L^p(\mathbb{R}^d)$, $\tilde{f} \in L^q(\mathbb{R}^d)$ and $\langle f, \tilde{f}\rangle \neq 0$. If Γ is not relatively uniformly separated, then, for any $0 < \varepsilon < |\langle f, \tilde{f}\rangle|$, we have*

$$\sup_{\beta\in\mathbb{R}^d} \#\{\gamma \in \Gamma : |\langle T_\gamma f, T_\beta \tilde{f}\rangle| > \varepsilon\} = \infty.$$

Proof. Consider the function $x \mapsto \langle T_x f, \tilde{f}\rangle$ for all $x \in \mathbb{R}^d$. Since the function is continuous, for any $0 < \varepsilon < |\langle f, \tilde{f}\rangle|$ there is a cube Q_h for some $h > 0$ such that

$$\inf_{x\in Q_h} |\langle T_x f, \tilde{f}\rangle| > \varepsilon.$$

Consider an arbitrary $N \in \mathbb{N}$, by Lemma 2.5, there is a cube $Q_h(\beta)$ for some $\beta \in \mathbb{R}^d$, which contains at least N elements from Γ. Then for any $\gamma \in Q_h(\beta)$, $\gamma - \beta \in Q_h$, we have

$$|\langle T_\gamma f, T_\beta \tilde{f}\rangle| = |\langle T_{\gamma-\beta} f, \tilde{f}\rangle| > \varepsilon.$$

It follows that

$$\#\big\{\gamma \in \Gamma : |\langle T_\gamma f, T_\beta \tilde{f}\rangle| > \varepsilon\big\} \geq \#\big(\Gamma \cap Q_h(\beta)\big) \geq N.$$

Since $N \in \mathbb{N}$ is arbitrary, the conclusion follows. □

For translate of one element, we get the following result.

Proposition 3.2. *Let $1 < p < \infty$, f be a nonzero function in $L^p(\mathbb{R}^d)$, and Γ be a sequence in $\mathbb{R}^d$. If $T_p(f, \Gamma)$ is a p'-Bessel sequence for some $1 < p' < \infty$, then Γ is relatively uniformly separated.*

Proof. Assume that Γ is not relatively uniformly separated. Then for any $N \in \mathbb{N}$, choose ε such that $0 < \varepsilon < \|f\|_p$. By Hahn-Banach Theorem, there is an $\tilde{f} \in L^q(\mathbb{R}^d)$ with $\|\tilde{f}\|_q = 1$ such that $\langle f, \tilde{f}\rangle = \|f\|_p > \varepsilon$. Then, by Lemma 3.1, there exists $\beta \in \mathbb{R}^d$ such that

$$\#\big\{\gamma \in \Gamma : |\langle T_\gamma f, T_\beta \tilde{f}\rangle| > \varepsilon\big\} \geq N.$$

Let $\Gamma_N = \{\gamma \in \Gamma : |\langle T_\gamma f, T_\beta \tilde{f}\rangle| > \varepsilon\}$. Then, we have

$$\sum_{\gamma\in\Gamma} |\langle T_\gamma f, T_\beta \tilde{f}\rangle|^{p'} \quad \geq \quad \sum_{\gamma\in\Gamma_N} |\langle T_\gamma f, T_\beta \tilde{f}\rangle|^{p'} > N\varepsilon^{p'}.$$

Since $N \in \mathbb{N}$ is arbitrary and $\|T_\beta \tilde{f}\|_q = \|\tilde{f}\|_q$ is fixed, $T_p(f, \Gamma)$ is not a p'-Bessel sequence, which leads to a contradiction. Thus Γ is relatively uniformly separated. □

The following equivalent form extends Lemma 1 in [15] by using a standard duality argument in Banach spaces.

