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COMPARISON OF ONE-SIDED MODULES

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Abstract. Given an inclusion $N \subset M$ of II_1 factors with trivial relative commutant, this paper lists all operators $x, y \in M$ such that the left N-module generated by x is equal to or contained in the right N-module generated by y.

1. Introduction

For an inclusion of finite von Neumann algebras $B \subset M$, the fundamental set of B is defined to be

$$N^f(B) = \{x \in M : Bx = xB\}.$$

This notion was defined in [2], and, it was shown there that $N^f(B)$ is crucial to the study of normalizers of masas (maximal abelian self-adjoint algebra) in II_1 factors, when B is a masa. Roughly speaking, some twisting or intertwining is involved whenever a left B-module is also a right-B module, and, it is often of interest to know the operators that do the twisting. In this paper, we calculate $N^f(\cdot)$ for inclusion of II_1 factors with trivial relative commutant. These results are generalizations of Dye-type results on masas.

Let M be a seperable II_1 factor equipped with its unique normal tracial state τ. The trace τ induces a Hilbert norm on M which is denoted by $\|\cdot\|_2$. We always assume that M acts on $L^2(M,\tau)$ (the Hilbert space completion of $(M, \|\cdot\|_2)$) via left multiplication. If $B \subset M$ is a von Neumann subalgebra, then the normalizer of B is the group

$$N(B) = \{u \in \mathcal{U}(M) : uBu^* = B\},$$

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where $\mathcal{U}(\cdot)$ denotes the unitary group of the associated algebra. Also let $\mathbb{E}_B$ denote the unique normal trace preserving conditional expectation onto B. Recall that for $x \in M$, if

$$K_B(x) = \overline{conv}^w \{uxu^* : u \in \mathcal{U}(B)\},$$

then $K_B(x)$ contains a unique element $\phi(x)$ of minimal $\|\cdot\|_2$; the closure in the above is with respect to the weak topology of $L^2(M, \tau)$. Moreover, $\phi(x) = \mathbb{E}_{B' \cap M}(x)$ (cf. [3]).

This writing is organized as follows. In §2 we describe $N^f(\cdot)$ of a subfactor with trivial relative commutant. §3 deals with intertwiners and inclusions or equality of modules with different cyclic vectors. In §4 we present a similar result related to masas to compare the changes with results of the previous sections.

2. Fundamental sets

Theorem 2.1. *Let $N \subset M$ be inclusion of II_1 factors. Let $0 \neq x \in N^f(N)$. Then $x \in N$, if and only if x is invertible in N.*

Proof. There are two cases to consider.
Case 1: Let $0 \neq x \in N^f(N) \cap N$ and $x \geq 0$. Choose $\epsilon > 0$ so small that $\epsilon \ll \tau(x)$. By Dixmier's approximation theorem choose unitaries $u_1, \cdots, u_k \in N$ and scalars $0 \leq \lambda_1, \cdots, \lambda_k \leq 1$, $\sum_{i=1}^k \lambda_i = 1$ such that

$$\left\| \sum_{i=1}^k \lambda_i u_i x u_i^* - \tau(x)1 \right\| < \epsilon.$$

Therefore, $\sum_{i=1}^k \lambda_i u_i x u_i^*$ is invertible in N. Now choose $n_1, \cdots, n_k \in N$ such that $u_i x = x n_i$ for all $1 \leq i \leq k$. Then

$$\sum_{i=1}^k \lambda_i u_i x u_i^* = \sum_{i=1}^k \lambda_i x n_i u_i^* = x \sum_{i=1}^k \lambda_i n_i u_i^*.$$

Therefore x has a right inverse in N. So x is invertible in N.
Case 2: Let $0 \neq x \in N^f(N) \cap N$. We also have $x^*N = Nx^*$. Then $Nxx^* = xNx^* = xx^*N$. So by Case 1, xx^* is invertible N. It follows that x is invertible in N. □

Theorem 2.2. *Let $N \subset M$ be inclusion of II_1 factors such that $N' \cap M = \mathbb{C}1$. Then $0 \neq x \in N^f(N)$ if and only if $x = wv$, where w is invertible in N and $v \in N(N)$.*

Proof. There are two cases to consider here as well.
Case 1: Let $0 \neq x \in N^f(N)$ and $x \geq 0$. Then $N\mathbb{E}_N(x) = \mathbb{E}_N(x)N$. So by Theorem 2.1, $\mathbb{E}_N(x)$ is invertible in N.

