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ON GENERALIZED (*M, N, L*)-JORDAN CENTRALIZERS OF SOME ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a unital algebra over a number field $\mathbb{K}$. A linear mapping δ from $\mathcal{A}$ into itself is called a generalized $(m,\ n,\ l)$-Jordan centralizer if it satisfies $(m+n+l)\delta(A^2) - m\delta(A)A - nA\delta(A) - lA\delta(I)A \in \mathbb{K}I$ for every $A \in \mathcal{A}$, where $m \geq 0, n \geq 0, l \geq 0$ are fixed integers with $m+n+l \neq 0$. In this paper, we study generalized $(m,\ n,\ l)$-Jordan centralizers on generalized matrix algebras and some reflexive algebras alg$\mathcal{L}$, where $\mathcal{L}$ is a CSL or satisfies $\vee\{L : L \in \mathcal{J}(\mathcal{L})\} = X$ or $\wedge\{L_- : L \in \mathcal{J}(\mathcal{L})\} = (0)$, and prove that each generalized $(m,\ n,\ l)$-Jordan centralizer of these algebras is a centralizer when $m+l \geq 1$ and $n+l \geq 1$.

1. Introduction

Let $\mathcal{A}$ be an algebra over a number field $\mathbb{K}$ and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. An additive (linear) mapping δ from $\mathcal{A}$ to $\mathcal{M}$ is called a *left (right) centralizer* if $\delta(AB) = \delta(A)B$ $(\delta(AB) = A\delta(B))$ for all $A, B \in \mathcal{A}$; it is called a *left (right) Jordan centralizer* if $\delta(A^2) = \delta(A)A$ $(\delta(A^2) = A\delta(A))$ for every $A \in \mathcal{A}$. We call δ a *centralizer* if δ is both a left centralizer and a right centralizer. Similarly, we can define a *Jordan centralizer*. It is clear that every centralizer is a Jordan centralizer, but the converse is not true in general. In [20], Zalar proved that each left Jordan centralizer of a semiprime ring is a left centralizer and each

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Jordan centralizer of a semiprime ring is a centralizer. For some other results, see [15, 16, 17, 18] and references therein.

Recently, Vukman[19] introduced a new type of Jordan centralizers, named $(m,\ n)$-*Jordan centralizer*, that is, an additive mapping δ from a ring $\mathcal{R}$ into itself satisfies

$$(m+n)\delta(A^2) = m\delta(A)A + nA\delta(A)$$

for every $A \in \mathcal{R}$, where $m \geq 0$, $n \geq 0$ are fixed integers with $m+n \neq 0$. Obviously, each $(1,\ 0)$-Jordan centralizer is a left Jordan centralizer and each $(0,\ 1)$-Jordan centralizer is a right Jordan centralizer. Moreover, each Jordan centralizer is an $(m,\ n)$-Jordan centralizer and $(1,\ 1)$-Jordan centralizer satisfies the relation $2\delta(A^2) = \delta(A)A + A\delta(A)$ for every $A \in \mathcal{R}$. The natural problem that one considers in this context is whether the converses are true. In [15], Vukman showed that each $(1,\ 1)$-Jordan centralizer of a 2-torsion free semiprime ring $\mathcal{R}$ is a centralizer. In [2], Guo and Li studied $(1,\ 1)$-Jordan centralizers of some reflexive algebras. In [19], Vukman investigated $(m,\ n)$-Jordan centralizers and proved that for $m \geq 1$ and $n \geq 1$, every $(m,\ n)$-Jordan centralizer of a prime ring $\mathcal{R}$ with $char(\mathcal{R}) \neq 6mn(m+n)$ is a centralizer. Furthermore, Qi and Hou in [12] showed that for a unital prime algebra $\mathcal{A}$ with center $\mathbb{K}I$, if δ is a linear mapping from $\mathcal{A}$ into itself such that $(m+n)\delta(AB) - mA\delta(B) - n\delta(A)B \in \mathbb{K}I$ for all $A, B \in \mathcal{A}$, then δ is a centralizer. Motivated by these facts, we define a new type of Jordan centralizers that generalizes all the types mentioned above, named *generalized $(m,\ n,\ l)$-Jordan centralizer*. A linear mapping δ from a unital algebra $\mathcal{A}$ into itself is called a generalized $(m,\ n,\ l)$-Jordan centralizer if it satisfies

$$(m+n+l)\delta(A^2) - m\delta(A)A - nA\delta(A) - lA\delta(I)A \in \mathbb{K}I$$

for every $A \in \mathcal{A}$, where $m \geq 0, n \geq 0, l \geq 0$ are fixed integers with $m+n+l \neq 0$. This is equivalent to say that for every $A \in \mathcal{A}$, there exists a $\lambda_A \in \mathbb{K}$ such that

$$(m+n+l)\delta(A^2) = m\delta(A)A + nA\delta(A) + lA\delta(I)A + \lambda_A I.$$

When $\lambda_A = 0$ for every $A \in \mathcal{A}$, we call such a δ an $(m,\ n,\ l)$-*Jordan centralizer*. It is clear that each $(m,\ n,\ l)$-Jordan centralizer is a generalized $(m,\ n,\ l)$-Jordan centralizer, each $(m,\ n,\ 0)$-Jordan centralizer is an $(m,\ n)$-Jordan centralizer and $(0,\ 0,\ 1)$-Jordan centralizer has the relation $\delta(A^2) = A\delta(I)A$ for every $A \in \mathcal{A}$. In this paper, we study (generalized) $(m,\ n,\ l)$-Jordan centralizers on some reflexive algebras and generalized matrix algebras.

Let X be a Banach space over $\mathbb{K}$ and $B(X)$ be the set of all bounded operators on X, where $\mathbb{K}$ is the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We use X^* to denote the set of all bounded linear functionals on X. For $A \in B(X)$, denote by A^* the adjoint of A. For any non-empty subset $L \subseteq X$, $L^\perp$ denotes its annihilator, that is, $L^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in L\}$. By a *subspace lattice* on X, we mean a collection $\mathcal{L}$ of closed subspaces of X with (0) and X in $\mathcal{L}$ such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\wedge M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\wedge M_r$ denotes the intersection of $\{M_r\}$, and $\vee M_r$ denotes the closed linear span of $\{M_r\}$. For a subspace lattice $\mathcal{L}$ of X, let alg$\mathcal{L}$ denote the algebra of all operators in $B(X)$ that leave members of $\mathcal{L}$ invariant; and for a subalgebra $\mathcal{A}$ of $B(X)$, let lat$\mathcal{A}$ denote the lattice of all closed subspaces of X that are invariant under all operators in $\mathcal{A}$. An algebra $\mathcal{A}$ is called *reflexive* if alglat$\mathcal{A} = \mathcal{A}$; and dually, a subspace lattice is called *reflexive* if latalg$\mathcal{L} = \mathcal{L}$. Every reflexive algebra is of the form alg$\mathcal{L}$ for some subspace lattice $\mathcal{L}$ and vice versa.

For a subspace lattice $\mathcal{L}$ and for $E \in \mathcal{L}$, define

$$E_- = \vee\{F \in \mathcal{L} : F \not\supseteq E\} \text{ and } E_+ = \wedge\{F \in \mathcal{L} : F \not\subseteq E\}.$$

Put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq (0) \text{ and } K_- \neq X\}.$$

For any non-zero vectors $x \in X$ and $f \in X^*$, the rank one operator $x \otimes f$ is defined by $x \otimes f(y) = f(y)x$ for $y \in X$. Several authors have studied the properties of the set of rank one operators in reflexive algebras (for example, see [4, 6]). It is well known (see [6]) that $x \otimes f \in \text{alg}\mathcal{L}$ if and only if there exists some $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$. When X is a separable Hilbert space over the complex field $\mathbb{C}$, we change it to H. In a Hilbert space, we disregard the distinction between a closed subspace and the orthogonal projection onto it. A subspace lattice $\mathcal{L}$ on a Hilbert space H is called a *commutative subspace lattice* (*CSL*), if all projections in $\mathcal{L}$ commute pairwise. If $\mathcal{L}$ is a CSL, then the corresponding algebra alg$\mathcal{L}$ is called a *CSL algebra*. By [1], we know that if $\mathcal{L}$ is a CSL, then $\mathcal{L}$ is reflexive. Let $\mathcal{L}$ be a subspace lattice on a Banach space X satisfying $\vee\{L : L \in \mathcal{J}(\mathcal{L})\} = X$ or $\wedge\{L_- : L \in \mathcal{J}(\mathcal{L})\} = (0)$. In [9], Lu considered this kind of reflexive algebras which have rich rank one operators. In Section 2, we prove that if δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from alg$\mathcal{L}$ into itself, where $\mathcal{L}$ is a CSL or satisfies $\vee\{L : L \in \mathcal{J}(\mathcal{L})\} = X$ or $\wedge\{L_- : L \in \mathcal{J}(\mathcal{L})\} = (0)$, then δ is a centralizer.

