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ON A J-POLAR DECOMPOSITION OF A BOUNDED OPERATOR AND MATRICES OF J-SYMMETRIC AND J-SKEW-SYMMETRIC OPERATORS

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Abstract. In this paper we study a possibility of a decomposition of a bounded operator in a Hilbert space H as a product of a J-unitary and a J-self-adjoint operators, where J is a conjugation (an antilinear involution). This decomposition shows an inner structure of a bounded operator in a Hilbert space. Some decompositions of J-unitary and unitary operators which generalize decompositions in the finite-dimensional case are also obtained. Matrix representations for J-symmetric and J-skew-symmetric operators are studied. Simple basic properties of J-symmetric, J-skew-symmetric and J-isometric operators are obtained.

1. Introduction and preliminaries

Complex symmetric, skew-symmetric and orthogonal matrices are classical objects of the finite-dimensional linear analysis [4]. In particular, the normal forms are known for them, see [4, Chapter XI]. Certainly, they have more complicated structures as for Hermitian matrices. However, in a certain sense complex symmetric matrices are more universal. Namely, an *arbitrary* square complex matrix is similar to a symmetric matrix [4, Chapter XI, p.321]. A generalization of complex symmetric, skew-symmetric and orthogonal matrices leads to the well-known J-symmetric, J-skew-symmetric and J-isometric operators.

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A general definition of a J-symmetric operator was given by I.M. Glazman in his paper [10]. A study of these operators had been continued in papers of N.A. Zhyhar and A. Galindo (see References in the monograph [11]). Later, an investigation of these operators had been performed by A.D. Makarova, L.A. Kamerina, V.P. Li, T.B. Kalinina, A.N. Kochubey, B.G. Mironov (a series of papers by these authors appeared in 70-th, 80-th of the 20-th century in Ulyanovskiy sbornik "Funkcionalniy analiz"), L.M. Rayh, E.R. Tsekanovskii, Sh. Asadi, I.E. Lutsenko, I. Knowles, D. Race, U.V. Riss and others (see, e.g. [3], [13], [14], [18], [20], [22], [23], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36]). Most of these papers were devoted to the questions of extensions of J-symmetric operators to J-self-adjoint operators and to a description of all such extensions. At the present time, J-self-adjoint operators are studied by S.R. Garcia, M. Putinar, E. Prodan (see, e.g., [5], [6], [7], [9] and References therein). In particular, a refined polar decomposition of a J-symmetric operator was established in [7]. In [5] some basic spectral properties of bounded J-symmetric operators were obtained. Among other results, a formula for computing the norm of a compact J-symmetric operator was given in [9].

A definition of a bounded J-skew-symmetric operator was given by Sh. Asadi and I.E. Lutsenko in their paper [3]. A general definition appeared in the paper of T.B. Kalinina [15]. She continued to study these operators in her papers [16], [17]. J-symmetric and J-skew-symmetric operators appeared in the book [12] in a study of Volterra operators context, as well.

J-Isometric, quasi-unitary operators and a notion of a quasi-unitary equivalence were introduced in papers of L.A. Kamerina [19],[21]. Another definition of a J-isometric operator (the operator in this definition should be densely defined) and a definition of a J-unitary operator (which is different from our definition below) were given by U.V. Riss in [33].

Consider a separable Hilbert space H. Recall that a conjugation (involution) is an operator J which is defined on the whole space H and satisfies the following conditions [2],[37]

$$J^2 = E, \qquad (Jx, Jy) = \overline{(x,y)}, \qquad x, y \in H,$$

where E is the identity operator in H, and $(\cdot,\cdot)$ is the inner product in H. For each conjugation there exists an orthonormal basis $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ in H such that

$$Jx = \sum_{k=0}^{\infty} \overline{x_k} f_k, \qquad x = \sum_{k=0}^{\infty} x_k f_k \in H. \tag{1.1}$$

This basis is not uniquely determined. It is determined up to a unitary transformation which commutes with J (recall that a transformation which commutes with J is called J-real). $\mathcal{F}$ will be called a *corresponding* basis to the involution J. Define the following bilinear functional (J-form):

$$[x, y]_J := (x, Jy), \qquad x, y \in H.$$

A linear operator A in H is said to be J*-symmetric*, if

$$[Ax, y]_J = [x, Ay]_J, \qquad x, y \in D(A), \tag{1.2}$$

and is said to be J-*skew-symmetric* if

$$[Ax, y]_J = -[x, Ay]_J, \qquad x, y \in D(A). \tag{1.3}$$

If

$$[Ax, Ay]_J = [x, y]_J, \qquad x, y \in D(A), \tag{1.4}$$

then the operator A is said to be J-*isometric*.

Let the domain of A be dense in H. The operator A is said to be J-*self-adjoint* if

$$A = JA^*J,$$

and is said to be J-*skew-self-adjoint* if

$$A = -JA^*J.$$

If

$$A^{-1} = JA^*J,$$

then A we shall call J-*unitary*. Notice that the operator $A^T = JA^*J$ is called *transposed* [2] (in some papers it was called J-adjoint but we shall use the latter word for the operator $\widetilde{A} = JAJ$).

For non-densely defined operators, one can also introduce a notion of J-symmetric and J-skew-symmetric linear relations, see, e.g., [31].

Let A be a linear bounded operator in H. In this case, conditions (1.2),(1.3), (1.4) mean that the matrix of the operator in an arbitrary basis $\mathcal{F}$, which is corresponding to J, will be symmetric, skew-symmetric or orthogonal, respectively. This remark and some properties of the J-form allow to obtain some simple properties of eigenvalues and eigenvectors of such matrices.