If $n \in N$ be such that $nx = 0$, then $n\mathbb{E}_N(x) = 0$, so $n = 0$. Similarly, $xn = 0$ for some $n \in N$ imply $n = 0$ as well. Thus there is a well defined map

$$\psi : N \mapsto N \text{ such that } nx = x\psi(n)$$

for all $n \in N$. Clearly, ψ is linear. Taking conditional expectations it follows that ψ is bounded.

Let $(N)_\psi$ denote the ball of radius $\|\psi\|$ in N. For each unitary $u \in N$, we have

$$uxu^* = x\psi(u)u^* \in x(N)_\psi. \tag{2.1}$$

As $N' \cap M = \mathbb{C}1$, so there is a sequence of convex combinations

$$y_n = \sum_{i=1}^{k_n} \lambda_i^{(n)} u_{i,n} x u_{i,n}^*$$

with $0 \leq \lambda_i^{(n)} \leq 1$ and $\sum_{i=1}^{k_n} \lambda_i^{(n)} = 1$ and $u_{i,n} \in \mathcal{U}(N)$ such that $y_n \to \tau(x)1$ in $\|\cdot\|_2$, and, hence $y_n \to \tau(x)1$ in $w.o.t$ as $n \to \infty$. On the other hand from Eq. (2.1),

$$y_n = \sum_{i=1}^{k_n} \lambda_i^{(n)} u_{i,n} x u_{i,n}^* = x \sum_{i=1}^{k_n} \lambda_i^{(n)} \psi(u_{i,n}) u_{i,n}^* \in x(N)_\psi.$$

By using Mazur's theorem on convex sets in Banach spaces and w^* compactness of the ball of a von Neumann algebra, there is a subsequence

$$\sum_{i=1}^{k_{n_l}} \lambda_i^{(n_l)} \psi(u_{i,n_l}) u_{i,n_l}^*$$

converging in $w.o.t$ to a $z \in (N)_\psi$. Thus $xz = \tau(x)1$. So x has a right inverse and hence is invertible in N. So $x \in N$.

Case 2: Let $0 \neq x \in N^f(N)$. Then $xx^*N = xNx^* = Nxx^*$. Apply the previous case, to conclude that xx^* is invertible in N. So x is invertible in M.

Let $x = (xx^*)^{\frac{1}{2}} v$ be the polar decomposition of x. Then v is a unitary. Replacing the roles of x by x^*, we also have $x^*x \in N$ and is invertible in N. Then $v^*xx^*v = x^*x \in N$. Now

$$Nx = N(xx^*)^{\frac{1}{2}} v = Nv.$$

One the other hand

$$xN = (xx^*)^{\frac{1}{2}} vN = vv^*(xx^*)^{\frac{1}{2}} vN = v(x^*x)^{\frac{1}{2}} N = vN.$$

So $Nv = vN$ and hence v is a normalizing unitary. Thus x has the form $x = wv$ where w is invertible in N and $v \in N(N)$. Since the other direction is obvious we are done. □

Remark 2.3. Note that in view of Theorem 2.2, $N^f(N) \setminus \{0\}$ is a group and clearly the invertibles in N is a normal subgroup of $N^f(N)$. While for masas $N^f(\cdot) \setminus \{0\}$ contains the groupoid normalizer [2].

3. Different cyclic vectors

Definition 3.1. Given an inclusion $B \subset M$ of von Neumann algebras satisfying $B' \cap M \subseteq B$, define the one-sided intertwiners of B to be the collection

$$\mathcal{GN}_M^{(1)}(B) = \{v \text{ a partial isometry in } M : vBv^* \subseteq B, v^*v \in B\}.$$

The superscript $^{(1)}$ is used in the above definition to indicate that even though $vBv^* \subseteq B$, there is no guarantee of the inclusion $v^*Bv \subseteq B$. For more on one-sided intertwiners check [1].