Lemma 3.3. *Let X be a separable reflexive Banach space and $1 < p, q < \infty$ with $1/p + 1/q = 1$. A system $\{f_n\} \subset X$ is a (C_q)-system in X with constant $K > 0$ if and only if*

$$\frac{1}{K}\|h\| \le \Big(\sum_{n=1}^{\infty} |\langle h, f_n\rangle|^p\Big)^{1/p} \quad \text{for all } h \in X^*.$$

Proof. For sufficiency, suppose that $\{f_n\}$ is not a (C_q)-system in X with constant $K > 0$. Let

$$A := \Big\{g = \sum a_n f_n : \big(\sum |a_n|^q\big)^{1/q} \le K\Big\}$$

be the set of finite linear combination and $\mathcal{C}$ be the closure of A in X. It is easy to prove that $\mathcal{C}$ is a closed convex subset of X. By assumption, $\mathcal{C}$ does not contain the closed unit ball B of X. That is, there exists an $f \in X$ with $\|f\| \le 1$, and f is not in $\mathcal{C}$. By the Hahn-Banach theorem, there is an $h \in X^*$ such that $|\langle h, f\rangle| = 1$ and $\sup_{g\in\mathcal{C}} |\langle h, g\rangle| < 1$. Hence, for sufficiently small $\varepsilon > 0$, we have $\sup_{g\in\mathcal{C}} |\langle h, g\rangle| < 1 - \varepsilon$. This implies for any $M \in \mathbb{N}$,

$$\begin{aligned}
\Big(\sum_{n=1}^{M} |\langle h, f_n\rangle|^p\Big)^{1/p} &= \sup_{(\sum_{n=1}^{M}|\alpha_n|^q)^{1/q}\le 1} \Big|\sum_{n=1}^{M}\langle h, f_n\rangle \alpha_n\Big| \\
&= \frac{1}{K}\sup_{(\sum_{n=1}^{M}|\alpha_n|^q)^{1/q}\le K} \Big|\langle h, \sum_{n=1}^{M}\alpha_n f_n\rangle\Big| \\
&= \frac{1}{K}\sup_{g\in\mathcal{C}} |\langle h, g\rangle| \\
&< \frac{1}{K}(1-\varepsilon).
\end{aligned}$$

By the arbitrary of M, we have

$$\Big(\sum_{n=1}^{\infty} |\langle h, f_n\rangle|^p\Big)^{1/p} \le \frac{1}{K}(1-\varepsilon) < \frac{1}{K} = \frac{1}{K}|\langle h, f\rangle| \le \frac{1}{K}\|h\|,$$

which leads to a contradiction.

For necessity, let $\{f_n\}$ be a (C_q)-system with constant $K > 0$ in X. For every $h \in X^*$ and $\varepsilon > 0$, there exists an $f \in X$, $\|f\| = 1$, and $|\langle h, f\rangle| = \|h\|$. Choose a linear combination $g = \sum a_n f_n$ such that $\|f - g\| < \varepsilon$ and

$$\Big(\sum |a_n|^q\Big)^{1/q} \le K\|f\| = K.$$

Then

$$\begin{aligned}
\|h\| &= |\langle h, f\rangle| \\
&\le |\langle h, f - g\rangle| + |\langle h, g\rangle| \\
&\le \varepsilon\|h\| + \sum |a_n||\langle h, f_n\rangle| \\
&\le \varepsilon\|h\| + \Big(\sum |a_n|^q\Big)^{1/q}\Big(\sum |\langle h, f_n\rangle|^p\Big)^{1/p} \\
&\le \varepsilon\|h\| + K\Big(\sum |\langle h, f_n\rangle|^p\Big)^{1/p}.
\end{aligned}$$

That is,

$$\frac{1-\varepsilon}{K}\|h\| \le \Big(\sum |\langle h, f_n\rangle|^p\Big)^{1/p} \quad (h \in X^*, \varepsilon > 0).$$

Since ε is arbitrarily small, take $\varepsilon \to 0$, we complete the proof. □

The following result is elementary but very useful.

Lemma 3.4. *Let $1 < p < \infty$, $f \in L^p(\mathbb{R}^d)$, and Γ be a sequence in $\mathbb{R}^d$. If Γ is relatively uniformly separated, then for all cubes $Q_h(x)$, for any $x \in \mathbb{R}^d$ and $h > 0$, we have*
(i) $\sum_{\gamma\in\Gamma} \|\chi_{Q_h(x)} T_\gamma f\|_p^p < \infty$. *(ii)* $\sum_{\gamma\in\Gamma} \|\chi_{Q_h(x)} T_\gamma f\|_p^p \to 0$, *as* $h \to 0$.