In this paper we shall obtain a J-polar decomposition for bounded operators (under some conditions). This decomposition is analogous to the polar decomposition of a bounded operator and to the J-polar decomposition in J-spaces [24]. Also, we obtain some other decompositions which are analogous to decompositions for finite-dimensional matrices in [4]. A possibility of the matrix representation for J-symmetric and J-skew-symmetric operators and its properties are studied.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}^2$ the sets of real numbers, complex numbers, positive integers, non-negative integers and the real plane, respectively. Everywhere in this paper, all Hilbert spaces are assumed to be separable, $(\cdot,\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in a Hilbert space, respectively.

For a set M in a Hilbert space H, by $\overline{M}$ we mean the closure of M in the norm $\|\cdot\|$. For $\{x_k\}_{k\in\mathbb{Z}_+}$, $x_k \in H$, we write $\mathrm{Lin}\{x_k\}_{k\in\mathbb{Z}_+} := \{y \in H : \ y = \sum_{j=0}^n \alpha_j x_j, \ \alpha_j \in \mathbb{C}, \ n \in \mathbb{Z}_+\}$; $\mathrm{span}\{x_k\}_{k\in\mathbb{Z}_+} := \overline{\mathrm{Lin}\{x_k\}_{k\in\mathbb{Z}_+}}$.

The identity operator in a Hilbert space H is denoted by E. For an arbitrary linear operator A in H, the operators $A^*, \overline{A}, A^{-1}$ mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator A, and by $\mathrm{Ker}\, A$ we mean the kernel of the operator A. By $\sigma(A)$, $\rho(A)$ we denote the spectrum of A and the resolvent set of A, respectively. We denote by $R_\lambda(A)$ the resolvent function of A, $\lambda \in \rho(A)$. Also, we denote $\Delta_A(\lambda) = (A-\lambda E)D(A)$. The norm of a bounded operator A is denoted by $\|A\|$.

By l_2 we denote the space of complex sequences $x = (x_0, x_1, x_2, ...)^T$, $x_k \in \mathbb{C}$, $k \in \mathbb{Z}_+$, with a finite norm $\|x\| = (\sum_{k=0}^{\infty} |x_k|^2)^{\frac{1}{2}}$ (the superscript T stands for the transposition).

2. Some basic properties of J-symmetric, J-skew-symmetric and J-orthogonal operators.

In this section we shall study some basic properties of J-symmetric, J-skew-symmetric and J-orthogonal operators. Some of this properties will be used in the next section, while the others could probably be useful in the future investigations. The most of this properties have their analogs in the theory of Hermitian operators. So, our aim was to develop some basic tools for the study of operators related to an antilinear involution.

2.1. **Properties of eigenvalues and eigenvectors. Properties of the J-adjoint operator.** Let J be a conjugation in a Hilbert space H. Vectors x and y are said to be *J-orthogonal*, if $[x, y]_J = 0$. The following proposition is true (for statement (i) of this Proposition see. Theorem 2 in the paper [25, p.86]).

Proposition 2.1. *Let A be a J-symmetric operator in a Hilbert space H. The following statements are true:*
(i) *Eigenvectors of the operator A which correspond to different eigenvalues are J-orthogonal;*
(ii) *If vectors x and Jx, $x \in D(A)$, are eigenvectors of the operator A, then they correspond to the same eigenvalue.*

Proof. In fact, we can write

$$\lambda_x [x, y]_J = [Ax, y]_J = [x, Ay]_J = \lambda_y [x, y]_J,$$

and therefore

$$(\lambda_x - \lambda_y)[x, y]_J = 0. \tag{2.1}$$

Suppose that $x, \overline{x} := Jx \in D(A)$ are eigenvectors of the operator A, which correspond to eigenvalues λ_x and $\lambda_{\overline{x}}$, respectively. Write (2.1) with $y = \overline{x}, \lambda_y = \lambda_{\overline{x}}$:

$$(\lambda_x - \lambda_{\overline{x}})[x, \overline{x}]_J = 0.$$

Since $[x, \overline{x}]_J = \|x\|^2 > 0$, we get $\lambda_x = \lambda_{\overline{x}}$. □

Define the following set:

$$H_{J;0} := \{x \in H : \ [x, x]_J = 0\}.$$

The following two propositions are obtained in a similar way.

Proposition 2.2. *If A is a J-skew-symmetric operator in a Hilbert space H, then the followings are true:*
(i) *Eigenvectors of the operator A, which correspond to non-zero eigenvalues, belong to the set $H_{J;0}$;*

(ii) *If λ_x, λ_y are eigenvalues of the operator A such that $\lambda_x \neq -\lambda_y$, then the corresponding eigenvectors are J-orthogonal;*
(iii) *Suppose that $x, \overline{x} := Jx \in D(A)$ are eigenvectors of the operator A, corresponding to the eigenvalues λ_x and $\lambda_{\overline{x}}$, respectively. Then $\lambda_x = -\lambda_{\overline{x}}$.*

Proposition 2.3. *Let A be a J-isometric operator in a Hilbert space H. Then the following statements are true:*
(i) *Eigenvectors of the operator A, which correspond to different from ± 1 eigenvalues belong to the set $H_{J;0}$;*
(ii) *If λ_x, λ_y are eigenvalues of the operator A such that $\lambda_x \neq \frac{1}{\lambda_y}$, then the corresponding eigenvectors are J-orthogonal;*
(iii) *Suppose that $x, \overline{x} := Jx \in D(A)$ are eigenvectors of the operator A, corresponding to the eigenvalues λ_x and $\lambda_{\overline{x}}$, respectively. Then $\lambda_x = \frac{1}{\lambda_{\overline{x}}}$.*

It is interesting to notice that in the finite-dimensional case the point 0 for a skew-symmetric matrix and points ± 1 for an orthogonal matrix are distinguished in a special manner in the spectrum, as well.

Consider a finite-dimensional Hilbert space H_n with dimension n, $n \in \mathbb{Z}_+$. In this case the conjugation J, the J-form, and J-orthogonality are defined similarly. Thus, the latter statements hold true for complex symmetric, skew-symmetric and orthogonal matrices.