A *Morita context* is a set $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N})$ and two mappings ϕ and φ, where $\mathcal{A}$ and $\mathcal{B}$ are two algebras over a number field $\mathbb{K}$, $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule and $\mathcal{N}$ is a $(\mathcal{B}, \mathcal{A})$-bimodule. The mappings $\phi : \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$ and $\varphi : \mathcal{N} \otimes_{\mathcal{A}} \mathcal{M} \to \mathcal{B}$ are two bimodule homomorphisms satisfying $\phi(M \otimes N)M' = M\varphi(N \otimes M')$ and $\varphi(N \otimes M)N' = N\phi(M \otimes N')$ for any $M, M' \in \mathcal{M}$ and $N, N' \in \mathcal{N}$. These conditions insure that the set

$$\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix} = \left\{ \begin{bmatrix} A & M \\ N & B \end{bmatrix} \mid\ A \in \mathcal{A}, M \in \mathcal{M}, N \in \mathcal{N}, B \in \mathcal{B} \right\}$$

forms an algebra over $\mathbb{K}$ under usual matrix operations. We call such an algebra a *generalized matrix algebra* and denote it by $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$, where $\mathcal{A}$ and $\mathcal{B}$ are two unital algebras and at least one of the two bimodules $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. This kind of algebra was first introduced by Sands in [14]. Obviously, when $\mathcal{M} = 0$ or $\mathcal{N} = 0$, $\mathcal{U}$ degenerates to the *triangular algebra*. In Section 3, we show that if δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{U}$ into itself, then δ is a centralizer. We also study $(m,\ n,\ l)$-Jordan centralizers on AF C^*-algebras. Throughout the paper, we assume $m, n, l \in \mathbb{N}$ are such that $m + l \geq 1$, $n + l \geq 1$.

2. Centralizers of certain reflexive algebras

In order to prove our main results, we need the following several lemmas.

Lemma 2.1. *Let $\mathcal{A}$ be a unital algebra with identity I. Suppose δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{A}$ into itself. Then for any $A, B \in \mathcal{A}$,*

$$\begin{aligned}(m+n+l)&\delta(AB+BA) \\ &= m\delta(A)B + m\delta(B)A + nA\delta(B) + nB\delta(A) \\ &\quad + lA\delta(I)B + lB\delta(I)A + (\lambda_{A+B} - \lambda_A - \lambda_B)I. \end{aligned} \tag{2.1}$$

In particular, for any $A \in \mathcal{A}$,

$$\delta(A) = \frac{m+l}{m+n+2l}\delta(I)A + \frac{n+l}{m+n+2l}A\delta(I) + \lambda(A), \tag{2.2}$$

where we set $\lambda(A) = \frac{1}{m+n+2l}(\lambda_{A+I} - \lambda_A)I$ for every $A \in \mathcal{A}$.

Proof. Since δ is a generalized $(m,\ n,\ l)$-Jordan centralizer, we have

$$(m+n+l)\delta(A^2) = m\delta(A)A + nA\delta(A) + lA\delta(I)A + \lambda_A I$$

for every $A \in \mathcal{A}$. Replacing A by $A+B$ in above equation, (2.1) holds. Letting $B = I$ in (2.1) gives (2.2), since $\lambda_I = 0$. □

Remark 2.2. For an $(m,\ n,\ l)$-Jordan centralizer, we could actually define it from a unital algebra $\mathcal{A}$ to an $\mathcal{A}$-bimodule. Hence when lemmas in this section are applied to an $(m,\ n,\ l)$-Jordan centralizer δ, we will take it for granted that δ is from a unital algebra $\mathcal{A}$ to its bimodule, since all the proofs remain true if we set $\lambda_A = 0$ for all $A \in \mathcal{A}$.

Remark 2.3. Obviously, each $(1,\ 0,\ 0)$-Jordan centralizer is a left Jordan centralizer and each $(0,\ 1,\ 0)$-Jordan centralizer is a right Jordan centralizer. So by Lemma 2.1, it follows that every left Jordan centralizer of unital algebras is a left centralizer and every right Jordan centralizer of unital algebras is a right centralizer. Therefore every Jordan centralizer of unital algebras is a centralizer.

Let f be a linear mapping from an algebra $\mathcal{A}$ to its bimodule $\mathcal{M}$. Recall that f is a *derivation* if $f(ab) = f(a)b + af(b)$ for all $a, b \in \mathcal{A}$; it is a *Jordan derivation* if $f(a^2) = f(a)a + af(a)$ for every $a \in \mathcal{A}$; it is a *generalized derivation* if $f(ab) = f(a)b + ad(b)$ for all $a, b \in \mathcal{A}$, where d is a derivation from $\mathcal{A}$ to $\mathcal{M}$; and it is a *generalized Jordan derivation* if $f(a^2) = f(a)a + ad(a)$ for every $a \in \mathcal{A}$, where d is a Jordan derivation from $\mathcal{A}$ to $\mathcal{M}$. From Remarks 2.2 and 2.3, we have the following corollary.

Corollary 2.4. *Let $\mathcal{L}$ be a subspace lattice on a Banach space X satisfying $\vee\{F : F \in \mathcal{J}(\mathcal{L})\} = X$ or $\wedge\{L_- : L \in \mathcal{J}(\mathcal{L})\} = (0)$. If f is a generalized Jordan derivation from alg$\mathcal{L}$ to $B(X)$, then f is a generalized derivation.*

Proof. Since f is a generalized Jordan derivation, we have the relation

$$f(A^2) = f(A)A + Ad(A)$$

for every $A \in \mathrm{alg}\mathcal{L}$, where d is a Jordan derivation of $\mathrm{alg}\mathcal{L}$. By [9, Theorem 2.1], one can conclude that d is a derivation. Let $\delta = f - d$. Then we have

$$\begin{aligned}\delta(A^2) &= f(A^2) - d(A^2)\\ &= f(A)A + Ad(A) - Ad(A) - d(A)A\\ &= f(A)A - d(A)A\\ &= \delta(A)A\end{aligned}$$

for every $A \in \mathrm{alg}\mathcal{L}$. This means that δ is a left Jordan centralizer. By Remark 2.3, δ is a left centralizer. Hence

$$f(AB) = d(AB) + \delta(AB) = d(A)B + Ad(B) + \delta(A)B = f(A)B + Ad(B)$$

for all $A, B \in \mathrm{alg}\mathcal{L}$. In other words, f is a generalized derivation. □

Since every Jordan derivation of CSL algebras is a derivation [10], we also have the following corollary.