Proof. (i) Since Γ is relatively uniformly separated, it is a disjoint finite union of δ_k-separated sequences Γ_k for $\delta_k > 0$ with $k = 1, ..., n$. Let $\delta = \min\limits_{1\le k\le n} \delta_k > 0$ be the relatively separated constant and choose $0 < \varepsilon < \delta/\sqrt{d}$. Because any cube $Q_h(x)$ is bounded, it must be contained in $Q_{2N\varepsilon}$ for some $N \in \mathbb{N}$. Thus, it is enough to prove that $\sum_{\gamma\in\Gamma} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p < \infty$ for all $N \in \mathbb{N}$. For any $x \in \mathbb{R}^d$ and $h > 0$, let

$$Q_h^+(x) = x + \prod_{j=1}^d [0, h) = \prod_{j=1}^d [x_j, x_j + h).$$

Then

$$\begin{aligned}
\sum_{\gamma\in\Gamma} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p &= \sum_{k=1}^n \sum_{\gamma\in\Gamma_k} \|\chi_{Q_{2N\varepsilon}} T_\gamma f\|_p^p \\
&= \sum_{k=1}^n \sum_{\gamma\in\Gamma_k} \sum_{a\in Q_{2N}\cap\mathbb{Z}^d} \|\chi_{Q_\varepsilon^+(\varepsilon a)} T_\gamma f\|_p^p \\
&= \sum_{k=1}^n \sum_{a\in Q_{2N}\cap\mathbb{Z}^d} \sum_{\gamma\in\Gamma_k} \|\chi_{Q_\varepsilon^+(\varepsilon a)} T_\gamma f\|_p^p \\
&= \sum_{k=1}^n \sum_{a\in Q_{2N}\cap\mathbb{Z}^d} \sum_{\gamma\in\Gamma_k} \|\chi_{Q_\varepsilon^+(\varepsilon a)-\gamma} f\|_p^p \\
&= \sum_{k=1}^n \sum_{a\in Q_{2N}\cap\mathbb{Z}^d} \sum_{\gamma\in\Gamma_k} \int_{Q_\varepsilon^+(\varepsilon a)-\gamma} |f(x)|^p \, dx.
\end{aligned}$$

Since $\operatorname{diam}\big(Q_\varepsilon^+(\varepsilon a)\big) = \sqrt{d}\varepsilon < \delta$ for any $a \in Q_{2N} \cap \mathbb{Z}^d$, we get

$$Q_\varepsilon^+(\varepsilon a) - \gamma = Q_\varepsilon^+(\varepsilon a - \gamma)$$

are mutually disjoint for $\gamma \in \Gamma_k$. Thus

$$\begin{aligned}\sum_{\gamma\in\Gamma}\|\chi_{Q_{2N\varepsilon}}T_\gamma f\|_p^p &= \sum_{k=1}^{n}\sum_{a\in Q_{2N}\cap\mathbb{Z}^d}\sum_{\gamma\in\Gamma_k}\int_{Q_\varepsilon^+(\varepsilon a)-\gamma}|f(x)|^p\,dx \\ &\le \sum_{k=1}^{n}\sum_{a\in Q_{2N}\cap\mathbb{Z}^d}\|f\|_p^p \\ &= n(2N)^d\|f\|_p^p \\ &< \infty.\end{aligned} \tag{3.1}$$

(ii) For each $k=1,...,n$, $a\in Q_{2N}\cap\mathbb{Z}^d$ and fixed $x\in\mathbb{R}^d$, we have

$$\chi_{Q_\varepsilon^+(\varepsilon a)-\Gamma_k}|f(x)|^p \to 0 \quad \text{as} \quad \varepsilon\to 0,$$

here $Q_\varepsilon^+(\varepsilon a)-\Gamma_k=\bigcup_{\gamma\in\Gamma_k}Q_\varepsilon^+(\varepsilon a)-\gamma$. Since