Example 2.4. Consider a complex numerical matrix $A = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$. Its eigenvalues are $\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, and the corresponding normalized eigenvectors are $f_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} - i \\ 2 \end{pmatrix}$, $f_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} -\sqrt{3} - i \\ 2 \end{pmatrix}$. Vectors f_1, f_2 are not orthogonal. However, they are J-orthogonal.

Let A be an arbitrary bounded linear operator and J be a conjugation in a Hilbert space H. As it easily follows from the properties of the involution, the norm of A can be calculated in the following way:

$$\|A\| = \sup_{x,y \in H:\ \|x\| = \|y\| = 1} |[Ax, y]_J|. \tag{2.2}$$

The following result can be found in [8, Theorem 1] but we present another proof of it.

Proposition 2.5. *If A is a bounded J-symmetric operator in a Hilbert space H, then its norm can be calculated as*

$$\|A\| = \sup_{x \in H:\ \|x\| = 1} |[Ax, x]_J|. \tag{2.3}$$

Proof. Set $C := \sup_{x \in H:\ \|x\|=1} |[Ax, x]_J|$. For arbitrary elements $x, y \in H:\ x \neq \pm y$ we can write

$$[A(x+y), x+y]_J - [A(x-y), x-y]_J = 4[Ax, y]_J;$$

and

$$|[Ax, y]_J| \leq \frac{1}{4} \left(|[A(x+y), x+y]_J| + |[A(x-y), x-y]_J| \right)$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\left| \left[A\left(\frac{x+y}{\|x+y\|}\right), \frac{x+y}{\|x+y\|} \right]_J \right| \|x+y\|^2 \right. \\
&\quad \left. + \left| \left[A\left(\frac{x-y}{\|x-y\|}\right), \frac{x-y}{\|x-y\|} \right]_J \right| \|x-y\|^2 \right) \\
&\leq \frac{1}{4} C(\|x+y\|^2 + \|x-y\|^2) = \frac{1}{2} C(\|x\|^2 + \|y\|^2). \qquad (2.4)
\end{aligned}
$$

Thus, by using (2.2) and (2.4) we get

$$\|A\| = \sup_{x,y\in H:\ \|x\|=\|y\|=1} |[Ax,y]_J| \leq C.$$

On the other hand, we can write

$$C = \sup_{x,y\in H:\ \|x\|=1} |[Ax,x]_J| \leq \sup_{x,y\in H:\ \|x\|=\|y\|=1} |[Ax,y]_J| = \|A\|.$$

Therefore $C = \|A\|$. $\square$

For a J-skew-symmetric operator A the norm can not be calculated by the formula (2.3). Moreover, the following characteristic property of a J-skew-symmetric operator is true.

Proposition 2.6. *A linear operator A in a Hilbert space H is J-skew-symmetric if and only if the following equality holds*

$$[Ax,x]_J = 0, \qquad \textit{for all } x \in D(A). \qquad (2.5)$$

Proof. We first notice that the properties of the involution imply that $[x,y]_J = [y,x]_J, \quad x,y \in H$. Let us check the necessity. From relation (1.3) it follows that

$$[Ax,x]_J = -[x,Ax]_J = -[Ax,x]_J,$$

and therefore (2.5) holds true.
Let us check the sufficiency. By using (2.5) we write

$$
\begin{aligned}
0 = [A(x+y), x+y]_J &= [Ax,x]_J + [Ax,y]_J + [Ay,x]_J + [Ay,y]_J \\
&= [Ax,y]_J + [Ay,x]_J, \qquad x,y \in D(A).
\end{aligned}
$$

From this relation we obtain that $[Ax,y]_J = -[Ay,x]_J = -[x,Ay]_J$. $\square$

Let J be a conjugation in a Hilbert space H and A be an arbitrary linear operator in H. The following operator: $\widetilde{A} := \widetilde{(A)}_J := JAJ$, we shall call J-*adjoint* to the operator A. We first notice that $\widetilde{\widetilde{A}} = A$.

Lemma 2.7. *For a linear operator A in a Hilbert space H equalities $\overline{D(A)} = H$ and $\overline{D(\widetilde{A})} = H$ are equivalent. Equalities $\overline{R(A)} = H$ and $\overline{R(\widetilde{A})} = H$ are equivalent, as well.*

Let us formulate some properties of the J-adjoint operator as propositions.

Proposition 2.8. *Let A be a linear operator in a Hilbert space H with a dense domain and J be a conjugation in H. Then*

$$\widetilde{A^*} = (\widetilde{A})^*.$$

Proof. Choose an arbitrary element $g \in D((\widetilde{A})^*)$. On the one hand, we have

$$(\widetilde{A}x, g) = (x, (\widetilde{A})^* g) = (JJx, JJ(\widetilde{A})^* g) = \overline{(Jx, J(\widetilde{A})^* g)}$$
$$= (J(\widetilde{A})^* g, Jx), \qquad x \in D(\widetilde{A}).$$

On the other hand, we can write

$$(\widetilde{A}x, g) = (JAJx, JJg) = \overline{(AJx, Jg)} = (Jg, AJx), \qquad x \in D(\widetilde{A}).$$

Comparing the right hand sides we obtain that

$$(AJx, Jg) = (Jx, J(\widetilde{A})^* g),$$

and therefore $Jg \in D(A^*)$, $A^* Jg = J(\widetilde{A})^* g$. Multiplying by J both sides of the latter equality we get $\widetilde{A^*} g = (\widetilde{A})^* g$. Therefore

$$(\widetilde{A})^* \subseteq \widetilde{A^*}. \tag{2.6}$$

To obtain the inverse inclusion, one should write the inclusion (2.6) with the operator $\widetilde{A}$, and then to calculate the J-adjoint operators for the both sides (the inclusion remains true). □

