Banach J. Math. Anal. 6 (2012), no. 1, 35–44

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

COMPLETE MONOTONICITY OF A FUNCTION INVOLVING THE RATIO OF GAMMA FUNCTIONS AND APPLICATIONS

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Communicated by T. Sugawa

ABSTRACT. In the paper, necessary and sufficient conditions are presented for a function involving a ratio of gamma functions to be logarithmically completely monotonic. This extends and generalizes the main result of Guo and Qi [Taiwanese J. Math. 7 (2003), no. 2, 239–247] and others. As applications, several inequalities involving the volume of the unit ball in $\mathbb{R}^n$ are derived, which refine, generalize and extend some known inequalities.

1. INTRODUCTION

Recall from [4, 25] that a positive real-valued function $f(x)$ is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$0 \le (-1)^k [\ln f(x)]^{(k)} < \infty$$

for $k \in \mathbb{N} = \{1, 2, \dots\}$ on I. For more properties of this class of functions, please refer to [6].

It is general knowledge that the classical Euler gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \,\mathrm{d}\, t.$$

Date: Received: 8 July 2011; Revised: 4 September 2011; Accepted: 2 September 2011.

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2010 *Mathematics Subject Classification.* Primary 33B15; Secondary 26A48, 26A51, 26D07.

Key words and phrases. Necessary and sufficient condition, logarithmically completely monotonic function, gamma function, volume of unit ball.

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function and the $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ are called the polygamma functions. It is well known that these functions are fundamental and that they have much extensive applications in mathematical sciences.

In [15, Theorem 2], the following monotonicity was established: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{1.1}$$

is decreasing with respect to $x\ge1$ for fixed $y\ge0$. Consequently, for positive real numbers $x\ge1$ and $y\ge0$, we have

$$\frac{x+y+1}{x+y+2}\le\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{1.2}$$

In [26], the function (1.1) was proved to be logarithmically completely monotonic with respect to $x\in(0,\infty)$ for $y\ge0$ and so is its reciprocal for $-1<y\le-\frac12$. Consequently, the inequality (1.2) is valid for $(x,y)\in(0,\infty)\times[0,\infty)$ and reversed for $(x,y)\in(0,\infty)\times\left(-1,-\frac12\right]$.

For $(x,y)\in(0,\infty)\times[0,\infty)$ and $\alpha\in[0,\infty)$, the function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{(x+y+1)^{\alpha}} \tag{1.3}$$

was proved in [44] to be strictly increasing (or decreasing, respectively) with respect to the single variable $x\in(0,\infty)$ if and only if $0\le\alpha\le\frac12$ (or $\alpha\ge1$, respectively), to be strictly increasing with respect to y on $[0,\infty)$ if and only if $0\le\alpha\le1$ and to be logarithmically concave with respect to the 2-variable $(x,y)\in(0,\infty)\times(0,\infty)$ if $0\le\alpha\le\frac14$.

For given $y\in(-1,\infty)$ and $\alpha\in(-\infty,\infty)$, let

$$h_{\alpha,y}(x)=\begin{cases}\dfrac{1}{(x+y+1)^{\alpha}}\left[\dfrac{\Gamma(x+y+1)}{\Gamma(y+1)}\right]^{1/x}, & x\in(-y-1,\infty)\setminus\{0\};\\[2ex] \dfrac{1}{(y+1)^{\alpha}}\exp[\psi(y+1)], & x=0.\end{cases} \tag{1.4}$$

It is clear that the ranges of x, y and α in the function $h_{\alpha,y}(x)$ extend the corresponding ones in the functions (1.1) and (1.3) which were ever discussed in [15, 26, 44].

The aim of this paper is to present necessary and sufficient conditions such that the function (1.4) or its reciprocal are logarithmically completely monotonic.

Our main results may be stated as follows.

Theorem 1.1. *For $y>-1$, we have the following statements:*

(1) *the function (1.4) is logarithmically completely monotonic with respect to $x\in(-y-1,\infty)$ if and only if $\alpha\ge\max\left\{1,\frac{1}{y+1}\right\}$;*
(2) *if $\alpha\le\min\left\{1,\frac{1}{2(y+1)}\right\}$, the reciprocal of the function (1.4) is logarithmically completely monotonic with respect to $x\in(-y-1,\infty)$;*
(3) *a necessary condition for the reciprocal of the function (1.4) to be logarithmically completely monotonic with respect to $x\in(-y-1,\infty)$ is $\alpha\le1$.*

As a ready consequence of monotonic results in Theorem 1.1, the following double inequality may be derived.