$$\chi_{Q_\varepsilon^+(\varepsilon a)-\Gamma_k}|f(x)|^p \le |f(x)|^p,$$

by the Lebesgue Dominated Convergence Theorem, it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{\gamma\in\Gamma_k}\int_{Q_\varepsilon^+(\varepsilon a)-\gamma}|f(x)|^p\,dx &= \lim_{\varepsilon\to 0}\int_{Q_\varepsilon^+(\varepsilon a)-\Gamma_k}|f(x)|^p\,dx \\ &= \lim_{\varepsilon\to 0\mathbb{R}^d}\int_{\mathbb{R}^d}\chi_{Q_\varepsilon^+(\varepsilon a)-\Gamma_k}|f(x)|^p\,dx \\ &= 0.\end{aligned}$$

Thus by (3.1),

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{\gamma\in\Gamma}\|\chi_{Q_{2N\varepsilon}}T_\gamma f\|_p^p &= \lim_{\varepsilon\to 0}\sum_{k=1}^{n}\sum_{a\in Q_{2N}\cap\mathbb{Z}^d}\sum_{\gamma\in\Gamma_k}\int_{Q_\varepsilon^+(\varepsilon a)-\gamma}|f(x)|^p\,dx \\ &= \sum_{k=1}^{n}\sum_{a\in Q_{2N}\cap\mathbb{Z}^d}\lim_{\varepsilon\to 0}\sum_{\gamma\in\Gamma_k}\int_{Q_\varepsilon^+(\varepsilon a)-\gamma}|f(x)|^p\,dx \\ &= 0.\end{aligned}$$

Thus, we obtain that if Γ is relatively uniformly separated, then for any $N\in\mathbb{N}$,

$$\lim_{\varepsilon\to 0}\sum_{\gamma\in\Gamma}\|\chi_{Q_{2N\varepsilon}}T_\gamma f\|_p^p = 0.$$

Since for all $x\in\mathbb{R}^d$, the translation $\Gamma - x = \{\gamma - x : \gamma\in\Gamma\}$ of Γ is relatively uniformly separated, then

$$\lim_{h\to 0}\sum_{\gamma\in\Gamma}\|\chi_{Q_h(x)}T_\gamma f\|_p^p = \lim_{h\to 0}\sum_{\gamma\in\Gamma}\|\chi_{Q_h}T_{\gamma-x}f\|_p^p = \lim_{h\to 0}\sum_{\gamma\in\Gamma-x}\|\chi_{Q_h}T_\gamma f\|_p^p = 0.$$

Now the conclusion follows. □

Now we prove our main result.

Theorem 3.5. *Let $1<p,q<\infty$ with $1/p+1/q=1$ and $n,d\in\mathbb{N}$. For each $k=1,...,n$, choose a nonzero function $f_k\in L^p(\mathbb{R}^d)$ and an arbitrary sequence $\Gamma_k\subset\mathbb{R}^d$. Let Γ be the disjoint union of $\Gamma_1,...,\Gamma_n$.*

(i) If for some $1 < p' < \infty$, $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ *is a* p'*-Bessel sequence, then*

$$D^+(\Gamma) < \infty.$$

(ii) If $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ *is a* (C_q)*-system, then* $D^+(\Gamma) = \infty$.

In particular, there is no p'*-Bessel* (C_q)*-system in* $L^p(\mathbb{R}^d)$ *of the form* $\bigcup_{k=1}^{n} T_p(f_k, \Gamma_k)$.

Proof. (i) Suppose that, for some $1 < p' < \infty$, $\bigcup_{k=1}^n T_p(f_k, \Gamma_k)$ is a p'-Bessel sequence. It is equivalent to that each $T_p(f_k, \Gamma_k)$ is a p'-Bessel sequence for $L^q(\mathbb{R}^d)$. Then, by Proposition 3.2, each Γ_k is relatively uniformly separated. By Lemma 2.5, Γ_k has finite upper Beurling density for each $1 \le k \le n$, i.e. $D^+(\Gamma_k) < +\infty$. Then by definition we have

$$\begin{aligned}
\nu_\Gamma^+(h) &= \sup_{x \in \mathbb{R}^d} \#(\Gamma \cap Q_h(x)) \\
&= \sup_{x \in \mathbb{R}^d} \#(\cup_{k=1}^n (\Gamma_k \cap Q_h(x))) \\
&\le \sum_{k=1}^n \sup_{x \in \mathbb{R}^d} \#(\Gamma_k \cap Q_h(x)) \\
&= \sum_{k=1}^n \nu_{\Gamma_k}^+(h).
\end{aligned}$$