Corollary 2.5. *Let $\mathcal{L}$ be a CSL on a Hilbert space H. If f is a generalized Jordan derivation from $\mathrm{alg}\mathcal{L}$ into itself, then f is a generalized derivation.*

Lemma 2.6. *Let $\mathcal{A}$ be a unital algebra and δ be a generalized (m, n, l)-Jordan centralizer from $\mathcal{A}$ into itself. Then for every idempotent $P \in \mathcal{A}$ and every $A \in \mathcal{A}$,*
(i) *$\delta(P) = P\delta(I) = \delta(I)P$;*
(ii) *$\delta(AP) = \delta(A)P + \lambda(AP) - \lambda(A)P$;*
(iii) *$\delta(PA) = P\delta(A) + \lambda(PA) - \lambda(A)P$.*

Proof. (i) Suppose P is an idempotent in $\mathcal{A}$. It follows from Lemma 2.1 that

$$(m + n + 2l)\delta(P) = (m + l)\delta(I)P + (n + l)P\delta(I) + (\lambda_{P+I} - \lambda_P)I. \tag{2.3}$$

Right and left multiplication of (2.3) by P gives

$$P\delta(P)P = P\delta(I)P + \frac{1}{m+n+2l}(\lambda_{P+I} - \lambda_P)P.$$

Since $(m+n+l)\delta(P) = m\delta(P)P + nP\delta(P) + lP\delta(I)P + \lambda_P I$, multiplying P from the right leads to

$$\begin{aligned}(n + l)\delta(P)P &= n(P\delta(I)P + \frac{1}{m+n+2l}(\lambda_{P+I} - \lambda_P)P) + lP\delta(I)P + \lambda_P P\\ &= (n + l)P\delta(I)P + (\frac{n}{m+n+2l}(\lambda_{P+I} - \lambda_P) + \lambda_P)P,\end{aligned}$$

whence

$$\delta(P)P = P\delta(I)P + \varepsilon_P P \tag{2.4}$$

for some $\varepsilon_P \in \mathbb{C}$. Similarly, $P\delta(P) = P\delta(I)P + \varepsilon'_P P$ for some $\varepsilon'_P \in \mathbb{C}$.
Hence $\delta(P)P - \varepsilon_P P = P\delta(P) - \varepsilon'_P P$. Right and left multiplication of P gives $\varepsilon_P = \varepsilon'_P$, which implies

$$\delta(P)P = P\delta(P). \tag{2.5}$$

Replacing P by $I - P$ in the above equation gives $\delta(I)P = P\delta(I)$.

Now, we have from (2.3)

$$\delta(P)=\delta(I)P+\frac{1}{m+n+2l}(\lambda_{P+I}-\lambda_P)I. \tag{2.6}$$

On the other hand, (2.4) and (2.5) yields

$$\begin{aligned}(m+n+l)\delta(P)&=m\delta(P)P+nP\delta(P)+lP\delta(I)P+\lambda_P I\\&=(m+n+l)\delta(P)P+\lambda_P I-l\varepsilon_P P,\end{aligned}$$

right multiplication of which by P gives $\lambda_P=l\varepsilon_P$. Hence

$$\delta(P)=\delta(P)P+\frac{1}{m+n+l}\lambda_P(I-P). \tag{2.7}$$

We then have from (2.6) that

$$\delta(P)P=\delta(I)P+\frac{1}{m+n+2l}(\lambda_{P+I}-\lambda_P)P. \tag{2.8}$$

Now (2.7) and (2.8) yield

$$\delta(P)=\delta(I)P+\frac{1}{m+n+2l}(\lambda_{P+I}-\lambda_P)P+\frac{1}{m+n+l}\lambda_P(I-P),$$

which together with (2.6) implies

$$\frac{1}{m+n+2l}(\lambda_{P+I}-\lambda_P)=\frac{1}{m+n+l}\lambda_P.$$

Thus we have

$$\delta(P)=\delta(I)P+\frac{1}{m+n+l}\lambda_P I, \tag{2.9}$$

while

$$\delta(P)=\frac{m+n}{m+n+l}\delta(P)P+\frac{l}{m+n+l}\delta(I)P+\frac{1}{m+n+l}\lambda_P I. \tag{2.10}$$

Comparing (2.9) and (2.10) gives

$$\delta(I)P=\delta(P)P.$$

This together with (2.8) gives

$$\lambda(P)=\frac{1}{m+n+2l}(\lambda_{P+I}-\lambda_P)I=\frac{1}{m+n+l}\lambda_P I=0,$$

whence

$$\delta(P)=\delta(I)P=P\delta(I).$$

(ii) By Lemma 2.1 and (i), we have

$$\begin{aligned}\delta(AP)&=\frac{m+l}{m+n+2l}\delta(I)AP+\frac{n+l}{m+n+2l}AP\delta(I)+\lambda(AP)\\&=(\frac{m+l}{m+n+2l}\delta(I)A+\frac{n+l}{m+n+2l}A\delta(I))P+\lambda(AP)\\&=(\delta(A)-\lambda(A))P+\lambda(AP)\\&=\delta(A)P+\lambda(AP)-\lambda(A)P.\end{aligned}$$

(iii) The proof is analogous to the proof of (ii). □

An subset $\mathcal{I}$ of an algebra $\mathcal{A}$ is called a *left separating set* of $\mathcal{A}$ if for every $A \in \mathcal{A}$, $A\mathcal{I} = 0$ implies $A = 0$. We have the following simple but noteworthy result.

Corollary 2.7. *Suppose $\mathcal{I}$ is a left separating left ideal of a unital algebra $\mathcal{A}$ and is contained in the algebra generated by all idempotents in $\mathcal{A}$. Then each generalized (m, n, l)-Jordan centralizer δ from $\mathcal{A}$ into itself is a centralizer.*

Proof. Since $\mathcal{I}$ is contained in the algebra generated by all idempotents in $\mathcal{A}$ and by (i) of Lemma 2.6, we have that $\delta(I) \in \mathcal{I}'$, where $\mathcal{I}'$ denotes the commutant of $\mathcal{I}$. Hence $\delta(A) = \delta(I)A + \lambda(A) = A\delta(I) + \lambda(A)$ for every $A \in \mathcal{I}$ according to (2.2). For any $A(\neq \mathbb{K}I) \in \mathcal{I}$, we have

$$\begin{aligned}
&(m+n+l)(\delta(I)A^2 + \lambda(A^2)) \\
&\qquad = (m+n+l)\delta(A^2) \\
&\qquad = m\delta(A)A + nA\delta(A) + lA\delta(I)A + \lambda_A I \\
&\qquad = m(\delta(I)A^2 + \lambda(A)A) + n(A^2\delta(I) + A\lambda(A)) + lA^2\delta(I) + \lambda_A I,
\end{aligned}$$

which implies $\lambda(A)A = kI$ for some $k \in \mathbb{K}$.

Hence $\lambda(A) = 0$ and $\delta(A) = \delta(I)A = A\delta(I)$ for every $A \in \mathcal{I}$. Then Lemma 2.6 yields $A\delta(I)B = AB\delta(I) = \delta(AB) = \delta(I)AB$ for every $B \in \mathcal{I}$, and since $\mathcal{I}$ is a separating left ideal, we have $A\delta(I) = \delta(I)A$ for every $A \in \mathcal{A}$. Therefore, $\delta(A) = \delta(I)A + \lambda(A) = A\delta(I) + \lambda(A)$ for every $A \in \mathcal{A}$. Now by the same argument as above, we have that $\delta(A) = \delta(I)A = A\delta(I)$ for every $A \in \mathcal{A}$ and this completes the proof. □

Remark 2.8. By [3, Proposition 2.2], [13, Example 6.2], we see that the class of algebras we discussed in Corollary 2.7 contains a lot of algebras and is therefore very large.

The proof of the following lemma is analogous to the proof of [8, Proposition 1.1]. For the sake of completeness, we present the proof here.

