



QUADRUPLE FIXED POINT THEOREMS FOR NONLINEAR CONTRACTIONS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. In this paper we obtain existence and uniqueness results for quadruple fixed points of operators $F : X^4 \rightarrow X$. We also give some examples to support our results.

1. INTRODUCTION AND PRELIMINARIES

The basic topological properties of ordered sets were discussed by Wolk [17] and Monjardet [15]. The existence of fixed points in partially ordered metric spaces was considered by Ran and Reurings [16]. Recently, many papers have been reported on partially ordered metric spaces (see e.g. [16, 1, 5, 13, 4, 2, 3]).

The notion of coupled fixed point appears to have been introduced by Guo and Lakshmikantham [8] in connexion to monotone operators. Recently, this concept has been reconsidered by Bhaskar and Lakshmikantham [5] in connexion to mixed monotone operators that satisfy a certain contractive type condition. More specifically, they proved the existence, as well as the existence and uniqueness of a coupled fixed point of an operator $F : X \times X \rightarrow X$ on a complete metric space (X, d) where X has an partial order. Later, several authors have devoted their efforts to the study of coupled fixed points or coupled coincidence points [14, 11, 10, 7, 6]. For example, Lakshmikantham and Ćirić in [13] who extended the results in [5] by considering g -monotone operators.

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Very recently, Berinde and Borcut [4] by continuing this trend have introduced the concept of tripled fixed point and proved some related theorems. In this manuscript, in a natural fashion, the quadruple fixed point is considered and by using the mixed g -monotone mapping, several existence, as well existence and uniqueness of quadruple fixed points are obtained. First we recall some basic definitions and results from which quadruple fixed point is inspired. The triple (X, d, \leq) is called partially ordered metric spaces (POMS) if (X, \leq) is a partially ordered set and (X, d) is a metric space. Further, if (X, d) is a complete metric space, the triple (X, d, \leq) is called partially ordered complete metric spaces (POCMS). Throughout the manuscript, we assume that $X \neq \emptyset$ and $X^k = \underbrace{X \times X \times \cdots \times X}_{k\text{-times}}$.

Then the mapping $\rho_k : X^k \times X^k \rightarrow [0, \infty)$ such that

$$\rho_k(\mathbf{x}, \mathbf{y}) := d(x_1, y_1) + d(x_2, y_2) + \cdots + d(x_k, y_k),$$

where $\mathbf{x} = (x_1, x_2, \dots, x_k), \mathbf{y} = (y_1, y_2, \dots, y_k) \in X^k$, is a metric on X^k .

An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if

$$F(x, y) = x \text{ and } F(y, x) = y. \tag{1.1}$$

Let (X, \leq) be partially ordered set and $F : X \times X \rightarrow X$. The operator F is said to have mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone non-increasing in y , that is, for any $x, y \in X$,

$$x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y), \text{ for } x_1, x_2 \in X, \text{ and} \tag{1.2}$$

$$y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1), \text{ for } y_1, y_2 \in X. \tag{1.3}$$

Throughout this paper, let (X, d, \leq) be a POCMS and consider on the product space $X \times X$ the following order:

$$(u, v) \leq (x, y) \Leftrightarrow u \leq x, y \leq v; \text{ for all } (x, y), (u, v) \in X \times X. \tag{1.4}$$

In [5] Bhaskar and Lakshmikantham proved the existence of coupled fixed points for an operator $F : X \times X \rightarrow X$ having the mixed monotone property on (X, d, \leq) by supposing that there exists a $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \text{ for all } u \leq x, y \leq v. \tag{1.5}$$

Inspired by the definition of mixed monotone property, the concept of a g -mixed monotone mapping introduced by Lakshmikantham and Ćirić [13] as follows:

On a partially ordered set (X, \leq) an operator $F : X^2 \rightarrow X$ is said to have the mixed g -monotone property if $F(x, y)$ is monotone g -non-decreasing in x and is monotone g -non-increasing in y , where $g : X \rightarrow X$ is a given self-mapping. It is clear that the definition of Lakshmikantham and Ćirić [13] reduces to the definition of Bhaskar and Lakshmikantham [5] when g is the identity map. In this context, an element $(x, y) \in X \times X$ is called a *coupled coincidence point* of the operators $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ if

$$F(x, y) = g(x), \quad F(y, x) = g(y). \tag{1.6}$$

Furthermore, (x, y) is called the *common coupled fixed point* of F and g if

$$F(x, y) = g(x) = x, \quad F(y, x) = g(y) = y. \quad (1.7)$$

Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ where $X \neq \emptyset$. The mappings F and g are said to *commute* if

$$g(F(x, y)) = F(g(x), g(y)), \quad \text{for all } x, y \in X. \quad (1.8)$$

Lakshmikantham and Ćirić [13] proved the existence of the common coupled fixed point of the operators sequentially continuous $g : X \rightarrow X$ and $F : X^2 \rightarrow X$, where F has as the mixed g -monotone property, $F(X \times X) \subset g(X)$ and F and g commutes, as an extension of the fixed point results in [5]. They also proved the uniqueness of the coupled common fixed point of F and g under some additional assumptions. Berinde and Borcut [4] introduced the following partial order on the product space $X^3 = X \times X \times X$:

$$(u, v, w) \leq (x, y, z) \text{ if and only if } x \geq u, \quad y \leq v, \quad z \geq w, \quad (1.9)$$

where $(u, v, w), (x, y, z) \in X^3$. Regarding this partial order, they introduced the following definition:

Definition 1.1. (See [4]) Let (X, \leq) be partially ordered set and $F : X^3 \rightarrow X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone non-decreasing in x and z , and it is monotone non-increasing in y , that is, for any $x, y, z \in X$

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X, \quad z_1 \leq z_2 &\Rightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned} \quad (1.10)$$

Theorem 1.2. (See [4]) Let (X, \leq) be partially ordered set and (X, d) be a complete metric space. Let $F : X \times X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Assume that there exist constants $a, b, c \in [0, 1)$ such that $a + b + c < 1$ for which

$$d(F(x, y, z), F(u, v, w)) \leq ad(x, u) + bd(y, v) + cd(z, w) \quad (1.11)$$

for all $x \geq u, y \leq v, z \geq w$. Assume that X has the following properties:

- (i) if non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all n ,

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(x_0, y_0, z_0)$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x \text{ and } F(y, x, y) = y \text{ and } F(z, y, x) = z$$

Starting from this rich background, the aim of this paper is to introduce the concept of quadruple fixed point and prove the existence and uniqueness of the common quadruple fixed point of $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ on a POCMS (X, d, \leq) under certain appropriate conditions. Regarding the paper of Haghi–Rezapour–Shahzad [9], we emphasize that our results are real generalization.