It follows that

$$\begin{aligned}
D^+(\Gamma) &= \limsup_{h \to \infty} \frac{\nu_\Gamma^+(h)}{h^d} \\
&\le \limsup_{h \to \infty} \frac{\sum_{k=1}^n \nu_{\Gamma_k}^+(h)}{h^d} \\
&\le \sum_{k=1}^n \limsup_{h \to \infty} \frac{\nu_{\Gamma_k}^+(h)}{h^d} \\
&= \sum_{k=1}^n D^+(\Gamma_k) \\
&< +\infty. \qquad (3.2)
\end{aligned}$$

Thus, Γ has finite upper Beurling density.

(ii) Since Γ is the disjoint union of sequences Γ_k, then, by formula (3.2), we have $D^+(\Gamma) < \infty$ if and only if $D^+(\Gamma_k) < \infty$ for each $k = 1, ..., n$. Assume that Γ has finite upper Beurling density. By Lemma 2.5, we know that Γ_k is relatively uniformly separated. Now consider the cube $Q_{2h} = \prod_{j=1}^{d}[-h, h)$ for $h > 0$. Then

$$\begin{aligned}\sum_{k=1}^{n}\sum_{\gamma\in\Gamma_k}|\langle\chi_{Q_{2h}}, T_\gamma f_k\rangle|^p &= \sum_{k=1}^{n}\sum_{\gamma\in\Gamma_k}|\langle\chi_{Q_{2h}}, \chi_{Q_{2h}}T_\gamma f_k\rangle|^p \\ &\le \sum_{k=1}^{n}\sum_{\gamma\in\Gamma_k}\|\chi_{Q_{2h}}\|_q^p\|\chi_{Q_{2h}}T_\gamma f_k\|_p^p \\ &\le \|\chi_{Q_{2h}}\|_q^p\sum_{k=1}^{n}\sum_{\gamma\in\Gamma_k}\|\chi_{Q_{2h}}T_\gamma f_k\|_p^p.\end{aligned}$$

By Lemma 3.4, we have for each $k = 1, ..., n$,

$$\sum_{\gamma\in\Gamma_k}\|\chi_{Q_{2h}}T_\gamma f_k\|_p^p \to 0 \text{ as } h \to 0.$$

Thus, by Lemma 3.3, it is easy to see that $\bigcup_k T(f_k, \Gamma_k)$ is not a (C_q)-system. Thus, we complete the proof. □

Remark 3.6. (i) The result due to Christensen, Deng and Heil [11] is a special case of Theorem 3.5 for $p = p' = 2$.

(ii) As a consequence of Theorem 3.5, for no function $g \in L^p(\mathbb{R}^d)$ and no constants $a, b > 0$, $p' > 1$ can be a collection of functions of the form $\{T_{na}E_{mb}g\}_{n\in\mathbb{Z},m=1,...,M}$ a p'-Bessel (C_q)-system in $L^p(\mathbb{R}^d)$, where the modulation operator E_{mb} on $L^p(\mathbb{R}^d)$ is defined by

$$(E_{mb}f)(x) = e^{2\pi imbx}f(x).$$

However, Hilbert frames of the infinite type $\{T_{na}E_{mb}g\}_{m,n\in\mathbb{Z}}$ exist in $L^2(\mathbb{R})$ (every Hilbert frame is a Bessel (C_2)-system). For more information on Gabor frames and density theorems, please see [7, 12].

4. Nonexistence of unconditional bases of translates in $L^p(\mathbb{R}^d)$

In this section, we will prove that there doesn't exist any unconditional basis of the form $\bigcup_{k=1}^{n} T_p(f_k, \Gamma_k)$ in $L^p(\mathbb{R}^d)$ for $1 < p \le 2$. We use standard Banach space notations as may be found in [13, 14]. Background material on bases, unconditional bases and such can be found there. For the benefit of those less familiar with these notions we recall some definitions and facts.