Theorem 1.2. *For $t > 0$, $y + 1 > 0$ and $x + y + 1 > 0$, the double inequality*

$$\left(\frac{x+y+1}{x+y+t+1}\right)^a < \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+t+1)/\Gamma(y+1)]^{1/(x+t)}} < \left(\frac{x+y+1}{x+y+t+1}\right)^b \tag{1.5}$$

holds if $a \ge \max\left\{1, \frac{1}{y+1}\right\}$ and $b \le \min\left\{1, \frac{1}{2(y+1)}\right\}$.

In order to show the applicability of Theorem 1.2, we derive the following double inequalities involving the n-dimensional volume

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$$

of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$.

Theorem 1.3. *For $n \in \mathbb{N}$, we have*

$$\sqrt{\frac{n+2}{n+4}} < \frac{\Omega_{n+2}^{1/(n+2)}}{\Omega_n^{1/n}} < \sqrt[4]{\frac{n+2}{n+4}}, \tag{1.6}$$

$$\frac{1}{\pi^{2/(n-2)n}}\sqrt{\frac{n+2}{n+4}} < \frac{\Omega_{n+2}^{1/n}}{\Omega_n^{1/(n-2)}} < \frac{1}{\pi^{2/(n-2)n}}\sqrt[8]{\frac{n+2}{n+4}}, \tag{1.7}$$

$$\sqrt{\frac{n+2}{n+3}} < \frac{\Omega_{n+1}^{1/(n+1)}}{\Omega_n^{1/n}} < \sqrt[4]{\frac{n+2}{n+3}}. \tag{1.8}$$

In the final section, we will give several remarks about these three theorems.

2. A Lemma

In order to prove our main results, the following lemma is needed.

Lemma 2.1 ([19, p. 107, Lemma 3])**.** *For $x \in (0, \infty)$ and $k \in \mathbb{N}$, we have*

$$\ln x - \frac{1}{x} < \psi(x) < \ln x - \frac{1}{2x} \tag{2.1}$$

and

$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}. \tag{2.2}$$

Remark 2.2. We remark that some minor errors in the proof of Lemma 2.1 were corrected in [14, p. 1212, Lemma 2.2]. The inequalities in Lemma 2.1 have been proved, derived, used and applied in several papers such as [11, p. 131], [17, Lemma 1], [18, p. 223, Lemma 2.3], [20, p. 853], [23, p. 55, Theorem 5.11], [26, p. 1625], [31, p. 79], and [37, p. 2155, Lemma 3].

3. Proofs of theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1.1. For $x \ne 0$, taking the logarithm of $h_{\alpha,y}(x)$ gives

$$\ln h_{\alpha,y}(x) = \frac{\ln\Gamma(x+y+1)-\ln\Gamma(y+1)}{x} - \alpha\ln(x+y+1).$$

A direct differentiation yields

$$\begin{aligned}[\ln h_{\alpha,y}(x)]^{(k)} ={}& \frac{k!}{x^{k+1}}\sum_{i=0}^{k}\frac{(-1)^{k-i}x^i\psi^{(i-1)}(x+y+1)}{i!}\\ &- \frac{(-1)^k k!\ln\Gamma(y+1)}{x^{k+1}} - \frac{(-1)^{k-1}(k-1)!\alpha}{(x+y+1)^k}\end{aligned} \tag{3.1}$$

for $k \in \mathbb{N}$, where $\psi^{(-1)}(x+y+1)$ and $\psi^{(0)}(x+y+1)$ stand for $\ln\Gamma(x+y+1)$ and $\psi(x+y+1)$ respectively. Furthermore, a simple calculation gives

$$\begin{aligned}\big\{x^{k+1}[\ln h_{\alpha,y}(x)]^{(k)}\big\}' = (-1)^{k-1}x^k\bigg[&(-1)^{k-1}\psi^{(k)}(x+y+1)\\ &- \frac{(k-1)!\alpha}{(x+y+1)^k} - \frac{k!(y+1)\alpha}{(x+y+1)^{k+1}}\bigg].\end{aligned}$$