Proposition 2.9. *Let A be a linear operator in a Hilbert space H and J be a conjugation in H. Suppose that operators A and $\widetilde{A}$ admit closures. Then the following equality is true*

$$\widetilde{\overline{A}} = \overline{\widetilde{A}}.$$

Proof. Choose an arbitrary element $g \in D(\overline{\widetilde{A}})$. Then there exists a sequence $x_n \in D(\widetilde{A}), n \in \mathbb{Z}_+$, such that $x_n \to x$, $\widetilde{A} x_n = JAJx_n \to \overline{\widetilde{A}} x$ as $n \to \infty$. By continuity of the operator J we obtain that

$$Jx_n \to Jx, \quad AJx_n \to J\overline{\widetilde{A}}x.$$

Consequently, we have $Jx \in D(\overline{A})$ and $\overline{A}Jx = J\overline{\widetilde{A}}x$. Therefore $x \in D(\widetilde{\overline{A}})$ and $\widetilde{\overline{A}}x = \overline{\widetilde{A}}x$. We conclude that

$$\overline{\widetilde{A}} \subseteq \widetilde{\overline{A}}. \tag{2.7}$$

In order to obtain the inverse inclusion, we write the inclusion (2.7) for the operator $\widetilde{A}$, and then calculate the J-adjoint operators for the both sides. □

Proposition 2.10. *Let A be a linear invertible operator in a Hilbert space H and J be a conjugation in H. Then the operator $\widetilde{A}$ is also invertible and the following equality is true*

$$\widetilde{A^{-1}} = (\widetilde{A})^{-1}. \tag{2.8}$$

Proof. Since $\widetilde{A^{-1}}\widetilde{A} = E|_{D(\widetilde{A})}$, and $D(\widetilde{A^{-1}}) = JD(A^{-1}) = JR(A) = R(\widetilde{A})$, the operator $\widetilde{A}$ is invertible and relation (2.8) is true. □

Notice that the property of a J-symmetric operator (1.2) by virtue of the J-adjoint operator can be written as follows:

$$(Ax, y) = (x, \widetilde{A}y), \qquad x \in D(A),\ y \in D(\widetilde{A}). \tag{2.9}$$

Properties of a J-skew-symmetric operator (1.3) and of a J-isometric operator(1.4) can be written as

$$(Ax, y) = -(x, \widetilde{A}y), \qquad x \in D(A),\ y \in D(\widetilde{A}), \tag{2.10}$$

and

$$(Ax, \widetilde{A}y) = (x, y), \qquad x \in D(A),\ y \in D(\widetilde{A}), \tag{2.11}$$

respectively.

Now we shall assume that the operator A is densely defined. In this case, from condition (2.11) it follows that the operator A is invertible. In fact, the equality $Ax = 0$ implies the equality $(x, y) = 0$ on a dense set $D(\widetilde{A})$. Thus, a densely defined J-isometric operator is always invertible.

In the case of a densely defined operator A, conditions (2.9), (2.10), (2.11) are equivalent to the following conditions

$$A \subseteq (\widetilde{A})^*,$$

$$A \subseteq -(\widetilde{A})^*,$$

and

$$A^{-1} \subseteq (\widetilde{A})^*,$$

respectively. From these relations it follows that densely defined J-symmetric and J-skew-symmetric operators admit the closures. Relations (1.2),(1.3) imply that their closures are J-symmetric or J-skew-symmetric operators, respectively. For a densely defined J-isometric operator one can only say that its inverse operator admits the closure. However, from relation (1.4) it follows that the inverse operator to a J-isometric operator is J-isometric, as well. Consequently, if the range of the original J-isometric operator is dense then it admits the closure. In this case, relation (1.4) implies that this closure will be J-isometric.

Proposition 2.11. *Let A be a linear operator in a Hilbert space H and J be a conjugation in H. If the operator A is J-symmetric, J-skew-symmetric or J-isometric, then the same is true for the operator $\widetilde{A} = JAJ$, as well.*

Proof. The statement about a J-symmetric (J-skew-symmetric, J-isometric) operator follows from relation (2.9) (respectively (2.10), (2.11)), if we take into account that $A = \widetilde{\widetilde{A}}$. $\square$

For an element $x \in H$ and a set $M \subseteq H$ we write $x \perp_J M$, if $x \perp_J y$, for all $y \in M$. For a set $M \subseteq H$ we denote $M_J^{\perp} = \{x \in H :\ x \perp_J y,\ y \in M\}$.

It is known that the residual spectrum of a J-self-adjoint operator is empty. It follows from the theorem below.

Theorem 2.12. *([25, Theorem 4, p.87]) Let A be a J-self-adjoint operator in a Hilbert space H. A complex number λ is an eigenvalue of A if and only if*

$$\overline{\Delta_A(\lambda)} \neq H.$$

In this case, $(\Delta_A(\lambda))_J^{\perp}$ will be the eigen subspace which corresponds to λ.

We shall obtain analogous results for J-skew-symmetric and J-isometric operators.

Theorem 2.13. *Let A be a J-skew-self-adjoint operator in a Hilbert space H. A complex value λ is an eigenvalue of A if and only if*

$$\overline{\Delta_A(-\lambda)} \neq H. \tag{2.12}$$

In this case, $(\Delta_A(-\lambda))_J^{\perp}$ will be the eigen subspace which corresponds to λ.

Proof. Necessity. Let x be an arbitrary eigenvector of the operator A which corresponds to an eigenvalue λ. Since A is skew-symmetric, we can write for an arbitrary $y \in D(A)$

$$0 = [(A - \lambda E)x, y]_J = -[x, (A + \lambda E)y]_J.$$

Therefore $x \perp_J \Delta_A(-\lambda)$ and by continuity of $[\cdot, \cdot]_J$ we get

$$x \perp_J \overline{\Delta_A(-\lambda)}. \tag{2.13}$$

Since $[x, Jx] = \|x\|^2 > 0$, then $Jx \notin \overline{\Delta_A(-\lambda)}$ and therefore $\overline{\Delta_A(\lambda)} \neq H$.
Sufficiency. Suppose that equality (2.12) is true. Then there exists $0 \neq y \in H$ such that

$$(z, y) = 0, \qquad z \in \overline{\Delta_A(-\lambda)}. \tag{2.14}$$

Therefore $((A+\lambda E)x, y) = 0$, and from this relation we get $(Ax, y) = (x, \overline{(-\lambda)}y)$, $x \in D(A)$. Thus, we have $y \in D(A^*)$ and

$$A^* y = -\overline{\lambda} y.$$

Since A is J-skew-self-adjoint, we have $A^* = -\widetilde{A}$, and we obtain

$$\widetilde{A} y = \overline{\lambda} y.$$

From this relation it follows that $Jy \neq 0$ is an eigenvector of the operator A with the eigenvalue λ.