A biorthogonal system is a sequence $\{x_n, f_n\} \subset X \times X^*$ where $f_n(x_m) = \delta_{nm}$.

$\{x_n\} \subset X$ is a (Schauder) basis for X if for all $x \in X$, there exists a unique sequence of scalars $\{a_n\}$ so that $x = \sum_{n=1}^{\infty} a_n x_n$. This is equivalent to saying that all $x_n \neq 0$, $\overline{\text{span}}\{x_n\} = X$ and for some $K < \infty$, all $m < l$ in $\mathbb{N}$ and all scalars $\{a_n\}_{n=1}^{l}$,

$$\left\|\sum_{n=1}^{m} a_n x_n\right\| \le K\left\|\sum_{n=1}^{l} a_n x_n\right\|.$$

The smallest such K is the basis constant of $\{x_n\}$.

$\{x_n\}$ is an unconditional basis for X if for all $x \in X$, there exists a unique sequence of scalars $\{a_n\}$ so that $x = \sum_{n=1}^{\infty} a_n x_n$ and the convergence is unconditional. i.e. $x = \sum_{n=1}^{\infty} a_{\pi(n)}x_{\pi(n)}$ for all permutations π of $\mathbb{N}$.

If $\{x_n\}$ is an unconditional basis for the Banach space X and $\theta = \{\theta_n\}_{n=1}^{\infty}$ is a sequence of ± 1's, define $S_\theta : X \to X$ by $S_\theta(\sum \alpha_n x_n) = \sum \theta_n \alpha_n x_n$. The supremum over all such $\|S_\theta\|$ is finite, and is called the unconditional constant of the basis [13].

The following lemma is easy to prove, which we leave to interested readers.

Lemma 4.1. *Let X be a separable reflexive Banach space with $\{x_n, f_n\} \subset X \times X^*$. Assume that $\{x_n, f_n\}$ is a biorthogonal system, that is, $\langle x_n, f_m\rangle = \delta_{nm}$ for $n, m \in \mathbb{N}$. Then $\{x_n\}$ is a seminormalized unconditional basis of X if and only if $\{f_n\}$ is a seminormalized unconditional basis of X^*.*

Recall the following known inequalities in L^p-space [1]. For $1 < p < \infty$, there exist constants A_p, $B_p > 0$ such that, if $\{f_k\}_{k=1}^{\infty}$ is a normalized C-unconditional basic sequence in $L^p(\mathbb{R}^d)$, then

$$(CA_p)^{-1}\Big(\sum_{k=1}^{\infty}|a_k|^2\Big)^{1/2} \le \Big\|\sum_{k=1}^{\infty} a_k f_k\Big\|_p \le C\Big(\sum_{k=1}^{\infty}|a_k|^p\Big)^{1/p}, \text{ if } 1 < p \le 2, \quad (4.1)$$

$$C^{-1}\Big(\sum_{k=1}^{\infty}|a_k|^p\Big)^{1/p} \le \Big\|\sum_{k=1}^{\infty} a_k f_k\Big\|_p \le CB_p\Big(\sum_{k=1}^{\infty}|a_k|^2\Big)^{1/2}, \text{ if } 2 \le p < \infty. \quad (4.2)$$

Proposition 4.2. *Given $p, q \in (1, \infty)$ with $1/p + 1/q = 1$. Then*

- (i) *If $1 < p \le 2$, then every seminormalized unconditional basis of $L^p(\mathbb{R}^d)$ is a q-Bessel (C_2)-system.*
- (ii) *If $2 \le p < \infty$, then every seminormalized unconditional basis of $L^p(\mathbb{R}^d)$ is a Bessel (C_p)-system.*

Proof. Let $\{f_i\}$ be a seminormalized unconditional basis of $L^p(\mathbb{R}^d)$, and $\{\tilde f_i\} \subset L^q(\mathbb{R}^d)$ be the biorthogonal functionals of $\{f_i\}$. Then, by Lemma 4.1, $\{\tilde f_i\}$ is a seminormalized C-unconditional basis of $L^q(\mathbb{R}^d)$. Let $C_1 = \inf \|\tilde f_i\|_p$ and $C_2 = \sup \|\tilde f_i\|_q$.