Utilizing (2.2) in the above equation leads to

$$\begin{aligned}\frac{(k-1)!(1-\alpha)}{(x+y+1)^k} + \frac{k![1/2-(y+1)\alpha]}{(x+y+1)^{k+1}} &\le \frac{(-1)^{k-1}}{x^k}\big\{x^{k+1}[\ln h_{\alpha,y}(x)]^{(k)}\big\}'\\ &\le \frac{(k-1)!(1-\alpha)}{(x+y+1)^k} + \frac{k![1-(y+1)\alpha]}{(x+y+1)^{k+1}}\end{aligned}$$

for $k \in \mathbb{N}$, $x \ne 0$, $y \in (-1,\infty)$ and $\alpha \in (-\infty,\infty)$. Therefore,

$$\frac{(-1)^{k-1}}{x^k}\big\{x^{k+1}[\ln h_{\alpha,y}(x)]^{(k)}\big\}' \begin{cases} \le 0, & \text{if } \alpha \ge 1 \text{ and } \alpha \ge \frac{1}{y+1}\\ \ge 0, & \text{if } \alpha \le 1 \text{ and } \alpha \le \frac{1}{2(y+1)}\end{cases} \tag{3.2}$$

for $k \in \mathbb{N}$, $y > -1$ and $x \ne 0$. For $x > 0$, the equation (3.2) means

$$\big\{x^{2k}[\ln h_{\alpha,y}(x)]^{(2k-1)}\big\}' \begin{cases} \le 0, & \text{if } \alpha \ge 1 \text{ and } \alpha \ge \frac{1}{y+1}\\ \ge 0, & \text{if } \alpha \le 1 \text{ and } \alpha \le \frac{1}{2(y+1)}\end{cases}$$

and

$$\big\{x^{2k+1}[\ln h_{\alpha,y}(x)]^{(2k)}\big\}' \begin{cases} \ge 0, & \text{if } \alpha \ge 1 \text{ and } \alpha \ge \frac{1}{y+1}\\ \le 0, & \text{if } \alpha \le 1 \text{ and } \alpha \le \frac{1}{2(y+1)}\end{cases}$$

for $k \in \mathbb{N}$. From (3.1), it is easy to see that

$$\lim_{x\to0}\big\{x^{k+1}[\ln h_{\alpha,y}(x)]^{(k)}\big\} = 0 \tag{3.3}$$

for $k \in \mathbb{N}$ and any given $y > -1$. As a result,

$$[\ln h_{\alpha,y}(x)]^{(2k-1)} \begin{cases} < 0, & \text{if } \alpha \ge 1 \text{ and } \alpha \ge \frac{1}{y+1}\\ > 0, & \text{if } \alpha \le 1 \text{ and } \alpha \le \frac{1}{2(y+1)}\end{cases} \tag{3.4}$$

and

$$[\ln h_{\alpha,y}(x)]^{(2k)} \begin{cases} > 0, & \text{if } \alpha \ge 1 \text{ and } \alpha \ge \frac{1}{y+1} \\ < 0, & \text{if } \alpha \le 1 \text{ and } \alpha \le \frac{1}{2(y+1)} \end{cases} \tag{3.5}$$

for $k \in \mathbb{N}$ and $x \in (0, \infty)$, that is,

$$(-1)^k [\ln h_{\alpha,y}(x)]^{(k)} \begin{cases} > 0, & \text{if } \alpha \ge 1 \text{ and } \alpha \ge \frac{1}{y+1} \\ < 0, & \text{if } \alpha \le 1 \text{ and } \alpha \le \frac{1}{2(y+1)} \end{cases} \tag{3.6}$$

for $k \in \mathbb{N}$ and $x \in (0, \infty)$. Hence, the function (1.4) is logarithmically completely monotonic with respect to x on $(0, \infty)$ if $\alpha \ge 1$ and $\alpha \ge \frac{1}{y+1}$ and so is the reciprocal of the function (1.4) if either $0 < \alpha \le 1$ and $\alpha \le \frac{1}{2(y+1)}$ or $\alpha \le 0$ and $y > -1$.