Let us show that the following set

$$V(\lambda) := (\Delta_A(-\lambda))_J^{\perp} \backslash \{0\},$$

is a set of all eigenvectors of the operator A, corresponding to an eigenvalue λ. Denote the latter set by $S(\lambda)$. By (2.13), the inclusion $S(\lambda) \subseteq V(\lambda)$ is true. On the other hand, if $x \in V(\lambda)$, then for $y := Jx$ relation (2.14) is true. Repeating the arguments which follow after this formula we conclude that x is an eigenvector of the operator A which corresponds to λ. Thus, the inverse inclusion is true, as well.

Finally, since $A = (\widetilde{A})^*$, then A is closed. Therefore $(\Delta_A(-\lambda))_J^{\perp}$ is the eigen subspace of the operator A, which corresponds to λ. $\square$

Corollary 2.14. *The point 0 can not belong to the residual spectrum of a J-skew-self-adjoint operator.*

In an analogous manner, the following result for J-unitary operators is established.

Theorem 2.15. *Let A be a J-unitary operator in a Hilbert space H. A complex number λ is an eigenvalue of A if and only if*

$$\overline{\Delta_A\left(\frac{1}{\lambda}\right)} \neq H.$$

In this case, $(\Delta_A(\frac{1}{\lambda}))^{\perp}_J$ is the eigen subspace which corresponds to λ.

Corollary 2.16. *Points ± 1 can not belong to the residual spectrum of a J-unitary operator.*

From relations (2.9),(2.10) we see that a J-symmetric (J-skew-symmetric) operator defined on the whole H is a bounded J-self-adjoint (respectively J-skew-self-adjoint) operator. The following statements are also true.

Proposition 2.17. *(*[25, Theorem 1, p.85-86], [15, Theorem 3, p.69]*) Let A be a linear densely defined operator in a Hilbert space H which is J-symmetric (J-skew-symmetric). Suppose that $R(A) = H$. Then the operator A is a J-self-adjoint (respectively a J-skew-self-adjoint) operator.*

Proposition 2.18. *Let A be a linear densely defined operator in a Hilbert space H which is J-symmetric (J-skew-symmetric). Suppose that $\overline{R(A)} = H$. Then the operator A is invertible and the operator A^{-1} is a J-symmetric (respectively a J-skew-symmetric) operator, as well.*

Proof. In view of analogous considerations, we shall only prove this Proposition for the case of a J-skew-symmetric operator A. Notice that $\operatorname{Ker} A^* = H \ominus \overline{R(A)} = \{0\}$. Thus, the operator A^* is invertible. Since A is J-skew-symmetric, the inclusion $\widetilde{A} \subseteq -A^*$ is true and therefore $\widetilde{A}$ is invertible, as well. By Proposition 2.10 we conclude that the operator A has the inverse. From the inclusion $\widetilde{A} \subseteq -A^*$ it follows that

$$(\widetilde{A})^{-1} \subseteq -(A^*)^{-1}. \tag{2.15}$$

Notice that $\overline{D(A^{-1})} = \overline{R(A)} = H$. Thus, we can state that $(A^*)^{-1} = (A^{-1})^*$. Using this equality and Proposition 2.10, by (2.15) we obtain the following inclusion

$$\widetilde{A^{-1}} \subseteq -(A^{-1})^*.$$

This means that the operator A^{-1} is J-skew-symmetric. $\square$

Proposition 2.19. *Let A be a J-self-adjoint (J-skew-self-adjoint) operator in a Hilbert space H. Suppose that $\overline{R(A)} = H$. Then the operator A is invertible and the operator A^{-1} is a J-self-adjoint (respectively a J-skew-self-adjoint) operator, as well.*

Proof. In view of analogous considerations, we shall give the proof only for the case of a J-self-adjoint operator A. By Proposition 2.18 the operator A is invertible. By Proposition 2.10 the operator $\widetilde{A}$ is invertible, as well. From Lemma 2.7 it follows that $\overline{R(\widetilde{A})} = H$ and $\overline{D(\widetilde{A})} = H$. Thus, we have $\overline{D((\widetilde{A})^{-1})} = H$. Consequently, the following equality is true $((\widetilde{A})^*)^{-1} = ((\widetilde{A})^{-1})^*$. Since the operator A is J-self-adjoint, the last equality can be written as $A^{-1} = ((\widetilde{A})^{-1})^*$. By Proposition 2.10 we obtain $A^{-1} = (\widetilde{A^{-1}})^*$. This shows that the operator A^{-1} is J-self-adjoint. $\square$

2.2. **Matrix representations of J-symmetric and J-skew-symmetric operators.** We shall study matrix representations of J-symmetric and J-skew-symmetric operators. We shall obtain properties which are analogous to the properties of symmetric operators. Let J be a conjugation in a Hilbert space H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ be an orthonormal basis in H which corresponds to J. Let A be a linear operator in H which is J-symmetric (J-skew-symmetric) and such that $\mathcal{F} \subset D(A)$.

Define the matrix of the operator A in the basis $\mathcal{F}$: $A_M := (a_{i,j})_{i,j\in\mathbb{Z}_+}$, $a_{i,j} = (Af_j, f_i)$. It is not hard to check that this matrix is complex symmetric (skew-symmetric) in the case of a J-symmetric (respectively a J-skew-symmetric) operator A. Notice that the columns of this matrix are square summable, i.e. belong to l^2.