We first prove (i). Since $1 < p \le 2$, we have $2 \le q < \infty$. By inequality (4.2), for all $\tilde f \in L^q(\mathbb{R}^d)$,

$$\begin{aligned}
\Big(\sum |\langle \tilde f, f_i\rangle|^q\Big)^{1/q} &= \Big(\sum \frac{1}{\|\tilde f_i\|_q^q}|\langle \tilde f, \|\tilde f_i\|_q f_i\rangle|^q\Big)^{1/q} \\
&\le \frac{1}{C_1}\Big(\sum |\langle \tilde f, \|\tilde f_i\|_q f_i\rangle|^q\Big)^{1/q} \\
&\le \frac{C}{C_1}\Big\|\sum \langle \tilde f, \|\tilde f_i\|_q f_i\rangle \frac{\tilde f_i}{\|\tilde f_i\|_q}\Big\|_q \\
&= \frac{C}{C_1}\Big\|\sum \langle \tilde f, f_i\rangle \tilde f_i\Big\|_q \\
&= \frac{C}{C_1}\|\tilde f\|_q.
\end{aligned}$$

Moreover, for the lower 2-frame bound, we have

$$\begin{aligned}\big(\sum|\langle\tilde f,f_i\rangle|^2\big)^{1/2} &= \Big(\sum\frac{1}{\|\tilde f_i\|_q^2}|\langle\tilde f,\|\tilde f_i\|_q f_i\rangle|^2\Big)^{1/2}\\ &\geq \frac{1}{C_2}\big(\sum|\langle\tilde f,\|\tilde f_i\|_q f_i\rangle|^2\big)^{1/2}\\ &\geq \frac{1}{B_qCC_2}\Big\|\sum\langle\tilde f,\|\tilde f_i\|_q f_i\rangle\frac{\tilde f_i}{\|\tilde f_i\|_q}\Big\|_q\\ &= \frac{1}{B_qCC_2}\big\|\sum\langle\tilde f,f_i\rangle\tilde f_i\big\|_q\\ &= \frac{1}{B_qCC_2}\|\tilde f\|_q^2.\end{aligned}$$

Now we prove (ii). Similarly, by inequality (4.1), for all $\tilde f\in L^q(\mathbb{R}^d)$, we get that

$$\big(\sum|\langle\tilde f,f_i\rangle|^2\big)^{1/2}\leq\frac{CA_q}{C_1}\|\tilde f\|_q.$$

For the lower q-frame bound, we have

$$\big(\sum|\langle\tilde f,f_i\rangle|^q\big)^{1/q}\geq\frac{1}{CC_2}\|\tilde f\|_q.$$

Thus, we complete the proof. □

The following is the main result in this section.

Theorem 4.3. *Let $1<p\leq 2$ and $n,d\in\mathbb{N}$. For each $k=1,\cdots,n$, choose a nonzero function $f_k\in L^p(\mathbb{R}^d)$ and an arbitrary sequence $\Gamma_k\subset\mathbb{R}^d$. Let Γ be the disjoint union of $\Gamma_1,\cdots,\Gamma_n$. Then $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ can at most be an unconditional basis for a proper subspace of $L^p(\mathbb{R}^d)$.*

Equivalently, there is no unconditional basis of $L^p(\mathbb{R}^d)$ of the form $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$.

Proof. If $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ is an unconditional basis of $L^p(\mathbb{R}^d)$, then, by Proposition 4.2, it is a q-Bessel (C_2)-system for $L^p(\mathbb{R}^d)$. Since $1<p\leq 2$, $2\leq q<\infty$ with $1/p+1/q=1$, we have $(\sum|a_n|^q)^{1/q}\leq(\sum|a_n|^2)^{1/2}$. Then, by (2.1) in Definition 2.2, $\bigcup_{k=1}^n T_p(f_k,\Gamma_k)$ is a q-Bessel (C_q)-system for the whole $L^p(\mathbb{R}^d)$. It leads to a contradiction by Theorem 3.5. □

Acknowledgement. This work was partially done while the second author was visiting the University of Texas at Austin and Texas A&M University, and in the Linear Analysis Workshop at Texas A&M University which was funded by NSF Workshop Grant. The second author would like to thank David Larson, Edward Odell and Thomas Schlumprecht for the invitation and great help. Both authors also expresses their appreciation to Qiyu Sun and Wenchang Sun for very helpful comments.