Lemma 2.9. *Let E and F be non-zero subspaces of X and X^* respectively. Let $\phi : E \times F \to B(X)$ be a bilinear mapping such that $\phi(x, f)X \subseteq \mathbb{K}x$ for all $x \in E$ and $f \in F$. Then there exists a linear mapping $S : F \to X^*$ such that $\phi(x, f) = x \otimes Sf$ for all $x \in E$ and $f \in F$.*

Proof. For any non-zero vectors $x \in E$ and $f \in F$, since $\phi(x, f)X \subseteq \mathbb{K}x$, there exists a continuous linear functional $h_{x,f}$ on X such that for each $z \in X$, $\phi(x, f)z = h_{x,f}(z)x$. That is, for all $x \in E$ and $f \in F$,

$$\phi(x, f) = x \otimes h_{x,f} \tag{2.11}$$

We claim that $h_{x,f}$ depends only on f. To see this, fix a non-zero functional f in F, and let x_1 and x_2 be non-zero vectors in E. Suppose that x_1 and x_2 are linearly independent. For all $z \in X$, by (2.11) we have

$$\begin{aligned}
h_{x_1+x_2,f}(z)(x_1 + x_2) &= \phi(x_1 + x_2, f)z \\
&= \phi(x_1, f)z + \phi(x_2, f)z \\
&= h_{x_1,f}(z)x_1 + h_{x_2,f}(z)x_2
\end{aligned}$$

from which we have

$$(h_{x_1+x_2,f}(z) - h_{x_1,f}(z))x_1 = (h_{x_2,f}(z) - h_{x_1+x_2,f}(z))x_2.$$

So $h_{x_1,f} = h_{x_1+x_2,f} = h_{x_2,f}$. Now suppose that x_1 and x_2 are linearly dependent. Let $x_2 = kx_1$. Then

$$x_2 \otimes h_{x_2,f} = \phi(x_2, f) = k\phi(x_1, f) = kx_1 \otimes h_{x_1,f} = x_2 \otimes h_{x_1,f},$$

which yields $h_{x_1,f} = h_{x_2,f}$. Thus $\phi(x, f) = x \otimes h_f$ for all $x \in E$ and $f \in F$. Hence there exists a linear mapping S from F to X^* such that $\phi(x, f) = x \otimes Sf$. It is easy to check that the mapping S is well defined and linear. □

Lemma 2.10. *Let $\mathcal{L}$ be a subspace lattice on a Banach space X and δ be a generalized (m, n, l)-Jordan centralizer from alg$\mathcal{L}$ into itself. Suppose that E and L are in $\mathcal{J}(\mathcal{L})$ such that $E_- \ngeq L$. Let x be in E and f be in $L_-^\perp$. Then $(\delta(x \otimes f) - \lambda(x \otimes f))X \subseteq \mathbb{K}x$.*

Proof. Since $E_- \ngeq L$, we have that $E \leq L$. So $x \otimes f \in \text{alg}\mathcal{L}$. Suppose $f(x) \neq 0$, it follows from Lemmas 2.1 and 2.6 that $\lambda(x \otimes f) = 0$ and $\delta(x \otimes f) = x \otimes f\delta(I)$. Thus $\delta(x \otimes f)X \subseteq \mathbb{K}x$.

Now we assume $f(x) = 0$. Choose z from L and g from $E_-^\perp$ such that $g(z) = 1$. Then

$$\begin{aligned}
&(m+n+2l)(m+n+l)\delta(x \otimes f)\\
&\quad=(m+n+2l)(m+n+l)\delta((x \otimes g)(z \otimes f) + (z \otimes f)(x \otimes g))\\
&\quad=(m+n+2l)(m\delta(x \otimes g)(z \otimes f) + n(x \otimes g)\delta(z \otimes f) + l(x \otimes g)\delta(I)(z \otimes f))\\
&\qquad+ (m+n+2l)(m\delta(z \otimes f)(x \otimes g) + n(z \otimes f)\delta(x \otimes g)\\
&\qquad+ l(z \otimes f)\delta(I)(x \otimes g)) + (m+n+2l)(\lambda_{x\otimes g+z\otimes f} - \lambda_{x\otimes g} - \lambda_{z\otimes f})I\\
&\quad=(m^2+ml)\delta(I)x \otimes f + (n^2+nl)x \otimes f\delta(I)\\
&\qquad+ 2(mn+ml+nl+l^2)(x \otimes g\delta(I)z \otimes f + z \otimes f\delta(I)x \otimes g) + \lambda_1 I
\end{aligned}$$

for some $\lambda_1 \in \mathbb{K}$.

On the other hand,

$$\begin{aligned}
&(m+2n+l)(m+n+l)\delta(x \otimes f)\\
&\quad=(m+n+l)((m+l)\delta(I)x \otimes f + (n+l)x \otimes f\delta(I) + (\lambda_{x\otimes f+I} - \lambda_{x\otimes f})I)\\
&\quad=(m^2+2ml+l^2+mn+nl)\delta(I)x \otimes f\\
&\qquad+ (ml+mn+l^2+2nl+n^2)x \otimes f\delta(I) + \lambda_2 I
\end{aligned}$$

for some $\lambda_2 \in \mathbb{K}$.

So

$$\delta(I)x \otimes f + x \otimes f\delta(I) = 2x \otimes g\delta(I)z \otimes f + 2z \otimes f\delta(I)x \otimes g + \lambda I \tag{2.12}$$

for some $\lambda \in \mathbb{K}$.

Notice that (2.12) is valid for all z in L satisfying $g(z) = 1$. Applying this equation to x, we have

$$f(\delta(I)x)x = 2g(x)f(\delta(I)x)z + \lambda x. \tag{2.13}$$

If $g(x)=0$ and $f(z)=0$, then $f(\delta(I)x)=\lambda$. Substituting $z+x$ for z in (2.12) gives

$$\delta(I)x\otimes f+x\otimes f\delta(I)=2x\otimes g\delta(I)(z+x)\otimes f+2\lambda(z+x)\otimes g+\lambda I. \quad (2.14)$$

Comparing (2.12) with (2.14) yields

$$g(\delta(I)x)x\otimes f+\lambda x\otimes g=0.$$

Applying this equation to z leads to $\lambda x=0$, which means $f(\delta(I)x)=\lambda=0$.

If $g(x)=0$ and $f(z)\neq 0$, from (2.13) we also have $f(\delta(I)x)=\lambda$, and it follows from Lemma 2.6 that

$$\begin{aligned}\delta(I)x\otimes f+x\otimes f\delta(I)&=2x\otimes g\delta(I)z\otimes f+2z\otimes f\delta(I)x\otimes g+\lambda I\\&=2(x\otimes g)(z\otimes f)\delta(I)+2\delta(I)(z\otimes f)(x\otimes g)+\lambda I\\&=2x\otimes f\delta(I)+\lambda I,\end{aligned}$$

whence

$$\delta(I)x\otimes f=x\otimes f\delta(I)+\lambda I.$$

Applying the above equation to x yields $f(\delta(I)x)=-\lambda$. Thus $f(\delta(I)x)=\lambda=0$.

If $g(x)\neq 0$, replacing z by $\frac{1}{g(x)}x$ in (2.13) gives $f(\delta(I)x)=-\lambda$, while

$$\begin{aligned}\delta(I)x\otimes f+x\otimes f\delta(I)&=2x\otimes g\delta(I)z\otimes f+2z\otimes f\delta(I)x\otimes g+\lambda I\\&=2\delta(I)(x\otimes g)(z\otimes f)+2(z\otimes f)(x\otimes g)\delta(I)+\lambda I\\&=2\delta(I)(x\otimes f)+\lambda I.\end{aligned}$$

Hence

$$x\otimes f\delta(I)=\delta(I)x\otimes f+\lambda I. \quad (2.15)$$

Applying (2.15) to x leads to $f(\delta(I)x)=\lambda$. Therefore, $f(\delta(I)x)=\lambda=0$.

So by (2.12), we obtain $\delta(I)x\otimes f=2g(\delta(I)z)x\otimes f-x\otimes f\delta(I)$. It follows from Lemma 2.1 that

$$\begin{aligned}\delta(x\otimes f)=&\frac{m+l}{m+n+2l}\delta(I)(x\otimes f)+\frac{n+l}{m+n+2l}(x\otimes f)\delta(I)+\lambda(x\otimes f)\\=&\frac{m+l}{m+n+2l}(2g(\delta(I)z)x\otimes f-x\otimes f\delta(I))\\&+\frac{n+l}{m+n+2l}(x\otimes f)\delta(I)+\lambda(x\otimes f)\\=&\frac{2(m+l)}{m+n+2l}g(\delta(I)z)x\otimes f+\frac{n-m}{m+n+2l}(x\otimes f)\delta(I)+\lambda(x\otimes f).\end{aligned}$$

Hence $(\delta(x\otimes f)-\lambda(x\otimes f))X\subseteq\mathbb{K}x$. $\square$

Theorem 2.11. *Let $\mathcal{L}$ be a subspace lattice on a Banach space X satisfying $\vee\{F : F\in\mathcal{J}(\mathcal{L})\}=X$. If δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from alg$\mathcal{L}$ into itself, then δ is a centralizer. In particular, the conclusion holds if $\mathcal{L}$ has the property $X_-\neq X$.*