2. QUADRUPLE FIXED POINT THEOREMS

Definition 2.1. (See [12]) Let (X, \leq) be partially ordered set and $F : X^4 \rightarrow X$. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z , and it is monotone non-increasing in y and w , that is, for any $x, y, z, w \in X$

$$\begin{aligned} x_1, x_2 \in X, x_1 \leq x_2 &\Rightarrow F(x_1, y, z, w) \leq F(x_2, y, z, w), \\ y_1, y_2 \in X, y_1 \leq y_2 &\Rightarrow F(x, y_1, z, w) \geq F(x, y_2, z, w), \\ z_1, z_2 \in X, z_1 \leq z_2 &\Rightarrow F(x, y, z_1, w) \leq F(x, y, z_2, w), \\ w_1, w_2 \in X, w_1 \leq w_2 &\Rightarrow F(x, y, z, w_1) \geq F(x, y, z, w_2). \end{aligned} \tag{2.1}$$

Definition 2.2. (See [12]) An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $F : X^4 \rightarrow X$ if

$$F(x, y, z, w) = x, F(x, w, z, y) = y, F(z, y, x, w) = z, F(z, w, x, y) = w. \tag{2.2}$$

Definition 2.3. Let (X, \leq) be partially ordered set and $F : X^4 \rightarrow X$. We say that F has the mixed g -monotone property if $F(x, y, z, w)$ is monotone g -non-decreasing in x and z , and it is monotone g -non-increasing in y and w , that is, for any $x, y, z, w \in X$

$$\begin{aligned} x_1, x_2 \in X, g(x_1) \leq g(x_2) &\Rightarrow F(x_1, y, z, w) \leq F(x_2, y, z, w), \\ y_1, y_2 \in X, g(y_1) \leq g(y_2) &\Rightarrow F(x, y_1, z, w) \geq F(x, y_2, z, w), \\ z_1, z_2 \in X, g(z_1) \leq g(z_2) &\Rightarrow F(x, y, z_1, w) \leq F(x, y, z_2, w), \\ w_1, w_2 \in X, g(w_1) \leq g(w_2) &\Rightarrow F(x, y, z, w_1) \geq F(x, y, z, w_2). \end{aligned} \tag{2.3}$$

Definition 2.4. An element $(x, y, z, w) \in X^4$ is called a quadruple coincidence point of $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ if

$$\begin{aligned} F(x, y, z, w) = g(x), \quad F(y, z, w, x) = g(y), \\ F(z, w, x, y) = g(z), \quad F(w, x, y, z) = g(w). \end{aligned} \tag{2.4}$$

Notice that if g is identity mapping, then Definition 2.3 and Definition 2.4 reduce to Definition 2.1 and Definition 2.2, respectively.

Definition 2.5. Let $F : X^4 \rightarrow X$ and $g : X \rightarrow X$. F and g are called commutative if

$$g(F(x, y, z, w)) = F(g(x), g(y), g(z), g(w)), \quad \text{for all } x, y, z, w \in X. \tag{2.5}$$

For a metric space (X, d) , the function $\rho : X^4 \times X^4 \rightarrow [0, \infty)$, given by,

$$\rho((x, y, z, w), (u, v, r, t)) := d(x, u) + d(y, v) + d(z, r) + d(w, t)$$

forms a metric space on X^4 , that is, (X^4, ρ) is a metric induced by (X, d) .

Let Φ denote the all functions $\phi : [0, \infty) \rightarrow [0, \infty)$ which are continuous and satisfy that

- (i) $\phi(t) < t$
- (i) $\lim_{r \rightarrow t^+} \phi(r) < t$ for each $r > 0$.

In order to shorten the statements of the new results in this paper, for the space X and mapping F appearing in the next theorem, consider the following:

Assumption 2.1. Suppose either

- (a) F is continuous, or
- (b) X has the following property:
 - (i) if non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
 - (ii) if non-increasing sequence $y_n \rightarrow y$, then $y_n \geq y$ for all n .

The main result of this paper is the following theorem.

Theorem 2.6. Let (X, d, \leq) be a POCMS. Suppose $F : X^4 \rightarrow X$ and there exists $\phi \in \Phi$ such that F has the mixed g -monotone property and

$$\begin{aligned} & d(F(x, y, z, w), F(u, v, r, t)) \\ & \leq \phi \left(\frac{d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(r)) + d(g(w), g(t))}{4} \right) \end{aligned} \quad (2.6)$$

for all x, u, y, v, z, r, w, t for which $g(x) \leq g(u)$, $g(y) \geq g(v)$, $g(z) \leq g(r)$ and $g(w) \geq g(t)$. Suppose there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} g(x_0) &\leq F(x_0, y_0, z_0, w_0), & g(y_0) &\geq F(x_0, w_0, z_0, y_0), \\ g(z_0) &\leq F(z_0, y_0, x_0, w_0), & g(w_0) &\geq F(z_0, w_0, x_0, y_0). \end{aligned} \quad (2.7)$$

Assume also that Assumption 2.1 holds, that $F(X^4) \subset g(X)$ and g commutes with F . Then there exist $x, y, z, w \in X$ such that

$$\begin{aligned} F(x, y, z, w) &= g(x), & F(x, w, z, y) &= g(y), \\ F(z, y, x, w) &= g(z), & F(z, w, x, y) &= g(w). \end{aligned}$$

that is, F and g have a quadruple coincidence point.