This work are supported by the National Natural Science Foundation of China 11001134 and 11126250, the Fundamental Research Funds for the Central Universities, and the TianjinScience & Technology Fund 20100820.

References

1. D. Alspach and E. Odell, *L_p spaces*, Hanbook of the Geometry of Banach Spaces, Vol. 1, W.B. Johnson and J. Lindenstrauss, eds, Elsevier, Amsterdam (2001), 123–160.
2. A. Atzmon and A. Olevskii, *Completeness of integer translates in function spaces on* $\mathbb{R}$, J. Approx. Theory **87** (1996), 291–327.
3. A. Aldroubi, Q. Sun and W. Tang, *p-frames and shift invariant subspaces of L^p*, J. Fourier Anal. Appl. **7** (2001), no. 1, 1–22.
4. P.G. Casazza, *The art of frame theory*, Taiwanese J. Math. **4** (2000), no. 2, 129–201.
5. P. Casazza, O. Christensen and D.T. Stoeva, *Frame expansions in separable Banach spaces*, J. Math. Anal. Appl. **307** (2005), 710–723.
6. O. Christensen, *Frames contaning a Riesz basis and approximation of the frame coefficients*, J. Math. Anal. Appl. **199** (1995), 256–270.
7. O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, 2003.
8. O. Christensen and D.T. Stoeva, *p-frames in separable Banach spaces*, Adv. Comp. Math. **18** (2003), 117–126.
9. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.
10. G. Edgar and J. Rosenblatt, *Difference equations over locally compact abelian groups*, Trans. Amer. Math. Soc. **253** (1979), 273–289.
11. O. Christensen, B. Deng and C. Heil, *Density of Gabor frames*, Appl. Comput. Harmon. Anal. **7** (1999), 292–304.
12. C. Heil, *History and evolution of the density theorems for Gabor frames*, J. Fourier Anal. Appl. **13** (2007), 113–166.
13. W.B. Johnson, J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, Hanbook of the Geometry of Banach Spaces, Vol. 1, W.B. Johnson and J. Lindenstrauss, eds, Elsevier, Amsterdam (2001), 1–84.
14. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: Sequence Spaces*, Springer-Verlag, Berlin (1977).
15. S. Nitzan and A. Olevskii, *Sparse exponential systems: completeness with estimates*, Israel J. Math. **158** (2007), 205–215.
16. S. Nitzan and A. Olevskii, *Quasi-frames of translates*, C. R. Math. Acad. Sci. Paris **347** (2009), 739–742.
17. S. Nitzan and J. Olsen, *From exact systems to Riesz bases in the Balian-Low Theorem*, J. Fourier Anal. Appl. **17** (2011), 567–603.
18. E. Odell, B. Sari, Th. Schlumprecht and B. Zheng, *Systems formed by translates of one element in $L_p(\mathbb{R})$*, Trans. Amer. Math. Soc. **363** (2011), 6505–6529.
19. A. Olevskii, *Completeness in $L_2(\mathbb{R})$ of almost integer translates*, C. R. Acad. Sci. Paris **324** (1997), 987–991.
20. T.E. Olson and R.A. Zalik, *Nonexistence of a Riesz basis of translates, in: Approximation Theory*, Leture Notes in Pure and Applied Math., 138, Dekker, New York (1992), 401–408.
21. N. Wiener, *The Fourier integral and certain of its applications*, Cambridge University Press, Cambridge, 1933, Reprint: Dover, New York, 1958.

[1] Department of Mathematics, Tianjin University of Technology, Tianjin 300384, P.R. China.
E-mail address: beiliu1101@gmail.com

[2] Department of Mathematics and LPMC, Nankai University, Tianjin 300071, P.R. China.
E-mail address: ruiliu@nankai.edu.cn