Theorem 3.2. *Let $N \subset M$ be inclusion of* II_1 *factors with trivial relative commutant. Let $x, y \in M$ be non zero operators such that $Nx \subseteq yN$. Let $y = v\,|y|$ denote the polar decomposition of y.*
(i) If $x = y$, then $|x|$ is invertible in N and $v^* \in \mathcal{GN}_M^{(1)}(N) \cap \mathcal{U}(M)$.
(ii) Else $xy^{-1} \in N$ and $|y| \in N$.
In both cases $Nx \subseteq vN$.

Proof. Let $\mathcal{I} = \{n \in N : nx = 0\}$ and $\mathcal{J} = \{n \in N : yn = 0\}$. Then $\mathcal{I}$ and $\mathcal{J}$ are respectively weakly closed left and right ideals in N. Thus there exist projections $p, q \in N$ such that $\mathcal{I} = N(1-p)$ and $\mathcal{J} = (1-q)N$. Thus $x = px$ and $y = yq$.

If $n_1, n_2 \in N$ be such that $yn_1 = yn_2$, then $yq(n_1 - n_2) = 0$. Thus $q(n_1 - n_2) \in \mathcal{J}$. Hence $qn_1 = qn_2$. Thus there is a well defined map $\psi : Np \mapsto qN$ such that

$$npx = yq\psi(np), n \in N.$$

Clearly, ψ is linear and by closed graph theorem ψ is bounded. It is also easy to see that ψ is injective. Thus

$$\begin{aligned} x^*n^* &= x^*pn^* = \psi(np)^*qy^* = \psi(np)^*y^*, \text{ and} \\ nxx^* &= npxx^* = yq\psi(np)x^* = y\psi(np)x^*, n \in N. \end{aligned} \tag{3.1}$$

So

$$\begin{aligned} uxx^*u^* &= y\psi(up)x^*u^* \\ &= y\psi(up)\psi(up)^*y^*, \ u \in \mathcal{U}(N) \text{ from Eq. (3.1)}. \end{aligned}$$

Averaging xx^* over $u \in \mathcal{U}(N)$ and using w^* compactness of the ball of radius $\|\psi\|$ in N, it follows that there is a $z \in N$ such that

$$yzy^* = \tau(xx^*)1.$$

Thus y is invertible and $(y^*y)^{-1} \in N$. Thus $|y| \in N$ and $q = 1$. This settles the case when $x = y$. The rest is obvious. ☐

Since the symmetry of inclusion is missing in Theorem 3.2, not much can be said. However, when the symmetry is present stronger statements can be made. We need an intermediate lemma.

Lemma 3.3. *Let $N \subset M$ be inclusion of* II_1 *factors with trivial relative commutant. Let $0 \neq z \in M$ be such that $Nz = z^*N$. Then $z = wv$, where w is a invertible in N and v is a unitary in M such that $vNv = N$.*

Proof. Since $Nz = z^*N$, so $z^* = nz$ for some $n \in N$. Thus $ker z \subseteq ker z^*$. Let p and q denote the projections $\chi_{(0,\infty)}(|z|)$ and $\chi_{(0,\infty)}(|z^*|)$ respectively. Hence $q \leq p$. If $q < p$, then $\tau(q) < \tau(p)$. But $p \sim q$ in M. So $q = p$. We claim that $p = 1$. Indeed, $z = zp = pz$. Thus

$$pNpz = pNz = pz^*N = z^*N = Nz = Npz.$$

Hence, $(1-p)Npz = 0$ and so $(1-p)Npzz^* = 0$. By functional calculus $(1-p)Np = 0$. Hence $pnp = np$ for each $n \in N$, which forces $p \in N' \cap M$. It follows that $p = 1$ and z is invertible in M.

Consequently, there is a bounded linear surjective map $\psi : N \mapsto N$ such that $\psi(n) = z^* n z^{-1}$, $n \in N$.