Proof. Let $x_0, y_0, z_0, w_0 \in X$ be such that (2.7). We construct the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ defined, for $n = 1, 2, 3, \dots$, as follows

$$\begin{bmatrix} g(x_n) \\ g(y_n) \\ g(z_n) \\ g(w_n) \end{bmatrix} = \begin{bmatrix} F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \\ F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}) \\ F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) \\ F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \end{bmatrix}. \quad (2.8)$$

We claim that, for all $n \geq 1$,

$$g(x_{n-1}) \leq g(x_n), g(y_{n-1}) \geq g(y_n), g(z_{n-1}) \leq g(z_n), g(w_{n-1}) \geq g(w_n). \quad (2.9)$$

Indeed, we shall use mathematical induction to prove (2.9). Due to (2.7), we have

$$\begin{aligned} g(x_0) &\leq F(x_0, y_0, z_0, w_0) = g(x_1), & g(y_0) &\geq F(x, w_0, z_0, y_0) = g(y_1), \\ g(z_0) &\leq F(z_0, y_0, x_0, w_0) = g(z_1), & g(w_0) &\geq F(z_0, w_0, x_0, y_0) = g(w_1). \end{aligned}$$

Thus, the inequalities in (2.9) hold for $n = 1$. Suppose now that the inequalities in (2.9) hold for some $n \geq 1$. By the mixed g -monotone property of F , together with (2.8) and (2.3) we have

$$\begin{aligned} g(x_n) &= F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \leq F(x_n, y_n, z_n, w_n) = g(x_{n+1}), \\ g(y_n) &= F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}) \geq F(x_n, w_n, z_n, y_n) = g(y_{n+1}), \\ g(z_n) &= F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) \leq F(z_n, y_n, x_n, w_n) = g(z_{n+1}), \\ g(w_n) &= F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \geq F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) = g(w_{n+1}), \end{aligned} \tag{2.10}$$

Thus, (2.9) holds for all $n \geq 1$. Hence, we have

$$\begin{aligned} \cdots g(x_n) &\geq g(x_{n-1}) \geq \cdots \geq g(x_1) \geq g(x_0), \\ \cdots g(y_n) &\leq g(y_{n-1}) \leq \cdots \leq g(y_1) \leq g(y_0), \\ \cdots g(z_n) &\geq g(z_{n-1}) \geq \cdots \geq g(z_1) \geq g(z_0), \\ \cdots g(w_n) &\leq g(w_{n-1}) \leq \cdots \leq g(w_1) \leq g(w_0), \end{aligned} \tag{2.11}$$

For the simplicity, we define δ_n as a product of the following matrices:

$$\delta_n = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_n), g(x_{n+1})) \\ d(g(y_n), g(y_{n+1})) \\ d(g(z_n), g(z_{n+1})) \\ d(g(w_n), g(w_{n+1})) \end{bmatrix}.$$

We shall show that, for all n ,

$$\delta_{n+1} \leq \phi(\delta_n). \tag{2.12}$$

Due to (2.6), (2.8) and (2.11), we have

$$\begin{aligned} &d(g(x_{n+1}), g(x_{n+2})) \\ &= d(F(x_n, y_n, z_n, w_n), F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1})) \\ &\leq \phi \left(d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) \right. \\ &\quad \left. + d(g(z_n), g(z_{n+1})) + d(g(w_n), g(w_{n+1})) \right) / 4 \\ &= \phi(\delta_n) \end{aligned} \tag{2.13}$$

and, similarly,

$$d(g(y_{n+1}), g(y_{n+2})) = d(F(y_n, z_n, w_n, x_n), F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1})) \leq \phi(\delta_n), \tag{2.14}$$

$$d(g(z_{n+1}), g(z_{n+2})) = d(F(z_n, w_n, x_n, y_n), F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1})) \leq \phi(\delta_n), \tag{2.15}$$

$$d(g(w_{n+1}), g(w_{n+2})) = d(F(w_n, x_n, y_n, z_n), F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1})) \leq \phi(\delta_n). \tag{2.16}$$

Now, by summing up (2.13)-(2.16) we get exactly (2.12). Since $\phi(t) < t$ for all $t > 0$, then $\delta_{n+1} \leq \delta_n$ for all n . Hence $\{\delta_n\}$ is a non-increasing sequence. Since it is bounded below, there is some $\delta \geq 0$ such that

$$\lim_{n \rightarrow \infty} \delta_n = \delta + . \quad (2.17)$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Taking the limit as $\delta_n \rightarrow \delta +$ of both sides of (2.12) and having in mind that we suppose $\lim_{t \rightarrow r} \phi(r) < t$ for all $t > 0$, we have

$$\delta = \lim_{n \rightarrow \infty} \delta_{n+1} \leq \lim_{n \rightarrow \infty} \phi(\delta_n) = \lim_{\delta_n \rightarrow \delta+} \phi(\delta_n) < \delta, \quad (2.18)$$

which is a contradiction. Thus, $\delta = 0$, that is,

$$\begin{aligned} & \lim_{n \rightarrow \infty} [d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1})) + d(g(z_n), g(z_{n-1})) + d(g(w_n), g(w_{n-1}))] \\ & = 0. \end{aligned} \quad (2.19)$$