If $x \in (-y-1, 0)$, the equation (3.2) means

$$\left\{x^{k+1}[\ln h_{\alpha,y}(x)]^{(k)}\right\}' \begin{cases} \ge 0, & \text{if } \alpha \ge 1 \text{ and } \alpha \ge \frac{1}{y+1} \\ \le 0, & \text{if } \alpha \le 1 \text{ and } \alpha \le \frac{1}{2(y+1)} \end{cases}$$

for $k \in \mathbb{N}$. By virtue of (3.3), it follows that

$$x^{k+1}[\ln h_{\alpha,y}(x)]^{(k)} \begin{cases} \le 0, & \text{if } \alpha \ge 1 \text{ and } \alpha \ge \frac{1}{y+1} \\ \ge 0, & \text{if } \alpha \le 1 \text{ and } \alpha \le \frac{1}{2(y+1)} \end{cases}$$

for $k \in \mathbb{N}$, which is equivalent to the fact that the equations (3.4) and (3.5) hold for $x \in (-y-1, 0)$. As a result, the equation (3.6) is valid for $k \in \mathbb{N}$ and $x \in (-y-1, 0)$. Therefore, the function $h_{\alpha,y}(x)$ has the same logarithmically complete monotonicity properties on $(-y-1, 0)$ as on $(0, \infty)$.

Conversely, if $h_{\alpha,y}(x)$ is logarithmically completely monotonic on $(-y-1, \infty)$, then $[\ln h_{\alpha,y}(x)]' < 0$ on $(-y-1, \infty)$, which can be simplified as

$$\alpha \ge (x+y+1)\left[\frac{1}{x^2}\sum_{i=0}^{1}\frac{(-1)^{1-i}x^i\psi^{(i-1)}(x+y+1)}{i!} + \frac{\ln\Gamma(y+1)}{x^2}\right] \tag{3.7}$$

$$\begin{aligned} &= \frac{1}{x^2}[(x+y+1)\ln\Gamma(y+1) - (y+1)(x+y+1)\psi(x+y+1) \\ &\quad + (x+y+1)^2\psi(x+y+1) - (x+y+1)\ln\Gamma(x+y+1)] \end{aligned} \tag{3.8}$$

$$= \frac{x+y+1}{x}\left[\frac{x\psi(x+y+1) - \ln\Gamma(x+y+1)}{x} + \frac{\ln\Gamma(y+1)}{x}\right]. \tag{3.9}$$

From (2.1), it is easy to see that

$$\lim_{x\to 0^+}\left[x^2\psi(x)\right] = 0. \tag{3.10}$$

It is common knowledge that

$$\Gamma(x+1) = x\Gamma(x) \tag{3.11}$$

for $x > 0$. Taking the logarithm on both sides of (3.11), rearranging and taking limit lead to

$$\lim_{x\to 0^+}[x\ln\Gamma(x)] = \lim_{x\to 0^+}[x\ln\Gamma(x+1)] - \lim_{x\to 0^+}(x\ln x) = 0. \tag{3.12}$$

Taking logarithmic derivatives on both sides of (3.11) yields

$$\psi(x+1)=\frac{1}{x}+\psi(x)$$

for $x>0$ and so

$$\lim_{x\to0^+}[x\psi(x)]=-1+\lim_{x\to0^+}[x\psi(x+1)]=-1. \tag{3.13}$$

Thus, by utilizing (3.10), (3.12) and (3.13), it is revealed that the limit of the function (3.8) as $x\to(-y-1)^+$, that is, as $x+y+1\to0^+$, equals $\frac{1}{y+1}$. By L'Hôspital's rule and the double inequality (2.2) for $k=1$, we have

$$\lim_{x\to\infty}\frac{x\psi(x+y+1)-\ln\Gamma(x+y+1)}{x}=\lim_{x\to\infty}[x\psi'(x+y+1)]=1.$$

Hence, the limit of the function (3.9) as $x\to\infty$ equals 1. In a word, a necessary condition for $h_{\alpha,y}(x)$ to be logarithmically completely monotonic is $\alpha\ge1$ and $\alpha\ge\frac{1}{y+1}$.

If the reciprocal of $h_{\alpha,y}(x)$ is logarithmically completely monotonic, then the inequality (3.7) is reversed. Since the limit of the function (3.9) equals 1 as $x\to\infty$, as showed above, then the necessary condition $\alpha\le1$ is obtained. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. This follows from the monotonicity properties established in Theorem 1.1. □

Proof of Theorem 1.3. Letting $t=1$, $y=0$ and $x=\frac{n}{2}$ for $n\in\mathbb{N}$ in (1.5) reveals that

$$\frac{n+2}{n+4}<\frac{[\Gamma(n/2+1)]^{2/n}}{[\Gamma((n+2)/2+1)]^{2/(n+2)}}<\sqrt{\frac{n+2}{n+4}}$$

which is equivalent to the inequality (1.6).