Proof. Let E be in $\mathcal{J}(\mathcal{L})$. By $\vee\{F : F \in \mathcal{J}(\mathcal{L})\} = X$, there is an element L in $\mathcal{J}(\mathcal{L})$ such that $E_- \not\geq L$. Let x be in E and f be in $(L_-)^\perp$. Let $\overline{\delta} = \delta - \lambda$. Then $\overline{\delta}(I) = \delta(I)$, and it follows from Lemmas 2.9 and 2.10 that there exists a linear mapping $S : (L_-)^\perp \to X^*$ such that

$$\overline{\delta}(x \otimes f) = x \otimes Sf.$$

This together with

$$\frac{m+l}{m+n+2l}\overline{\delta}(I)x \otimes f + \frac{n+l}{m+n+2l}x \otimes f\overline{\delta}(I) = \overline{\delta}(x \otimes f)$$

leads to

$$x \otimes (Sf - \frac{n+l}{m+n+2l}\overline{\delta}(I)^* f) = \frac{m+l}{m+n+2l}\overline{\delta}(I)x \otimes f.$$

Thus there exists a constant λ_E in $\mathbb{K}$ such that $\overline{\delta}(I)x = \lambda_E x$ for every $x \in E$. Similarly, for every $y \in L$, we have $\overline{\delta}(I)y = \lambda_L y$.

If $f(x) \neq 0$, it follows from Lemma 2.6 that $\overline{\delta}(x \otimes f) = \overline{\delta}(I)x \otimes f = x \otimes f\overline{\delta}(I)$.

If $f(x) = 0$, according to the proof of Lemma 2.10, we can choose z from L and g from $E_-^\perp$ such that $g(z) = 1$ and $\overline{\delta}(I)x \otimes f = 2g(\overline{\delta}(I)z)x \otimes f - x \otimes f\overline{\delta}(I)$. Since $x \in E \leq L$, we have $\overline{\delta}(I)x = \lambda_L x$. Thus

$$\overline{\delta}(I)x \otimes f = 2\lambda_L x \otimes f - x \otimes f\overline{\delta}(I) = 2\overline{\delta}(I)x \otimes f - x \otimes f\overline{\delta}(I).$$

Hence $\overline{\delta}(x \otimes f) = \overline{\delta}(I)x \otimes f = x \otimes f\overline{\delta}(I)$.

Therefore, for any $x \in E$, $f \in (L_-)^\perp$ and $A \in \mathrm{alg}\mathcal{L}$, we have

$$A\overline{\delta}(I)x \otimes f = Ax \otimes f\overline{\delta}(I) = \overline{\delta}(I)Ax \otimes f,$$

which yields $A\overline{\delta}(I)x = \overline{\delta}(I)Ax$ for any $x \in E$.

Now by $\vee\{F : F \in \mathcal{J}(\mathcal{L})\} = X$, we have $\overline{\delta}(A) = A\overline{\delta}(I) = \overline{\delta}(I)A$ for any $A \in \mathrm{alg}\mathcal{L}$, this means $\delta(A) = A\delta(I) + \lambda(A) = \delta(I)A + \lambda(A)$. The remaining part goes along the same line as the proof of Corollary 2.7 and this completes the proof. $\square$

Remark 2.12. By [7], a subspace lattice $\mathcal{L}$ is said to be *completely distributive* if $L = \vee\{E \in \mathcal{L} : E_- \not\geq L\}$ and $L = \wedge\{E_- : E \in \mathcal{L} \text{ and } E \not\leq L\}$ for all $L \in \mathcal{L}$. It follows that completely distributive subspace lattices satisfy the condition $\vee\{E : E \in \mathcal{J}(\mathcal{L})\} = X$. Thus Theorem 2.11 applies to completely distributive subspace lattice algebras. A subspace lattice $\mathcal{L}$ is called a $\mathcal{J}$*-subspace lattice* on X if $\vee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$, $\wedge\{K_- : K \in \mathcal{J}(\mathcal{L})\} = (0)$, $K \vee K_- = X$ and $K \wedge K_- = (0)$ for any $K \in \mathcal{J}(\mathcal{L})$. Note also that the condition $\vee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$ is part of the definition of $\mathcal{J}$-subspace lattices, thus Theorem 2.11 also applies to $\mathcal{J}$-subspace lattice algebras.

With a proof similar to the proof of Theorem 2.11, we have the following theorem.

Theorem 2.13. *Let $\mathcal{L}$ be a subspace lattice on a Banach space X satisfying $\wedge\{L_- : L \in \mathcal{J}(\mathcal{L})\} = (0)$. If δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from alg$\mathcal{L}$ into itself, then δ is a centralizer. In particular, the conclusion holds if $\mathcal{L}$ has the property $(0)_+ \neq (0)$.*

As for the cases of (m,n,l)-Jordan centralizers, we have from Remark 2.2 , Theorem 2.11 and Theorem 2.13 the following theorem.

Theorem 2.14. *Let $\mathcal{L}$ be a subspace lattice on a Banach space X satisfying $\vee\{F : F \in \mathcal{J}(\mathcal{L})\} = X$ or $\wedge\{L_- : L \in \mathcal{J}(\mathcal{L})\} = (0)$. If δ is an $(m,\ n,\ l)$-Jordan centralizer from alg$\mathcal{L}$ to $B(X)$, then δ is a centralizer.*

In the rest of this section we will investigate generalized (m,n,l)-Jordan centralizers on CSL algebras. Let H be a complex separable Hilbert space and $\mathcal{L}$ be a CSL on H. Let $\mathcal{L}^\perp$ be the lattice $\{I-E : E \in \mathcal{L}\}$ and $\mathcal{L}'$ be the commutant of $\mathcal{L}$. It is easy to verify that $(\text{alg}\mathcal{L})^* = \text{alg}\mathcal{L}^\perp$ for any lattice $\mathcal{L}$ on H and the diagonal $(\text{alg}\mathcal{L}) \cap (\text{alg}\mathcal{L})^* = \mathcal{L}'$ is a von Neumann algebra. Given a CSL $\mathcal{L}$ on a Hilbert space H, we define $G_1(\mathcal{L})$ and $G_2(\mathcal{L})$ to be the projections onto the closures of the linear spans of $\{EA(I-E)x : E \in \mathcal{L}, A \in alg\mathcal{L}, x \in H\}$ and $\{(I-E)A^*Ex : E \in \mathcal{L}, A \in alg\mathcal{L}, x \in H\}$, respectively. For simplicity, we write G_1 and G_2 for $G_1(\mathcal{L})$ and $G_2(\mathcal{L})$. Since CSL is reflexive, it is easy to verify that $G_1 \in \mathcal{L}$ and $G_2 \in \mathcal{L}^\perp$. In [10], Lu showed that $G_1 \vee G_2 \in \mathcal{L} \cap \mathcal{L}^\perp$ and $\text{alg}\mathcal{L}(I - G_1 \vee G_2) \subseteq \mathcal{L}'$.

Theorem 2.15. *Let $\mathcal{L}$ be a CSL on a complex separable Hilbert space H. If δ is a bounded generalized $(m,\ n,\ l)$-Jordan centralizer from alg$\mathcal{L}$ into itself, then δ is a centralizer.*

Proof. We divide the proof into two cases.
Case 1: Suppose $G_1 \vee G_2 = I$.

Let $A \in \text{alg}\mathcal{L}$. For any $T \in \text{alg}\mathcal{L}$ and $P \in \mathcal{L}$, since

$$PT(I-P) = P - (P - PT(I-P)),$$

which is a difference of two idempotents, it follows from Lemma 2.6 that

$$\begin{aligned}
\delta(I)APT(I-P) &= A\delta(I)PT(I-P)\\
&= \delta(APT(I-P))\\
&= \delta(A)PT(I-P) - \lambda(A)PT(I-P).
\end{aligned}$$

By arbitrariness of P and T, we have $A\delta(I)G_1 = \delta(I)AG_1 = (\delta(A) - \lambda(A))G_1$. That is,

$$\delta(A)G_1 = (A\delta(I) + \lambda(A))G_1 = (\delta(I)A + \lambda(A))G_1,$$

whence

$$\begin{aligned}
\delta(AG_1) &= \delta(A)G_1 + \lambda(AG_1) - \lambda(A)G_1\\
&= \delta(I)AG_1 + \lambda(AG_1)\\
&= A\delta(I)G_1 + \lambda(AG_1).
\end{aligned} \tag{2.16}$$

Define $\delta^*(A^*) = \delta(A)^*$ for every $A^* \in \text{alg}\mathcal{L}^\perp$. So

$$\begin{aligned}
(m+n+l)\delta^*((A^*)^2) &= ((m+n+l)\delta(A^2))^*\\
&= (m\delta(A)A + nA\delta(A) + lA\delta(I)A + \lambda_A I)^*\\
&= mA^*\delta^*(A^*) + n\delta^*(A^*)A^* + lA^*\delta^*(I)A^* + \lambda_{A^*},
\end{aligned}$$

where $\lambda_{A^*} = \overline{\lambda_A}$.