Now, we shall prove that $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}$ and $\{g(w_n)\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}$ and $\{g(w_n)\}$ is not Cauchy. So, there exists an $\varepsilon > 0$ for which we can find subsequences $\{g(x_{n(k)})\}, \{g(x_{m(k)})\}$ of $\{g(x_n)\}$ and $\{g(y_{n(k)})\}, \{g(y_{m(k)})\}$ of $\{g(y_n)\}$ and $\{g(z_{n(k)})\}, \{g(z_{m(k)})\}$ of $\{g(z_n)\}$ and $\{g(w_{n(k)})\}, \{g(w_{m(k)})\}$ of $\{g(w_n)\}$ with $n(k) > m(k) \geq k$ such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)}), g(x_{m(k)})) \\ d(g(y_{n(k)}), g(y_{m(k)})) \\ d(g(z_{n(k)}), g(z_{m(k)})) \\ d(g(w_{n(k)}), g(w_{m(k)})) \end{bmatrix} \geq \varepsilon. \quad (2.20)$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.20) and $n(k) > m(k) \geq k$. Thus,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)-1}), g(x_{m(k)})) \\ d(g(y_{n(k)-1}), g(y_{m(k)})) \\ d(g(z_{n(k)-1}), g(z_{m(k)})) \\ d(g(w_{n(k)-1}), g(w_{m(k)})) \end{bmatrix} < \varepsilon. \quad (2.21)$$

By using triangle inequality and having (2.20), (2.21) in mind

$$\begin{aligned}
 \varepsilon \leq t_k &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)}), g(x_{m(k)})) \\ d(g(y_{n(k)}), g(y_{m(k)})) \\ d(g(z_{n(k)}), g(z_{m(k)})) \\ d(g(w_{n(k)}), g(w_{m(k)})) \end{bmatrix} \\
 &\leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)}), g(x_{n(k)-1})) + d(g(x_{n(k)-1}), g(x_{m(k)})) \\ d(g(y_{n(k)}), g(y_{n(k)-1})) + d(g(y_{n(k)-1}), g(y_{m(k)})) \\ d(g(z_{n(k)}), g(z_{n(k)-1})) + d(g(z_{n(k)-1}), g(z_{m(k)})) \\ d(g(w_{n(k)}), g(w_{n(k)-1})) + d(g(w_{n(k)-1}), g(w_{m(k)})) \end{bmatrix} \quad (2.22) \\
 &\leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)}), g(x_{n(k)-1})) \\ d(g(y_{n(k)}), g(y_{n(k)-1})) \\ d(g(z_{n(k)}), g(z_{n(k)-1})) \\ d(g(w_{n(k)}), g(w_{n(k)-1})) \end{bmatrix} + \varepsilon.
 \end{aligned}$$

Letting $k \rightarrow \infty$ in (2.22) and using (2.19)

$$\lim_{k \rightarrow \infty} t_k = \lim_{k \rightarrow \infty} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)}), g(x_{m(k)})) \\ d(g(y_{n(k)}), g(y_{m(k)})) \\ d(g(z_{n(k)}), g(z_{m(k)})) \\ d(g(w_{n(k)}), g(w_{m(k)})) \end{bmatrix} = \varepsilon + \quad (2.23)$$

Again by triangle inequality,

$$\begin{aligned}
 t_k &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)}), g(x_{m(k)})) \\ d(g(y_{n(k)}), g(y_{m(k)})) \\ d(g(z_{n(k)}), g(z_{m(k)})) \\ d(g(w_{n(k)}), g(w_{m(k)})) \end{bmatrix} \\
 &\leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)}), g(x_{n(k)+1})) + d(g(x_{n(k)+1}), g(x_{m(k)+1})) \\ d(g(y_{n(k)}), g(y_{n(k)+1})) + d(g(y_{n(k)+1}), g(y_{m(k)+1})) \\ d(g(z_{n(k)}), g(z_{n(k)+1})) + d(g(z_{n(k)+1}), g(z_{m(k)+1})) \\ d(g(w_{n(k)}), g(w_{n(k)+1})) + d(g(w_{n(k)+1}), g(w_{m(k)+1})) \end{bmatrix} \quad (2.24) \\
 &\quad \begin{bmatrix} +d(g(x_{m(k)+1}), g(x_{m(k)})) \\ +d(g(y_{m(k)+1}), g(y_{m(k)})) \\ +d(g(z_{m(k)+1}), g(z_{m(k)})) \\ +d(g(w_{m(k)+1}), g(w_{m(k)})) \end{bmatrix} \\
 &\leq \delta_{n(k)+1} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)+1}), g(x_{m(k)+1})) \\ d(g(y_{n(k)+1}), g(y_{m(k)+1})) \\ d(g(z_{n(k)+1}), g(z_{m(k)+1})) \\ d(g(w_{n(k)+1}), g(w_{m(k)+1})) \end{bmatrix} + \delta_{m(k)+1}.
 \end{aligned}$$

Since $n(k) > m(k)$, then

$$\begin{aligned}
 g(x_{n(k)}) &\geq g(x_{m(k)}) \text{ and } g(y_{n(k)}) \leq g(y_{m(k)}), \\
 g(z_{n(k)}) &\geq g(z_{m(k)}) \text{ and } g(w_{n(k)}) \leq g(w_{m(k)}).
 \end{aligned} \quad (2.25)$$

Hence from (2.25), (2.8) and (2.6), we have,

$$\begin{aligned}
& d(g(x_{n(k)+1}), g(x_{m(k)+1})) \\
&= d(F(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)})) \\
&\leq \phi \left(\frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)}), g(x_{m(k)})) \\ d(g(y_{n(k)}), g(y_{m(k)})) \\ d(g(z_{n(k)}), g(z_{m(k)})) \\ d(g(w_{n(k)}), g(w_{m(k)})) \end{bmatrix} \right) \tag{2.26}
\end{aligned}$$

$$\begin{aligned}
& d(g(y_{n(k)+1}), g(y_{m(k)+1})) \\
&= d(F(y_{n(k)}, z_{n(k)}, w_{n(k)}, x_{n(k)}), F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)})) \\
&\leq \phi \left(\frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(y_{n(k)}), g(y_{m(k)})) \\ d(g(z_{n(k)}), g(z_{m(k)})) \\ d(g(w_{n(k)}), g(w_{m(k)})) \\ d(g(x_{n(k)}), g(x_{m(k)})) \end{bmatrix} \right) \tag{2.27}
\end{aligned}$$