If taking $y=1$, $t=1$ and $x=\frac{n}{2}-1$ for $n\in\mathbb{N}$ in (1.5), then the inequality (1.7) follows.

Replacing t by $\frac{1}{2}$, y by 0 and x by $\frac{n}{2}$ in (1.5) and simplifying result in (1.8). □

4. Remarks

After proving our theorems, we give several remarks about them.

Remark 4.1. Theorem 1.1 extends and generalizes the logarithmically complete monotonicity of the function (1.1) established in [26] and a part of the results in [44].

Remark 4.2. The inequality (1.5) generalizes and extends the inequality (1.2) and the main results in [14, 43]: For $x+y>0$ and $y+1>0$ the inequality

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}<\left(\frac{x+y}{x+y+1}\right)^{1/2}$$

is valid if $x>1$ and reversed if $x<1$ and that the power $\frac{1}{2}$ is the best possible.

Remark 4.3. When $n > 2$, the inequality (1.8) refines the following double inequality in [2, Theorem 1]:

$$\frac{2}{\sqrt{\pi}}\Omega_{n+1}^{n/(n+1)} \le \Omega_n < \sqrt{e}\,\Omega_{n+1}^{n/(n+1)}, \quad n \in \mathbb{N}.$$

For more information on inequalities for the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, please refer to [3, 7, 8, 12, 33], Section 7.5 in [23, pp. 72–73] and related references therein.

Remark 4.4. Theorem 1.1 may be restated as follows: For $y \in (0, \infty)$ and $\alpha \in (-\infty, \infty)$, the function

$$H_y(x) = \begin{cases} \dfrac{[\Gamma(x+y)/\Gamma(y)]^{1/x}}{(x+y)^\alpha}, & x \in (-y, \infty) \setminus \{0\} \\ \dfrac{e^{\psi(y)}}{y^\alpha}, & x = 0 \end{cases}$$

is logarithmically completely monotonic with respect to $x \in (-y, \infty)$ if and only if $\alpha \ge \max\left\{1, \frac{1}{y}\right\}$ and so is its reciprocal if $\alpha \le \min\left\{1, \frac{1}{2y}\right\}$ and only if $\alpha \le 1$.

Remark 4.5. We conjecture that when $y > -\frac{1}{2}$ the condition $\alpha \le \frac{1}{2(y+1)}$ is also necessary for the reciprocal of the function (1.4) to be logarithmically completely monotonic with respect to $x \in (-y-1, \infty)$. In other words, the necessary and sufficient condition for the reciprocal of the function (1.4) to be logarithmically completely monotonic with respect to $x \in (-y-1, \infty)$ is $\alpha \le \min\left\{1, \frac{1}{2(y+1)}\right\}$.

Remark 4.6. For more information on the history, background, motivations and recent developments of the topic in this paper, please refer to [1, 5, 9, 15, 16, 24, 26, 29, 36, 39, 40] and related references therein.

Remark 4.7. In passing, we survey the history of the notion "logarithmically completely monotonic function". By searching for the term "logarithmically completely monotonic function" in the database MathSciNet, it is found that this phrase was probably first used in [4], but without an explicit definition. Thereafter, it seems to have not been used by the mathematical community. In early 2004, this terminology was again used in [28] (the preprint of [25, 32]) and it was immediately referenced in [10] and [35] (the preprint of [34]). In [28, Theorem 4], it was proved that a logarithmically completely monotonic function $f(x)$ on I must be completely monotonic (i.e., the inequality

$$0 \le (-1)^k f^{(k)}(x) < \infty$$

holds for all $k \ge 0$ on I), but not conversely. This result was announced while revising [25]. This conclusion and its proofs were presented once and again in [6] and [38] (the preprint of [13]). More importantly, in the paper [6], the logarithmically completely monotonic functions on $(0, \infty)$ were characterized as the infinitely divisible completely monotonic functions studied in [21] and all Stieltjes transforms were proved to be logarithmically completely monotonic on $(0, \infty)$. For information on the completely monotonic functions, please refer to [22, Chapter XIII] and [42, Chapter IV], especially to the recently published monograph [41].

Remark 4.8. This paper is a main part and a slightly modified version of the preprint [30]. Another part was rearranged as [27].

Acknowledgements. The authors appreciate anonymous referees for their helpful and valuable comments on this paper. The first author was partially supported by the China Scholarship Council and the Science Foundation of Tianjin Polytechnic University.

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