There is a sequence $z_n = \sum_{i=1}^{k_n} \lambda_i^{(n)} u_{i,n} zz^* u_{i,n}^*$ with $0 \leq \lambda_i^{(n)} \leq 1$ for $1 \leq i \leq k_n$, $\sum_{i=1}^{k_n} \lambda_i^{(n)} = 1$ for all n, $u_{i,n} \in \mathcal{U}(N)$ such that $z_n \to \tau(zz^*)1$ in $\|\cdot\|_2$. Write $z^* u_{i,n}^* = v_{i,n} z$ with $v_{i,n} = \psi(u_{i,n}^*) \in N$. Thus

$$z_n = \sum_{i=1}^{k_n} \lambda_i^{(n)} u_{i,n} z v_{i,n} z = z^* \left(\sum_{i=1}^{k_n} \lambda_i^{(n)} v_{i,n}^* v_{i,n} \right) z.$$

But $y_n = \sum_{i=1}^{k_n} \lambda_i^{(n)} v_{i,n}^* v_{i,n} \in (N)_\psi$ (the ball of radius $\|\psi\|$ in N); thus there is a subsequence y_{n_k} converging weakly to some $x \in N$. Consequently, $z^* x z = \tau(zz^*)1$ i.e., $x = \tau(zz^*)(z^*)^{-1} z^{-1} = \tau(zz^*)(zz^*)^{-1} \in N$. Thus $zz^* \in N$ is invertible.

Let $z = (zz^*)^{\frac{1}{2}} v$ denote the polar decomposition of z. Then v is a unitary in M. Now $Nz = N(zz^*)^{\frac{1}{2}} v = Nv$. Again $z^* N = v^*(zz^*)^{\frac{1}{2}} N = v^* N$. Thus $Nv = v^* N$. $\square$

Theorem 3.4. *Let $N \subset M$ be inclusion of II_1 factors with trivial relative commutant. Let $x, y \in M$ be non zero operators such that $Nx = yN$. Let $x = (xx^*)^{\frac{1}{2}} v$ and $y = w(y^*y)^{\frac{1}{2}}$ denote the polar decompositions of x, y, respectively. Then $(xx^*)^{\frac{1}{2}}, (y^*y)^{\frac{1}{2}} \in N^f(N) \cap N$. Moreover, $Nv = wN$ and $v, w \in N(N)$.*

Proof. From Theorem 3.2 it follows that $|y|$ is invertible in N. A similar argument using averaging technique shows that $|x|$ is also invertible in N.

We have $y = mx$ for some $m \in N$. Thus $Nx = mxN$. So

$$Nxx^*m^* = mxNx^*m^* = mxx^*N. \tag{3.2}$$

Write $z = xx^*m^*$. Then $Nz = z^*N$. From Lemma 3.3, $z = w_0 v_0$, where w_0 is a invertible in N and v_0 is a unitary in M such that $v_0 N v_0 = N$. Now $w_0 v_0 = xx^*m^*$ implies that xx^*, m are invertible. From Eq. (3.2) it follows that $Nxx^* = xx^*N$. From Theorem 2.2 it follows that xx^* is invertible in N. (A similar argument shows that yy^* is also invertible in N.)

Note that v, w are unitaries. We have

$$N(xx^*)^{\frac{1}{2}} v = Nv \text{ and } w(y^*y)^{\frac{1}{2}} N = wN.$$

Thus $Nv = wN$. So $w^*Nv = N$, in particular $w^*v \in \mathcal{U}(N)$. Let $N \ni u = w^*v$. Note that $v^*Nv = v^*wN = u^*N = N$. Thus $v \in N(N)$. Finally, check that $w \in N(N)$ as well. $\square$

4. Commutative Case

When the subfactor in the previous sections is replaced by a masa the conclusions change, although the techniques to arrive at the conclusion mostly remain the same.