$$\begin{aligned}
& d(g(z_{n(k)+1}), g(z_{m(k)+1})) \\
&= d(F(z_{n(k)}, w_{n(k)}, x_{n(k)}, y_{n(k)}), F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)})) \\
&\leq \phi \left(\frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(z_{n(k)}), g(z_{m(k)})) \\ d(g(w_{n(k)}), g(w_{m(k)})) \\ d(g(x_{n(k)}), g(x_{m(k)})) \\ d(g(y_{n(k)}), g(y_{m(k)})) \end{bmatrix} \right) \tag{2.28}
\end{aligned}$$

$$\begin{aligned}
& d(g(w_{n(k)+1}), g(w_{m(k)+1})) \\
&= d(F(w_{n(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)}), F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})) \\
&\leq \phi \left(\frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(w_{n(k)}), g(w_{m(k)})) \\ d(g(x_{n(k)}), g(x_{m(k)})) \\ d(g(y_{n(k)}), g(y_{m(k)})) \\ d(g(z_{n(k)}), g(z_{m(k)})) \end{bmatrix} \right) \tag{2.29}
\end{aligned}$$

Combining (2.24) with (2.26)-(2.29), we obtain that

$$t_k \leq \delta_{n(k)+1} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(x_{n(k)+1}), g(x_{m(k)+1})) \\ d(g(y_{n(k)+1}), g(y_{m(k)+1})) \\ d(g(z_{n(k)+1}), g(z_{m(k)+1})) \\ d(g(w_{n(k)+1}), g(w_{m(k)+1})) \end{bmatrix} + \delta_{m(k)+1} \tag{2.30}$$

$$\leq \delta_{n(k)+1} + \delta_{m(k)+1} + 4\phi \left(\frac{t_k}{4} \right) < \delta_{n(k)+1} + \delta_{m(k)+1} + 4\frac{t_k}{4}.$$

Letting $k \rightarrow \infty$, we get a contradiction. This shows that $\{g(x_n)\}, \{g(y_n)\}, \{g(z_n)\}$ and $\{g(w_n)\}$ are Cauchy sequences. Since X is complete metric space, there exists $x, y, z, w \in X$ such that

$$\begin{aligned}
\lim_{n \rightarrow \infty} g(x_n) &= x \text{ and } \lim_{n \rightarrow \infty} g(y_n) = y, \\
\lim_{n \rightarrow \infty} g(z_n) &= z \text{ and } \lim_{n \rightarrow \infty} g(w_n) = w.
\end{aligned} \tag{2.31}$$

Since g is continuous, (2.31) implies that

$$\begin{aligned}\lim_{n \rightarrow \infty} g(g(x_n)) &= g(x) \text{ and } \lim_{n \rightarrow \infty} g(g(y_n)) = g(y), \\ \lim_{n \rightarrow \infty} g(g(z_n)) &= g(z) \text{ and } \lim_{n \rightarrow \infty} g(g(w_n)) = g(w).\end{aligned}\quad (2.32)$$

From (2.10) and by regarding commutativity of F and g ,

$$\begin{aligned}g(g(x_{n+1})) &= g(F(x_n, y_n, z_n, w_n)) = F(g(x_n), g(y_n), g(z_n), g(w_n)), \\ g(g(y_{n+1})) &= g(F(x_n, w_n, z_n, y_n)) = F(g(x_n), g(w_n), g(z_n), g(y_n)), \\ g(g(z_{n+1})) &= g(F(z_n, y_n, x_n, w_n)) = F(g(z_n), g(y_n), g(x_n), g(w_n)), \\ g(g(w_{n+1})) &= g(F(z_n, w_n, x_n, y_n)) = F(g(z_n), g(w_n), g(x_n), g(y_n)),\end{aligned}\quad (2.33)$$

We shall show that

$$\begin{aligned}F(x, y, z, w) &= g(x), & F(x, w, z, y) &= g(y), \\ F(z, y, x, w) &= g(z), & F(z, w, x, y) &= g(w).\end{aligned}$$

Suppose now (a) holds. Then by (2.8), (2.33) and (2.31), we have

$$\begin{aligned}g(x) &= \lim_{n \rightarrow \infty} g(g(x_{n+1})) = \lim_{n \rightarrow \infty} g(F(x_n, y_n, z_n, w_n)) \\ &= \lim_{n \rightarrow \infty} F(g(x_n), g(y_n), g(z_n), g(w_n)) \\ &= F(\lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(y_n), \lim_{n \rightarrow \infty} g(z_n), \lim_{n \rightarrow \infty} g(w_n)) = F(x, y, z, w)\end{aligned}\quad (2.34)$$

Analogously, we also observe that

$$\begin{aligned}g(y) &= \lim_{n \rightarrow \infty} g(g(y_{n+1})) = \lim_{n \rightarrow \infty} g(F(x_n, w_n, z_n, y_n)) \\ &= \lim_{n \rightarrow \infty} F(g(x_n), g(w_n), g(z_n), g(y_n)) \\ &= F(\lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(w_n), \lim_{n \rightarrow \infty} g(z_n), \lim_{n \rightarrow \infty} g(y_n)) \\ &= F(x, w, z, y)\end{aligned}\quad (2.35)$$

$$\begin{aligned}g(z) &= \lim_{n \rightarrow \infty} g(g(z_{n+1})) = \lim_{n \rightarrow \infty} g(F(z_n, y_n, x_n, w_n)) \\ &= \lim_{n \rightarrow \infty} F(g(z_n), g(y_n), g(x_n), g(w_n)) \\ &= F(\lim_{n \rightarrow \infty} g(z_n), \lim_{n \rightarrow \infty} g(y_n), \lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(w_n)) \\ &= F(z, y, x, w)\end{aligned}\quad (2.36)$$

$$\begin{aligned}g(w) &= \lim_{n \rightarrow \infty} g(g(w_{n+1})) = \lim_{n \rightarrow \infty} g(F(z_n, w_n, x_n, y_n)) \\ &= \lim_{n \rightarrow \infty} F(g(z_n), g(w_n), g(x_n), g(y_n)) \\ &= F(\lim_{n \rightarrow \infty} g(z_n), \lim_{n \rightarrow \infty} g(w_n), \lim_{n \rightarrow \infty} g(x_n), \lim_{n \rightarrow \infty} g(y_n)) \\ &= F(z, w, x, y)\end{aligned}\quad (2.37)$$