In the case when the set $D(A) \cap D(A^*)$ is dense in H an arbitrary linear operator A in a Hilbert space H can be represented by the matrix multiplication [37]. In particular, it is true for symmetric operators. As far as we know, such a possibility for other classes of operators was not established. This property is true for J-symmetric and J-skew-symmetric operators.

Theorem 2.20. *Let J be a conjugation in a Hilbert space H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ be an orthonormal basis in H which corresponds to J. Let A be a linear operator in H which is J-symmetric (J-skew-symmetric) and such that $\mathcal{F} \subset D(A)$. Let $A_M = (a_{i,j})_{i,j\in\mathbb{Z}_+}$ be the matrix of the operator A in the basis $\mathcal{F}$. Then*

$$Ag = \sum_{i=0}^{\infty} y_i f_i, \quad y_i = \sum_{k=0}^{\infty} a_{i,k} g_k, \; g = \sum_{k=0}^{\infty} g_k f_k \in D(A).$$

Proof. Let us prove the statement of the Theorem for a J-skew-symmetric operator. For the case of a J-symmetric operator the proof is similar. Choose an arbitrary element $g = \sum_{k=0}^{\infty} g_k f_k \in D(A)$. Since the matrix A_M is skew-symmetric, by (1.3) we get

$$y_i = (Ag, Jf_i) = -(Af_i, Jg) = -(\sum_{k=0}^{\infty} (Af_i, f_k) f_k, \sum_{l=0}^{\infty} \overline{g_l} f_l)$$

$$= -\sum_{k=0}^{\infty} (Af_i, f_k) g_k = -\sum_{k=0}^{\infty} a_{k,i} g_k = \sum_{k=0}^{\infty} a_{i,k} g_k.$$

$\square$

Let us study how strong the matrix A_M of the operator A (which was considered above) determines this operator. Since J-symmetric and J-skew-symmetric operators admit J-symmetric (respectively J-skew-symmetric) closures, we shall suppose that the operator A is closed. By the matrix multiplication, the matrix A_M defines the operator T on $L := \mathrm{Lin}\,\mathcal{F}$. It is easy to check that this operator is J-symmetric (J-skew-symmetric) in the case of a J-symmetric (respectively a J-skew-symmetric) operator A. This operator admits the closure $\overline{T}$ which is also a J-symmetric (J-skew-symmetric) operator. If $A = \overline{T}$, then the basis $\mathcal{F}$ is called a *basis of the matrix representation* of the operator A.

Theorem 2.21. *Let an arbitrary complex semi-infinite symmetric matrix (skew-symmetric matrix) $M = (m_{i,j})_{i,j\in\mathbb{Z}_+}$ with columns in l^2 be given. Then there exist a Hilbert space H, a conjugation J in H, a J-symmetric (respectively a J-skew-symmetric) operator in H and a corresponding orthonormal basis $\mathcal{F}$ in H, $\mathcal{F} \subset D(A)$, such that the matrix M is the matrix of the operator A in the basis $\mathcal{F}$ and $\mathcal{F}$ is a basis of the matrix representation for A.*

Proof. For an arbitrary complex semi-infinite symmetric (skew-symmetric) matrix M with columns in l^2 we choose an arbitrary Hilbert space H and an arbitrary orthonormal basis $\mathcal{F}$ in H. Then we define a conjugation in H by formula (1.1). Using the above procedure we construct the operator $\overline{T}$. This is the required operator. □

If $\mathcal{F}$ is a basis of the matrix representation for a closed J-symmetric (J-skew-symmetric) operator A then $\mathcal{F}$ is a basis of the matrix representation for the J-adjoint operator $\widetilde{A} = JAJ$, as well. In fact, by Proposition 2.11 the operator $\widetilde{A}$ is J-symmetric (respectively J-skew-symmetric). By continuity of the operator J it follows that $\widetilde{A}$ is closed. If we choose an arbitrary element $x \in D(\widetilde{A})$ then $Jx \in D(A)$ and there exists a sequence $\widehat{x}_n \in L := \mathrm{Lin}\{f_k\}_{k\in\mathbb{Z}_+}$, $n \in \mathbb{Z}_+$: $\widehat{x}_n \to Jx$, $A\widehat{x}_n \to AJx$, $n \to \infty$. Then we have $J\widehat{x}_n \in L$, $J\widehat{x}_n \to x$, $JA\widehat{x}_n = \widetilde{A}J\widehat{x}_n \to JAJx = \widetilde{A}x$, $n \to \infty$.

Theorem 2.22. *Let J be a conjugation in a Hilbert space H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ be a corresponding orthonormal basis in H. Suppose that A is a closed J-symmetric (J-skew-symmetric) operator in H, $\mathcal{F} \subset D(A)$, and $\mathcal{F}$ is a basis of the matrix representation for the operator A. Let $a_{i,j} = (Af_j, f_i)$, $i,j \in \mathbb{Z}_+$. Define an operator B in the following way:*

$$Bg = \sum_{i=0}^{\infty} y_i f_i, \qquad y_i = \sum_{k=0}^{\infty} a_{i,k} g_k, \quad g = \sum_{k=0}^{\infty} g_k f_k \in D_B, \tag{2.16}$$

on a set $D_B = \{g = \sum_{k=0}^{\infty} g_k f_k \in H :\ \sum_{i=0}^{\infty} |\sum_{k=0}^{\infty} a_{i,k} g_k|^2 < \infty\}$.
Then $A \subseteq A^T = B$ (respectively $A \subseteq -A^T = B$).

Without conditions that A is closed and $\mathcal{F}$ is a basis of the matrix representation for A, one can only state that $A \subseteq A^T \subseteq B$ (respectively $A \subseteq -A^T \subseteq B$).