Recall that a partial isometry $v \in M$ is said to be a groupoid normalizer of a diffuse (one that has no minimal projections) abelian algebra $A \subset M$ if $v^*v, vv^* \in A$ and $vAv^* = Avv^* = vv^*A$. Let $\mathcal{GN}(A)$ denote the groupoid normalizers of A. A result similar to Theorem 4.1 was proved in [2]. The proof of Theorem 4.1 has a parallel measure theoretic explanation using the language of discrete bimodules and disintegration of measures.

Theorem 4.1. *Let $A \subset M$ be a masa. Let $x \in A$ be such that $Ax \subset xA$ and $\|x\| = 1$. Let $x = |x^*|\, v$ be the polar decomposition of x. Then $Av \subset vA$ and $|x^*| \in A$. In addition, if x is self-adjoint or $Ax^* \subset x^*A$, then $v \in \mathcal{GN}(A)$ and $|x| \in A$.*

Proof. Let $\mathcal{I} = \{a \in A : ax = 0\}$ and $\mathcal{J} = \{a \in A : xa = 0\}$. Then $\mathcal{I}, \mathcal{J}$ are weakly closed ideals in A and so has the form $A(1-p)$ and $A(1-q)$ for some projections $p, q \in A$, respectively. Thus $x = px = xq$.

If $xa_1q = xa_2q$ for some $a_1, a_2 \in A$, then $x(a_1 - a_2)q = 0$. So $(a_1 - a_2)q \in \mathcal{J}$. Hence, $(a_1 - a_2)q = (a_1 - a_2)q(1-q) = 0$. Therefore, $a_1q = a_2q$. Consequently, there is a well defined linear map $\psi : Ap \mapsto Aq$ such that

$$x\psi(ap) = apx = ax, \text{ for all } a \in A. \tag{4.1}$$

So $x^*a^* = \psi(ap)^*x^*$, $a \in A$. Thus

$$uxx^*u^* = x\psi(up)\psi(up)^*x^*, \text{ for all } u \in \mathcal{U}(A).$$

Therefore averaging over $\mathcal{U}(A)$ and noting that $A' \cap M = A$, one finds a $0 \leq a_0 \in Aq$ such that $a_0 \neq 0$ and $\mathbb{E}_A(xx^*) = xa_0x^*$. Thus

$$\tau(x^*x) = \tau(xx^*) = \tau(\mathbb{E}_A(xx^*)) = \tau(xa_0x^*) = \tau(x^*xa_0).$$

So

$$\left\|(1-a_0)^{\frac{1}{2}}\,|x|\right\|_2 = \tau(|x|\,(1-a_0)\,|x|) = \tau(x^*x(1-a_0)) = 0.$$

Hence $x^*x = a_0x^*x = x^*xa_0$ (as $x^*x \geq 0$). Therefore, $x^*x(1-a_0) = 0$. Equivalently, $x(1-a_0) = 0$. Thus $1 - a_0 \in \mathcal{J}$. So

$$1 - a_0 = (1-a_0)(1-q).$$

Thus $q = a_0q = a_0$. Consequently,

$$A \ni \mathbb{E}_A(xx^*) = xa_0x^* = xqx^* = xx^*.$$

We have $x = (xx^*)^{\frac{1}{2}}v$. Let $v^*v = r$ and $vv^* = s$. So $x^* = v^*(xx^*)^{\frac{1}{2}}$. Use functional calculus to see that $s = vv^* \in A$. Since weakly closed ideals are uniquely determined by projections so $s = p$. Then from Eq. (4.1) we have

$$(xx^*)^{\frac{1}{2}}v\psi(ap) = ap(xx^*)^{\frac{1}{2}}v = (xx^*)^{\frac{1}{2}}apv.$$

Consider

$$\mathcal{J}_0 = \{b \in A : bv\psi(ap) = bapv \ \forall a \in A\}.$$

Then $\mathcal{J}_0$ is a weakly closed ideal in A and contains $(xx^*)^{\frac{1}{2}}$. Thus $p \in \mathcal{J}_0$. So

$$v\psi(ap) = pv\psi(ap) = apv = av.$$

Thus $Av \subset vA$ and $v^*av = \psi(ap)r$. The final statement follows easily by replacing the role of x by x^* in the above. □

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