Thus, we have

$$\begin{aligned}F(x, y, z, w) &= g(x), & F(y, z, w, x) &= g(y), \\ F(z, , w, x, y) &= g(z), & F(w, x, y, z) &= g(w).\end{aligned}$$

Suppose now the assumption (b) holds. Since $\{g(x_n)\}$, $\{g(z_n)\}$ is non-decreasing and $g(x_n) \rightarrow x$, $g(z_n) \rightarrow z$ and also $\{g(y_n)\}$, $\{g(w_n)\}$ is non-increasing and $g(y_n) \rightarrow y$, $g(w_n) \rightarrow$, then by assumption (b) we have

$$g(x_n) \geq x, \quad g(y_n) \leq y, \quad g(z_n) \geq z, \quad g(w_n) \leq w \quad (2.38)$$

for all n . Thus, by triangle inequality and (2.33)

$$g(x_n) \geq x, \quad g(y_n) \leq y, \quad g(z_n) \geq z, \quad g(w_n) \leq w \quad (2.39)$$

for all n . Thus, by triangle inequality and (2.33)

$$\begin{aligned} d(g(x), F(x, y, z, w)) &\leq d(g(x), g(g(x_{n+1}))) + d(g(g(x_{n+1})), F(x, y, z, w)) \\ &\leq d(g(x), g(g(x_{n+1}))) \\ &\quad + \phi \left(\frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} d(g(g(x_n), g(x))) \\ d(g(g(y_n), g(y))) \\ d(g(g(z_n), g(z))) \\ d(g(g(w_n), g(w))) \end{bmatrix} \right) \end{aligned} \quad (2.40)$$

Letting $n \rightarrow \infty$ implies that $d(g(x), F(x, y, z, w)) \leq 0$. Hence, $g(x) = F(x, y, z, w)$. Analogously we can get that

$$F(y, z, w, x) = g(y), F(z, w, x, y) = g(z) \text{ and } F(w, x, y, z) = g(w).$$

Thus, we proved that F and g have a quadruple coincidence point. \square

Corollary 2.7. *Let (X, \leq) be partially ordered set and (X, d) be a complete metric space. Suppose $F : X^4 \rightarrow X$ and there exists $\phi \in \phi$ such that F has the mixed g -monotone property and there exists a $k \in [0, 1)$ with*

$$\begin{aligned} &\psi(d(F(x, y, z, w), F(u, v, r, t))) \\ &\leq \frac{k}{4} [d(g(x), g(u)) + d(g(y), g(v)) + d(g(z), g(r)) + d(g(w), g(t))] \end{aligned} \quad (2.41)$$

for all x, u, y, v, z, r, w, t for which $g(x) \leq g(u)$, $g(y) \geq g(v)$, $g(z) \leq g(r)$ and $g(w) \geq g(t)$. Suppose there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} g(x_0) &\leq F(x_0, y_0, z_0, w_0), & g(y_0) &\geq F(x_0, w_0, z_0, y_0), \\ g(z_0) &\leq F(z_0, y_0, x_0, w_0), & g(w_0) &\geq F(z_0, w_0, x_0, y_0). \end{aligned} \quad (2.42)$$

Assume also that Assumption 2.1 holds, that $F(X^4) \subset g(X)$ and g commutes with F . Then there exist $x, y, z, w \in X$ such that

$$\begin{aligned} F(x, y, z, w) &= g(x), & F(x, w, z, y) &= g(y), \\ F(z, y, x, w) &= g(z), & F(z, w, x, y) &= g(w). \end{aligned}$$

that is, F and g have a quadruple coincidence point.

Proof. It is sufficient to take $\phi = kt$ with $k \in [0, 1)$ in previous theorem. \square

3. UNIQUENESS OF QUADRUPLE FIXED POINT

In this section we shall prove the uniqueness of quadruple fixed point for a product X^4 of a partial ordered set (X, \leq) where partial order is defined as follow: For all $(x, y, z, w), (u, v, r, t) \in X^4$,

$$(x, y, z, w) \leq (u, v, r, t) \Leftrightarrow x \leq u, \quad y \geq v, \quad z \leq r, \quad w \geq t. \quad (3.1)$$

We say that (x, y, z, w) is equal (u, v, r, t) if and only if $x = u$, $y = v$, $z = r$ and $w = t$.

Theorem 3.1. *In addition to hypothesis of Theorem 2.6, assume that for all $(x, y, z, w), (u, v, r, t) \in X^4$, there exists $(a, b, c, d) \in X^4$ such that $(F(a, b, c, d),$*

$F(a, d, c, b), F(c, b, a, d), F(c, d, a, b)$ is comparable to

$$(F(x, y, z, w), F(x, w, z, y), F(z, y, x, w), F(z, w, x, y)) \text{ and} \\ (F(u, v, r, t), F(u, t, r, v), F(z, v, u, r), F(r, t, u, v)).$$

Then, F and g have a unique quadruple common fixed point, that is, there exists a unique $(p, q, s, o) \in X^4$ such that

$$g(p) = F(p, q, s, o), g(q) = F(p, o, s, q), g(s) = F(s, q, p, o), g(o) = F(s, o, p, q).$$