With the proof similar to the proof of (2.16), we have

$$G_2\delta(I)A = G_2A\delta(I) = G_2(\delta(A) - \lambda(A)).$$

So by $G_1 \vee G_2 = I$,

$$(I - G_1)\delta(I)A = (I - G_1)A\delta(I) = (I - G_1)(\delta(A) - \lambda(A)),$$

whence

$$\begin{aligned}
\delta((I - G_1)A) &= (1 - G_1)\delta(A) + \lambda((I - G_1)A) - \lambda(A)(I - G_1) \\
&= (1 - G_1)(\delta(A) - \lambda(A)) + \lambda((I - G_1)A) \\
&= (1 - G_1)\delta(I)A + \lambda((I - G_1)A) \\
&= (I - G_1)A\delta(I) + \lambda((I - G_1)A). \qquad (2.17)
\end{aligned}$$

Hence by (2.16) and (2.17),

$$\begin{aligned}
\delta(A) &= \delta(AG_1 + G_1A(I - G_1) + (I - G_1)A) \\
&= A\delta(I)G_1 + \lambda(AG_1) + G_1A(I - G_1)\delta(I) \\
&\quad + (I - G_1)A\delta(I) + \lambda((1 - G_1)A) \\
&= G_1A\delta(I)G_1 + G_1A\delta(I)(I - G_1) + (I - G_1)A\delta(I) \\
&\quad + \lambda(AG_1) + \lambda((1 - G_1)A) + \lambda(G_1A(1 - G_1)) \\
&= A\delta(I) + \lambda(A).
\end{aligned}$$

Similarly, $\delta(A) = \delta(I)A + \lambda(A)$. The remaining part goes along the same line as the proof of Corollary 2.7 and we conclude that δ is a centralizer in this case.
Case 2: Suppose $G_1 \vee G_2 < I$.

Let $G = G_1 \vee G_2$. Since $G \in \mathcal{L} \cap \mathcal{L}^\perp$ and $\mathrm{alg}\mathcal{L}(I - G) \subseteq \mathcal{L}'$, so $(I - G)\mathrm{alg}\mathcal{L}(I - G)$ is a von Neumann algebra. The algebra $\mathrm{alg}\mathcal{L}$ can be written as the direct sum

$$\mathrm{alg}\mathcal{L} = \mathrm{alg}(G\mathcal{L}G) \oplus \mathrm{alg}((I - G)\mathcal{L}(I - G)).$$

By Lemma 2.6 we have that

$$\delta(GAG) = G\delta(A)G \text{ and } \delta((I - G)A(I - G)) = (I - G)\delta(A)(I - G)$$

for every $A \in \mathrm{alg}\mathcal{L}$. Therefore δ can be written as $\delta^{(1)} \oplus \delta^{(2)}$, where $\delta^{(1)}$ is a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathrm{alg}(G\mathcal{L}G)$ into itself and $\delta^{(2)}$ is a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathrm{alg}((I - G)\mathcal{L}(I - G))$ into itself. It is easy to show that $G_1(G\mathcal{L}G) \vee G_2(G\mathcal{L}G) = G$. So it follows from Case 1 that $\delta^{(1)}$ is a centralizer on $\mathrm{alg}(G\mathcal{L}G)$. $(I - G)\mathrm{alg}\mathcal{L}(I - G)$ is a von Neumann algebra and $\delta^{(2)}$ is continuous, so by Corollary 2.7, $\delta^{(2)}$ is a centralizer on $\mathrm{alg}((I - G)\mathcal{L}(I - G))$. Consequently, δ is a centralizer on $\mathrm{alg}\mathcal{L}$. □

3. Centralizers of generalized matrix algebras

Let $\mathcal{A}$ be a unital algebra over a number field $\mathbb{K}$. We call $\mathcal{M}$ a *unital $\mathcal{A}$-bimodule* if $\mathcal{M}$ is an $\mathcal{A}$-bimodule and satisfies $I_{\mathcal{A}}M = MI_{\mathcal{A}} = M$ for every $M \in \mathcal{M}$. We call

$\mathcal{M}$ a *faithful left* $\mathcal{A}$*-module* if for any $A \in \mathcal{A}$, $A\mathcal{M} = 0$ implies $A = 0$. Similarly, we can define a *faithful right* $\mathcal{A}$*-module*.

Throughout this section, we denote the generalized matrix algebra originated from the Morita context $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \phi_{\mathcal{MN}}, \varphi_{\mathcal{NM}})$ by $\mathcal{U} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{bmatrix}$, where $\mathcal{A}, \mathcal{B}$ are two unital algebras over a number field $\mathbb{K}$ and $\mathcal{M}, \mathcal{N}$ are two unital bimodules, and at least one of $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. We use the symbols $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ to denote the unit element in $\mathcal{A}$ and $\mathcal{B}$, respectively. Moreover, we make no difference between $\lambda(A) = \frac{1}{m+n+2l}(\lambda_{A+I} - \lambda_A)I$ and $\frac{1}{m+n+2l}(\lambda_{A+I} - \lambda_A) \in \mathbb{K}$.

Lemma 3.1. *Let δ be a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{U}$ into itself. Then δ is of the form*

$$\delta\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) = \begin{bmatrix} a_{11}(A) + \lambda\left(\begin{bmatrix} 0 & M \\ N & B \end{bmatrix}\right) I_{\mathcal{A}} & c_{12}(M) \\ d_{21}(N) & b_{22}(B) + \lambda\left(\begin{bmatrix} A & M \\ N & 0 \end{bmatrix}\right) I_{\mathcal{B}} \end{bmatrix}$$

for any $A \in \mathcal{A}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$, $B \in \mathcal{B}$, where $a_{11} : \mathcal{A} \to \mathcal{A}$, $c_{12} : \mathcal{M} \to \mathcal{M}$, $d_{21} : \mathcal{N} \to \mathcal{N}$, $b_{22} : \mathcal{B} \to \mathcal{B}$ are all linear mappings satisfying

$$c_{12}(M) = a_{11}(I_{\mathcal{A}})M = Mb_{22}(I_{\mathcal{B}}) \text{ and } d_{21}(N) = Na_{11}(I_{\mathcal{A}}) = b_{22}(I_{\mathcal{B}})N.$$

Proof. Assume that δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{U}$ into itself. Because δ is linear, for any $A \in \mathcal{A}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$, $B \in \mathcal{B}$, we can write

$$\begin{aligned}
&\delta\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) \\
&= \begin{bmatrix} a_{11}(A) + b_{11}(B) + c_{11}(M) + d_{11}(N) & a_{12}(A) + b_{12}(B) + c_{12}(M) + d_{12}(N) \\ a_{21}(A) + b_{21}(B) + c_{21}(M) + d_{21}(N) & a_{22}(A) + b_{22}(B) + c_{22}(M) + d_{22}(N) \end{bmatrix}
\end{aligned}$$

where a_{ij}, b_{ij}, c_{ij}, d_{ij} are linear mappings, $i, j \in \{1, 2\}$.