Proof. The proof will be given for the case of a J-skew-self-adjoint operator A. The case of a J-symmetric operator is considered similarly. We first show that

$-A^T = -(\widetilde{A})^* \subseteq B$. Choose an arbitrary $g \in D(-(\widetilde{A})^*)$ and set $-(\widetilde{A})^* g = g^*$. Let $g = \sum_{k=0}^\infty g_k f_k$, $g^* = \sum_{i=0}^\infty \widehat{y}_i f_i$. We can write

$$\widehat{y}_i = (g^*, f_i) = (-(\widetilde{A})^* g, f_i) = -(g, \widetilde{A} f_i) = -(\sum_{k=0}^\infty g_k f_k, \sum_{j=0}^\infty (\widetilde{A} f_i, f_j) f_j)$$

$$= -\sum_{k=0}^\infty g_k \overline{(\widetilde{A} f_i, f_k)} = -\sum_{k=0}^\infty a_{k,i} g_k = \sum_{k=0}^\infty a_{i,k} g_k, \quad i \in \mathbb{Z}_+.$$

Therefore $\sum_{i=0}^\infty |\sum_{k=0}^\infty a_{i,k} g_k|^2 < \infty$ and, hence, we get $g \in D_B$. Also we have $-(\widetilde{A})^* g = g^* = Bg$. Thus, we obtain an inclusion $-(\widetilde{A})^* \subseteq B$. We have not used that A is closed and that $\mathcal{F}$ is a basis of the matrix representation for A. The inclusion $A \subseteq -(\widetilde{A})^*$ is obvious.

Let us prove the inclusion $B \subseteq -A^T$. As it was shown above, the operator $\widetilde{A}$ is closed and $\mathcal{F}$ is a basis of the matrix representation for $\widetilde{A}$, as well. Choose an arbitrary $g \in D_B$, $g = \sum_{k=0}^\infty g_k f_k$. Since the matrix of the operator A is skew-symmetric, we can write

$$(\widetilde{A} f_i, g) = (\sum_{j=0}^\infty (\widetilde{A} f_i, f_j) f_j, \sum_{k=0}^\infty g_k f_k) = \sum_{k=0}^\infty (\widetilde{A} f_i, f_k) \overline{g_k}$$

$$= \sum_{k=0}^\infty \overline{a_{k,i} g_k} = -\sum_{k=0}^\infty \overline{a_{i,k} g_k} = -\overline{\sum_{k=0}^\infty a_{i,k} g_k} = -\overline{y_i}, \quad i \in \mathbb{Z}_+;$$

$$(Bg, f_i) = y_i, \quad i \in \mathbb{Z}_+.$$

Therefore

$$-(\widetilde{A} f_i, g) = \overline{(Bg, f_i)} = (f_i, Bg),$$

and

$$-(\widetilde{A} f, g) = (f, Bg), \qquad f \in \mathrm{Lin}\{f_k\}_{k \in \mathbb{Z}_+} =: L.$$

For an arbitrary $f \in D(\widetilde{A})$ there exists a sequence $\{f^k\}_{k \in \mathbb{Z}_+}$, $f^k \in L$: $f^k \to f$, $\widetilde{A} f^k \to \widetilde{A} f$, as $k \to \infty$. Passing to the limit as $k \to \infty$ in the equality

$$-(\widetilde{A} f^k, g) = (f^k, Bg)$$

and using the continuity of the scalar product we obtain

$$-(\widetilde{A} f, g) = (f, Bg), \qquad f \in D(\widetilde{A}).$$

Thus, we have $g \in D((\widetilde{A})^*)$ and $(\widetilde{A})^* g = -Bg$. Therefore we get an inclusion $B \subseteq -(\widetilde{A})^*$. $\square$

Let J be a conjugation in a Hilbert space H and $\mathcal{F} = \{f_k\}_{k \in \mathbb{Z}_+}$ be a corresponding orthonormal basis in H. Let A be a closed J-symmetric (J-skew-symmetric) operator in H and $\mathcal{F} \subset D(A)$. Set $a_{i,j} = (Af_j, f_i)$, $i, j \in \mathbb{Z}_+$, and define an operator B by formula (2.16). Is the operator B J-symmetric (J-skew-symmetric)? We first notice that the domain of an operator $\widetilde{B} = JBJ$ is a set

$$D(\widetilde{B}) = \{h = \sum_{k=0}^\infty h_k f_k \in H : \sum_{i=0}^\infty |\sum_{k=0}^\infty \overline{a_{i,k}} h_k|^2 < \infty\}.$$

If $h = \sum_{k=0}^{\infty} h_k f_k \in D(\widetilde{B})$, then

$$\widetilde{B}h = \sum_{i=0}^{\infty} (\sum_{k=0}^{\infty} \overline{a_{i,k}} h_k) f_i.$$

Choose arbitrary elements $g = \sum_{k=0}^{\infty} g_k f_k \in D_B$ and $h = \sum_{k=0}^{\infty} h_k f_k \in D(\widetilde{B})$. Using relations (2.9),(2.10) it is easy to check that the operator B is J-symmetric (J-skew-symmetric), if the following equalities are true (for all $g \in D_B, h \in D(\widetilde{B})$)

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{i,k} g_k \overline{h_i} = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} a_{i,k} g_k \overline{h_i}.$$

In the latter case, the last theorem can be applied with the operator B to obtain that the operator B is J-self-adjoint (J-skew-self-adjoint).

When does there exist a basis of the matrix representation for a closed J-symmetric (J-skew-symmetric) operator? For an arbitrary closed operator there exists an orthonormal basis such that the operator is the closure of its values on the linear span of this basis. The proof for symmetric operators can be found in [2]. It remains valid in the general case, as well. The difficulty for the case of J-symmetric (J-skew-symmetric) operators is that this new basis can be a basis which does not correspond to the conjugation J. So, the above question remains open.