Proof. The set of quadruple common fixed point of F and g is not empty due to Theorem 2.6. Assume, now, (x, y, z, w) and (u, v, r, t) are the quadruple common fixed point of F and g , that is,

$$F(x, y, z, w) = g(x), F(u, v, r, t) = g(u), F(x, w, z, y) = g(y), F(u, t, r, v) = g(v), \\ F(z, y, x, w) = g(z), F(r, v, u, t) = g(r), F(z, w, x, y) = g(w), F(r, t, u, v) = g(t). \quad (3.2)$$

We shall show that $(g(x), g(y), g(z), g(w))$ and $(g(u), g(v), g(r), g(t))$ are equal. By assumption, there exists $(a, b, c, d) \in X \times X \times X \times X$ such that

$$(F(a, b, c, d), F(a, d, c, b), F(c, b, a, d), F(c, d, a, b)) \quad (3.3)$$

is comparable to

$$(F(x, y, z, w), F(x, w, z, y), F(z, y, x, w), F(z, w, x, y)) \text{ and} \quad (3.4)$$

$$(F(u, v, r, t), F(u, t, r, v), F(z, v, u, r), F(r, t, u, v)). \quad (3.5)$$

Define sequences $\{g(a_n)\}, \{g(b_n)\}, \{g(c_n)\}$ and $\{g(d_n)\}$ such that $a = a_0, b = b_0, c = c_0, d = d_0$ for all n , and

$$g(a_n) = F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}), g(b_n) = F(a_{n-1}, d_{n-1}, c_{n-1}, b_{n-1}), \\ g(c_n) = F(c_{n-1}, b_{n-1}, a_{n-1}, d_{n-1}), g(d_n) = F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}). \quad (3.6)$$

Since (3.4) is comparable with (3.3), we may assume that

$$(g(x), g(y), g(z), g(w)) \geq (g(a), g(b), g(c), g(d)) = (g(a_0), g(b_0), g(c_0), g(d_0)).$$

Recursively, we get that

$$(g(x), g(y), g(z), g(w)) \geq (g(a_n), g(b_n), g(c_n), g(d_n)) \quad \text{for all } n. \quad (3.7)$$

By (3.7) and (2.6), we have

$$d(g(x), (a_{n+1})) \\ = d(F(g(x), g(y), g(z), g(w)), F(g(a_n), g(b_n), g(c_n), g(d_n))) \\ \leq \phi \left(\frac{1}{4} [d(g(x), g(a_n)) + d(g(y), g(b_n)) + d(g(z), g(c_n)) + d(g(w), g(d_n))] \right) \quad (3.8)$$

$$d(g(b_{n+1}), g(y)) \\ = d(F(a_n, d_n, c_n, b_n), F(x, w, z, y)) \\ \leq \phi \left(\frac{1}{4} [d(g(a_n), g(x)) + d(g(d_n), g(w)) + d(g(c_n), g(z)) + d(g(b_n), g(y))] \right) \quad (3.9)$$

$$\begin{aligned}
& d(g(z), g(c_{n+1})) \\
&= d(F(z, y, x, w), F(c_n, b_n, a_n, d_n)) \\
&\leq \phi \left(\frac{1}{4} [d(g(z), g(c_n)) + d(g(y), g(b_n)) + d(g(x), g(a_n)) + d(g(w), g(d_n))] \right)
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
d(g(d_{n+1}), g(w)) &= d(F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}), F(z, w, x, y)) \\
&\leq \phi \left(\frac{1}{4} [d(g(c_n), g(z)) + d(g(d_n), g(w)) \right. \\
&\quad \left. + d(g(a_n), g(x)) + d(g(b_n), g(y))] \right)
\end{aligned} \tag{3.11}$$

Set $\gamma_n = d(g(x), g(a_n)) + d(g(y), g(b_n)) + d(g(z), g(c_n)) + d(g(w), g(d_n))$. Then, due to (3)-(3), we have

$$\frac{\gamma_{n+1}}{4} \leq \phi \left(\frac{\gamma_n}{4} \right), \quad \text{for all } n.$$

which implies

$$\frac{\gamma_{n+1}}{4} \leq \phi^n \left(\frac{\gamma_1}{4} \right) \quad \text{for all } n. \tag{3.12}$$

Since $\phi(t) < t$ and $\lim_{r \rightarrow t^+} \phi(r) < t$ then $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for each $t > 0$. Hence (3.12) implies that

$$\begin{aligned}
\lim_{n \rightarrow \infty} d(g(x), g(a_n)) &= 0, & \lim_{n \rightarrow \infty} d(g(y), g(b_n)) &= 0, \\
\lim_{n \rightarrow \infty} d(g(z), g(c_n)) &= 0, & \lim_{n \rightarrow \infty} d(g(w), g(d_n)) &= 0.
\end{aligned} \tag{3.13}$$

Analogously, we show that

$$\begin{aligned}
\lim_{n \rightarrow \infty} d(g(u), g(a_n)) &= 0, & \lim_{n \rightarrow \infty} d(g(v), g(b_n)) &= 0, \\
\lim_{n \rightarrow \infty} d(g(r), g(c_n)) &= 0, & \lim_{n \rightarrow \infty} d(g(s), g(d_n)) &= 0.
\end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14) and by using the triangle inequality,

$$\begin{aligned}
d(g(u), g(x)) &\leq d(g(u), g(a_n)) + d(g(a_n), g(x)) \rightarrow 0, \text{ as } n \rightarrow \infty, \\
d(g(v), g(y)) &\leq d(g(v), g(b_n)) + d(g(b_n), g(y)) \rightarrow 0, \text{ as } n \rightarrow \infty, \\
d(g(r), g(z)) &\leq d(g(r), g(c_n)) + d(g(c_n), g(z)) \rightarrow 0, \text{ as } n \rightarrow \infty, \\
d(g(s), g(w)) &\leq d(g(s), g(d_n)) + d(g(d_n), g(w)) \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned} \tag{3.15}$$