Let $P = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}$ and for any $A \in \mathcal{A}$, $S = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$. By Lemma 2.6, $\delta(PS) = P\delta(S) + \lambda(PS) - \lambda(S)P$ and $\delta(SP) = \delta(S)P + \lambda(SP) - \lambda(S)P$, so we have

$$\begin{aligned}
&\begin{bmatrix} a_{11}(A) & a_{12}(A) \\ a_{21}(A) & a_{22}(A) \end{bmatrix} \\
&\quad = \delta\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\quad = \delta\left(\begin{bmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\quad = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\delta\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) + \begin{bmatrix} \lambda(PS)I_{\mathcal{A}} & 0 \\ 0 & \lambda(PS)I_{\mathcal{B}} \end{bmatrix} - \begin{bmatrix} \lambda(S)I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad = \begin{bmatrix} a_{11}(A) & a_{12}(A) \\ 0 & \lambda\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) I_{\mathcal{B}} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\begin{bmatrix} a_{11}(A) & a_{12}(A) \\ a_{21}(A) & a_{22}(A) \end{bmatrix} \\
&\qquad = \delta\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\qquad = \delta\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\qquad = \delta\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda(SP)I_{\mathcal{A}} & 0 \\ 0 & \lambda(SP)I_{\mathcal{B}} \end{bmatrix} - \begin{bmatrix} \lambda(S)I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix} \\
&\qquad = \begin{bmatrix} a_{11}(A) & 0 \\ a_{21}(A) & \lambda\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) I_{\mathcal{B}} \end{bmatrix}.
\end{aligned}$$

So we have

$$a_{12}(A) = 0, a_{21}(A) = 0 \text{ and } a_{22}(A) = \lambda\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) I_{\mathcal{B}}\,.$$

Similarly, by considering $S = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}$, we obtain that

$$c_{11}(M) = \lambda\left(\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}\right) I_{\mathcal{A}}, c_{21}(M) = 0 \text{ and } c_{22}(M) = \lambda\left(\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}\right) I_{\mathcal{B}}$$

for every $M \in \mathcal{M}$.

By considering $S = \begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}$ and $P = \begin{bmatrix} I_{\mathcal{A}} & 0 \\ 0 & 0 \end{bmatrix}$, we obtain $d_{11}(N) = \lambda\left(\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}\right) I_{\mathcal{A}}$, $d_{12}(N) = 0$ and $d_{22}(N) = \lambda\left(\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}\right) I_{\mathcal{B}}$ for every $N \in \mathcal{N}$.

By considering $S = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{B}} \end{bmatrix}$, we obtain

$$b_{11}(B) = \lambda\left(\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}\right) I_{\mathcal{A}},\ b_{12}(B) = 0 \text{ and } b_{21}(B) = 0$$

for every $B \in \mathcal{B}$.

For any $A \in \mathcal{A}$, $M_1 \in \mathcal{M}$, $M_2 \in \mathcal{M}$ and $B \in \mathcal{B}$, let $S = \begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix}$ and

$T = \begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix}$. Then by Lemma 2.1 we have

$$
\begin{aligned}
&(m+n+l)\begin{bmatrix} \lambda(ST)I_{\mathcal{A}} & c_{12}(AM_2+M_1B) \\ 0 & \lambda(ST)I_{\mathcal{B}} \end{bmatrix} \\
&\quad = (m+n+l)\delta(ST) = (m+n+l)\delta(ST+TS) \\
&\quad = m\begin{bmatrix} a_{11}(A)+\lambda\left(\begin{bmatrix} 0 & M_1 \\ 0 & 0 \end{bmatrix}\right)I_{\mathcal{A}} & c_{12}(M_1) \\ 0 & \lambda\left(\begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix}\right)I_{\mathcal{B}} \end{bmatrix}\begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix} \\
&\qquad + m\begin{bmatrix} \lambda\left(\begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix}\right)I_{\mathcal{A}} & c_{12}(M_2) \\ 0 & b_{22}(B)+\lambda\left(\begin{bmatrix} 0 & M_2 \\ 0 & 0 \end{bmatrix}\right)I_{\mathcal{B}} \end{bmatrix}\begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix} \\
&\qquad + n\begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \lambda\left(\begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix}\right)I_{\mathcal{A}} & c_{12}(M_2) \\ 0 & \lambda\left(\begin{bmatrix} 0 & M_2 \\ 0 & 0 \end{bmatrix}\right)I_{\mathcal{B}}+b_{22}(B) \end{bmatrix} \\
&\qquad + n\begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix}\begin{bmatrix} a_{11}(A)+\lambda\left(\begin{bmatrix} 0 & M_1 \\ 0 & 0 \end{bmatrix}\right)I_{\mathcal{A}} & c_{12}(M_1) \\ 0 & \lambda\left(\begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix}\right)I_{\mathcal{B}} \end{bmatrix} \\
&\qquad + l\begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11}(I_{\mathcal{A}}) & 0 \\ 0 & b_{22}(I_{\mathcal{B}}) \end{bmatrix}\begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix} \\
&\qquad + \begin{bmatrix} (\lambda_{S+T}-\lambda_S-\lambda_T)I_{\mathcal{A}} & 0 \\ 0 & (\lambda_{S+T}-\lambda_S-\lambda_T)I_{\mathcal{B}} \end{bmatrix}.
\end{aligned}
$$

The above matrix equation implies

$$
\begin{aligned}
&(m+n+l)c_{12}(AM_2+M_1B) \\
&\quad = ma_{11}(A)M_2 + m\lambda\left(\begin{bmatrix} 0 & M_1 \\ 0 & 0 \end{bmatrix}\right)M_2 + mc_{12}(M_1)B + nM_1b_{22}(B) \\
&\qquad + m\lambda\left(\begin{bmatrix} 0 & M_2 \\ 0 & B \end{bmatrix}\right)M_1 + nAc_{12}(M_2) + n\lambda\left(\begin{bmatrix} 0 & M_2 \\ 0 & 0 \end{bmatrix}\right)M_1 \\
&\qquad + n\lambda\left(\begin{bmatrix} A & M_1 \\ 0 & 0 \end{bmatrix}\right)M_2 + lAa_{11}(I_{\mathcal{A}})M_2 + lM_1b_{22}(I_{\mathcal{B}})B. \qquad (3.1)
\end{aligned}
$$

Taking $B=0$, $A=I_{\mathcal{A}}$ and $M_1=0$ in (3.1), we have $c_{12}(M)=a_{11}(I_{\mathcal{A}})M$ for every $M\in\mathcal{M}$. Taking $A=0$, $B=I_{\mathcal{B}}$ and $M_2=0$ in (3.1), we have $c_{12}(M)=Mb_{22}(I_{\mathcal{B}})$ for every $M\in\mathcal{M}$.

Symmetrically, $d_{21}(N)=b_{22}(I_{\mathcal{B}})N=Na_{11}(I_{\mathcal{A}})$ for every $N\in\mathcal{N}$. □

Theorem 3.2. *Let δ be a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{U}$ into itself. Suppose that one of the following conditions holds:*
(1) *$\mathcal{M}$ is a faithful left $\mathcal{A}$-module and a faithful right $\mathcal{B}$-module;*

(2) *$\mathcal{M}$ is a faithful left $\mathcal{A}$-module and $\mathcal{N}$ is a faithful left $\mathcal{B}$-module;*
(3) *$\mathcal{N}$ is a faithful right $\mathcal{A}$-module and a faithful left $\mathcal{B}$-module;*
(4) *$\mathcal{N}$ is a faithful right $\mathcal{A}$-module and $\mathcal{M}$ is a faithful right $\mathcal{B}$-module.*
Then δ is a centralizer.

Proof. Let δ be a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{U}$ into itself. By Lemma 3.1, we have

$$c_{12}(M) = a_{11}(I_{\mathcal{A}})M = Mb_{22}(I_{\mathcal{B}}) \tag{3.2}$$

for every $M \in \mathcal{M}$, and

$$d_{21}(N) = Na_{11}(I_{\mathcal{A}}) = b_{22}(I_{\mathcal{B}})N \tag{3.3}$$

for every $N \in \mathcal{N}$.

We assume that (1) holds. The proofs for the other cases are analogous.