Hence,

$$(g(x), g(y), g(z), g(w)) = (g(u), g(v), g(r), g(t)). \tag{3.16}$$

By commutativity of F and g , the identities (3.2)

$$\begin{aligned}
g(g(x)) &= g(F(x, y, z, w)) = F(g(x), g(y), g(z), g(w)) \\
g(g(y)) &= g(F(x, w, z, y)) = F(g(x), g(w), g(z), g(y)), \\
g(g(z)) &= g(F(z, y, x, w)) = F(g(z), g(y), g(x), g(w)), \\
g(g(w)) &= g(F(z, w, x, y)) = F(g(z), g(w), g(x), g(y)).
\end{aligned} \tag{3.17}$$

Set

$$g(x) = p, \quad g(y) = q, \quad g(z) = s, \quad g(w) = o \tag{3.18}$$

Then (3.17) turn into,

$$g(p) = F(p, q, s, o), g(q) = F(p, o, s, q), g(s) = F(s, q, p, o), g(o) = F(s, o, p, q). \quad (3.19)$$

Thus, (p, q, s, o) is a quadruple common coincidence point of F and g . By taking $(u, v, r, t) = (p, q, s, o)$ (3.16) we have

$$g(x) = g(p), g(y) = g(q), g(z) = g(s), g(w) = g(o). \quad (3.20)$$

Combining (3.19), (3.20) and (3.18) we have

$$\begin{aligned} p &= g(p) = F(p, q, s, o), q = g(q) = F(p, o, s, q), \\ s &= g(s) = F(s, q, p, o), o = g(o) = F(s, o, p, q). \end{aligned} \quad (3.21)$$

Thus, (p, q, s, o) is a quadruple common fixed point of F and g . Due to (3.16), it is unique. \square

Corollary 3.2. *Let (X, d, \leq) be a POCMS. Suppose $F : X^4 \rightarrow X$ and there exists $\phi \in \Phi$ such that F has the mixed monotone property and*

$$\psi(d(F(x, y, z, w), F(u, v, r, t))) \leq \phi \left(\frac{d(x, u) + d(y, v) + d(z, r) + d(w, t)}{4} \right) \quad (3.22)$$

for all x, u, y, v, z, r, w, t for which $x \leq u, y \geq v, z \leq r$ and $w \geq t$ where $\phi \in \Phi$. Suppose there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} x_0 &\leq F(x_0, y_0, z_0, w_0), & y_0 &\geq F(x_0, w_0, z_0, y_0), \\ z_0 &\leq F(z_0, y_0, x_0, w_0), & w_0 &\geq F(z_0, w_0, x_0, y_0). \end{aligned} \quad (3.23)$$

Assume also that Assumption 2.1 holds. Then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = x, F(x, w, z, y) = y, F(z, y, x, w) = z, F(z, w, x, y) = w,$$

that is, F has a quadruple fixed point.

Proof. Take $g(x) = x$, then the assumptions of Theorem 2.6 are satisfied. Thus, we get the result. \square

Corollary 3.3. *Let (X, d, \leq) be a POCMS. Suppose $F : X^4 \rightarrow X$ and there exists $\phi \in \Phi$ such that F has the mixed g -monotone property and there exists a $k \in [0, 1)$ with*

$$\psi(d(F(x, y, z, w), F(u, v, r, t))) \leq \frac{k}{4} [d(x, u) + d(y, v) + d(z, r) + d(w, t)] \quad (3.24)$$

for all x, u, y, v, z, r, w, t for which $x \leq u, y \geq v, z \leq r$ and $w \geq t$. Suppose there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} x_0 &\leq F(x_0, y_0, z_0, w_0), & y_0 &\geq F(x_0, w_0, z_0, y_0), \\ z_0 &\leq F(z_0, y_0, x_0, w_0), & w_0 &\geq F(z_0, w_0, x_0, y_0). \end{aligned} \quad (3.25)$$

Assume also that Assumption 2.1 holds. Then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = x, F(x, w, z, y) = y, F(z, y, x, w) = z, F(z, w, x, y) = w,$$

that is, F and g have a quadruple coincidence point.

4. EXAMPLES

In this sections we give some examples to show that our results are effective.

Example 4.1. Let $X = [0, \infty)$ with the metric $d(x, y) = |x - y|$, for all $x, y \in X$ and the following order relation:

$$x, y \in X, \quad x \preceq y \Leftrightarrow x = y = 0 \text{ or } (x, y \in (0, \infty) \text{ and } x \leq y),$$

where \leq be the usual ordering. Let $F : X^4 \rightarrow X$ be given by

$$F(x, y, z, w) = \begin{cases} 1, & \text{if } xyzw \neq 0 \\ 0, & \text{if } xyzw = 0 \end{cases}$$

for all $x, y, z, w \in X$ and let $\phi : [0, \infty) \rightarrow [0, \infty)$ be given by $\phi(t) = \frac{9t}{10}$ for all $t \in [0, \infty)$.

It is easy to check that all the conditions of Corollary 2.7 are satisfied. Applying Corollary 2.7 we conclude that F has at least a quadruple fixed point. In fact, F has two quadruple fixed points, which are $(0, 0, 0, 0)$ and $(1, 1, 1, 1)$. Therefore, the conditions of Corollary 2.7 are not sufficient for the uniqueness of a quadruple fixed point.

Example 4.2. Let $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$, for all $x, y \in X$ and the usual ordering.

Let $F : X^4 \rightarrow X$ be given by

$$F(x, y, z, w) = \frac{x - y + z - w}{16}, \text{ for all } x, y, z, w \in X,$$

and let $\phi : [0, \infty) \rightarrow [0, \infty)$ be given by $\phi(t) = \frac{t}{2}$ for all $t \in [0, \infty)$.

It is easy to check that all the conditions of Corollary 3.2 are satisfied and $(0, 0, 0, 0)$ is the unique quadruple fixed point of F .

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