For any $A \in \mathcal{A}$ and $M \in \mathcal{M}$, $a_{11}(I_{\mathcal{A}})AM = AMb_{22}(I_{\mathcal{B}}) = Aa_{11}(I_{\mathcal{A}})M$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we have

$$a_{11}(I_{\mathcal{A}})A = Aa_{11}(I_{\mathcal{A}}),$$

whence

$$a_{11}(A) = Aa_{11}(I_{\mathcal{A}}) + \lambda\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) I_{\mathcal{A}} = a_{11}(I_{\mathcal{A}})A + \lambda\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) I_{\mathcal{A}}. \tag{3.4}$$

For any $B \in \mathcal{B}$ and $M \in \mathcal{M}$, $MBb_{22}(I_{\mathcal{B}}) = a_{11}(I_{\mathcal{A}})MB = Mb_{22}(I_{\mathcal{B}})B$. Since $\mathcal{M}$ is a faithful right $\mathcal{B}$-module, we have

$$b_{22}(B) = b_{22}(I_{\mathcal{B}})B + \lambda\left(\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}\right) I_{\mathcal{B}} = Bb_{22}(I_{\mathcal{B}}) + \lambda\left(\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}\right) I_{\mathcal{B}}. \tag{3.5}$$

For any $A \in \mathcal{A}$, $M \in \mathcal{M}$, $N \in \mathcal{N}$ and $B \in \mathcal{B}$,

$$\delta\left(\begin{bmatrix} A & M \\ N & B \end{bmatrix}\right) = \begin{bmatrix} a_{11}(A) + \lambda\left(\begin{bmatrix} 0 & M \\ N & B \end{bmatrix}\right) I_{\mathcal{A}} & c_{12}(M) \\ d_{21}(N) & b_{22}(B) + \lambda\left(\begin{bmatrix} A & M \\ N & 0 \end{bmatrix}\right) I_{\mathcal{B}} \end{bmatrix},$$

$$\delta(I)\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \begin{bmatrix} a_{11}(I_{\mathcal{A}})A & a_{11}(I_{\mathcal{A}})M \\ b_{22}(I_{\mathcal{B}})N & b_{22}(I_{\mathcal{B}})B \end{bmatrix}$$

and

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix}\delta(I) = \begin{bmatrix} Aa_{11}(I_{\mathcal{A}}) & Mb_{22}(I_{\mathcal{B}}) \\ Na_{11}(I_{\mathcal{A}}) & Bb_{22}(I_{\mathcal{B}}) \end{bmatrix}.$$

So by (3.2)–(3.5), we have for every $S \in \mathcal{U}$,

$$\delta(S) = \delta(I)S + \lambda(S) = S\delta(I) + \lambda(S).$$

The remaining part goes along the same line as the proof of Corollary 2.7 and this completes the proof. □

Note that a unital prime ring $\mathcal{A}$ with a non-trivial idempotent P can be written as the matrix form $\begin{bmatrix} P\mathcal{A}P & P\mathcal{A}(I-P) \\ (I-P)\mathcal{A}P & (I-P)\mathcal{A}(I-P) \end{bmatrix}$. Moreover, for any $A \in \mathcal{A}$, $PAP\mathcal{A}(I-P) = 0$ implies $PAP = 0$ and $P\mathcal{A}(I-P)A(I-P) = 0$ implies $(I-P)A(I-P) = 0$.

Corollary 3.3. *Let $\mathcal{A}$ be a unital prime ring with a non-trivial idempotent P. If δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{A}$ into itself, then δ is a centralizer.*

As von Neumann algebras have rich idempotent elements and factor von Neumann algebras are prime, the following corollary is obvious.

Corollary 3.4. *Let $\mathcal{A}$ be a factor von Neumann algebra. If δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{A}$ into itself, then δ is a centralizer.*

Obviously, when $\mathcal{N} = 0$, $\mathcal{U}$ degenerates to an upper triangular algebra. Thus we have the following corollary.

Corollary 3.5. *Let $\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be an upper triangular algebra such that $\mathcal{M}$ is a faithful ($\mathcal{A}$, $\mathcal{B}$)-bimodule. If δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{A}$ into itself, then δ is a centralizer.*

Let $\mathcal{N}$ be a nest on a Hilbert space H and alg$\mathcal{N}$ be the associated algebra. If $\mathcal{N}$ is trivial, then alg$\mathcal{N}$ is $B(H)$. If $\mathcal{N}$ is nontrivial, take a nontrivial projection $P \in \mathcal{N}$. Let $\mathcal{A} = P\text{alg}\mathcal{N}P$, $\mathcal{M} = P\text{alg}\mathcal{N}(I-P)$ and $\mathcal{B} = (I-P)\text{alg}\mathcal{N}(I-P)$. Then $\mathcal{M}$ is a faithful ($\mathcal{A}$, $\mathcal{B}$)-bimodule, and alg$\mathcal{N}$=Tri($\mathcal{A}$, $\mathcal{M}$, $\mathcal{B}$) is an upper triangular algebra. Thus as an application of Corollaries 3.4 and 3.5, we have the following corollary.

Corollary 3.6. *Let $\mathcal{N}$ be a nest on a Hilbert space H and* alg$\mathcal{N}$ *be the associated algebra. If δ is a generalized $(m,\ n,\ l)$-Jordan centralizer from* alg$\mathcal{N}$ *into itself, then δ is a centralizer.*

In the following, we study $(m,\ n,\ l)$-Jordan centralizers on AF C^*-algebras. A unital C^*-algebra $\mathcal{B}$ is called *approximately finite* (AF) if $\mathcal{B}$ contains an increasing chain $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$ of finite-dimensional C^*-subalgebra, all containing the unit I of $\mathcal{B}$, such that $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ is dense in $\mathcal{B}$. For more details and related terms, we refer the readers to [5, 11].

Lemma 3.7. *Let $\mathcal{M}_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices, $\mathcal{A}$ be a CSL subalgebra of $\mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_k}(\mathbb{C})$, and $\mathcal{B}$ be an algebra such that $\mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_k}(\mathbb{C}) \subseteq \mathcal{B}$ as an embedding. If δ is an $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{A}$ into $\mathcal{B}$, then δ is a centralizer.*

Proof. Let $\mathcal{A}$ be the linear span of its matrix units $\{E_{ij}\}$, and since δ is linear, we only need to show that for any i, j,

$$\delta(E_{ij}) = E_{ij}\delta(I) = \delta(I)E_{ij}. \tag{3.6}$$

If $i = j$, by Lemma 2.4, (3.6) is clear.

Next, we will prove (3.6) for $i \neq j$. By Lemma 2.1 and Remark 2.2, we have

$$\begin{aligned}(m+n+l)\delta(E_{ij}) &= (m+n+l)\delta(E_{ii}E_{ij}+E_{ij}E_{ii}) \\ &= m\delta(E_{ii})E_{ij} + nE_{ii}\delta(I)E_{ij} + lE_{ii}\delta(I)E_{ij} \\ &= (m+n+l)\delta(E_{ii})E_{ij},\end{aligned}$$

Hence $\delta(E_{ij}) = \delta(E_{ii})E_{ij}$ for any i, j.

Similarly, we have $\delta(E_{ij}) = E_{ij}\delta(E_{jj})$ for any i, j.

Hence for any i, j,

$$E_{ij}\delta(I) = E_{ij}\sum_{k=1}^{n}\delta(E_{kk}) = E_{ij}\sum_{k=1}^{n}E_{kk}\delta(E_{kk}) = E_{ij}\delta(E_{jj}) = \delta(E_{ij}).$$

Similarly, we have for any i, j, $\delta(I)E_{ij} = \delta(E_{ij})$ and the proof is complete. □

Theorem 3.8. *Let $\mathcal{A}$ be a canonical subalgebra of an AF C^*-algebra $\mathcal{B}$. If δ is a bounded $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{A}$ into $\mathcal{B}$, then δ is a centralizer.*

Proof. Suppose δ is a bounded $(m,\ n,\ l)$-Jordan centralizer from $\mathcal{A}$ into $\mathcal{B}$. Since $\mathcal{A}_n$ is a CSL algebra, $\delta|_{\mathcal{A}_n}$ is a centralizer by Lemma 3.7; that is, for any S in $\mathcal{A}_n$,

$$\delta(S) = \delta(I)S = S\delta(I).$$

Since δ is norm continuous and $\cup_{i=1}^{\infty}A_n$ is dense in A, it follows that δ is a centralizer.

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