3. A J-polar decomposition of bounded operators.

We shall extend to the case of J-symmetric, J-skew-symmetric and J-isometric operators a series of properties of finite-dimensional complex symmetric, skew-symmetric and orthogonal matrices (see [4]). We shall follow the main ideas of the proofs in the finite-dimensional case with some necessary changes. We start with the following lemma which generalizes [4, Chapter XI, Lemma 1]. In this lemma (and in the lemma which follows below) we shall use spectral representations of commuting self-adjoint operators and their spectral measures instead of using of the canonical quasi-diagonal form of commuting normal matrices which was used in the finite-dimensional case.

Lemma 3.1. *Let A be a bounded self-adjoint and J-isometric operator in a Hilbert space H. Then the operator A admits the following representation:*

$$A = Ie^{iK}, \tag{3.1}$$

where I is a bounded self-adjoint J-real involutory ($I^2 = E$) operator in H, and K is a commuting with I bounded skew-self-adjoint J-real operator in H.

If $A \geq 0$ then one can choose $I = E$.

Proof. Since the operator A is J-isometric and bounded, from (1.4) we obtain $A^*JA = J$, and $A^*\widetilde{A} = E$. Since A is self-adjoint, then

$$A\widetilde{A} = E. \tag{3.2}$$

For the operator A we can write

$$A = S + iT,$$

where $S = \frac{1}{2}(A + \widetilde{A})$, $T = \frac{1}{2i}(A - \widetilde{A})$. Thus, the operators S and T are J-real, the operator S is self-adjoint and J-self-adjoint, and the operator T is skew-self-adjoint and J-skew-self-adjoint. Since $\widetilde{A} = S - iT$, from relation (3.2) we get

$$E = A\widetilde{A} = (S + iT)(S - iT) = S^2 + T^2 + i(TS - ST).$$

From this relation it follows that operators T and S commute and

$$S^2 + T^2 = E. \tag{3.3}$$

Since operators S and iT are commuting bounded self-adjoint operators, they admit spectral representations

$$S = \int_L \lambda dE_\lambda, \qquad iT = \int_L z dF_z,$$

where $E_\lambda,\ F_z$ are commuting resolutions of unity of the operators, and $L = (l_1, l_2]$, $l_1, l_2 \in \mathbb{R}$, is a finite interval of the real line which contains the spectra of the operators S and T. From equality (3.3), by using spectral resolutions we get

$$\int_L \int_L (\lambda^2 - z^2 - 1) dE_\lambda dF_z = 0,$$

where the integral means a limit in the norm of H of the corresponding Riemann-Stieltjes type sums (in the plane).

A point $(\lambda_0, z_0) \in \mathbb{R}^2$ we call *a point of increase* for the measure $dE_\lambda dF_z$, if for an arbitrary number $\varepsilon > 0$, there exists an element $x \in H$ such that

$$(E_{\lambda_0+\varepsilon} - E_{\lambda_0-\varepsilon})(F_{z_0+\varepsilon} - F_{z_0-\varepsilon})x \neq 0,$$

or, equivalently,

$$((E_{\lambda_0+\varepsilon} - E_{\lambda_0-\varepsilon})(F_{z_0+\varepsilon} - F_{z_0-\varepsilon})x, x) > 0. \tag{3.4}$$

For a point of increase $(\lambda_0, z_0) \in \mathbb{R}^2$ of the measure $dE_\lambda dF_z$

$$\lambda_0^2 - z_0^2 - 1 = 0.$$

In fact, if the latter equality is not true for a point of increase $u_0 = (\lambda_0, z_0) \in \mathbb{R}^2$, then $|\lambda^2 - z^2 - 1| \geq a,\ a > 0$, in a neighborhood

$$U = U(\lambda_0, z_0; \varepsilon) = \{(\lambda, z) \in \mathbb{R}^2 :\ \lambda_0 - \varepsilon < \lambda \leq \lambda_0 + \varepsilon,\ z_0 - \varepsilon < z \leq z_0 + \varepsilon\}, \quad \varepsilon > 0,$$

of the point u_0. For this number ε, there exists an element $x \in H$ such that (3.4) is true. But

$$0 = \left\| \int_L \int_L (\lambda^2 - z^2 - 1) dE_\lambda dF_z x \right\|^2 = \int_L \int_L |\lambda^2 - z^2 - 1|^2 (dE_\lambda dF_z x, x)$$

$$\geq \int \int_U |\lambda^2 - z^2 - 1|^2 (dE_\lambda dF_z x, x) \geq a^2((E_{\lambda_0+\varepsilon} - E_{\lambda_0-\varepsilon})(E_{z_0+\varepsilon} - E_{z_0-\varepsilon})x, x) > 0.$$

If two continuous functions $\varphi(\lambda, z)$ and $\widetilde{\varphi}(\lambda, z)$ on $L^2 = \{(\lambda, z) \in \mathbb{R}^2 :\ \lambda, z \in L\}$ coincide in the points of increase of the measure $dE_\lambda dF_z$, then

$$\int_L \int_L \varphi(\lambda, z) dE_\lambda dF_z = \int_L \int_L \widetilde{\varphi}(\lambda, z) dE_\lambda dF_z.$$

In fact,

$$\left\| \int_L \int_L (\varphi(\lambda,z) - \widetilde{\varphi}(\lambda,z)) dE_\lambda dF_z x \right\|^2 = \int_L \int_L |\varphi(\lambda,z) - \widetilde{\varphi}(\lambda,z)|^2 (dE_\lambda dF_z x, x),$$

and it remains to notice that $(dE_\lambda dF_z x, x)$ is a positive measure on L^2, and the function under the integral is equal to zero in all points of increase of this measure.

Consider a set $\Gamma \subset \mathbb{R}^2$, which consists of points $(\lambda, z) \in \mathbb{R}^2$, such that

$$\lambda^2 - z^2 - 1 = 0. \tag{3.5}$$

From (3.5) it follows that for all points of the set Γ $|\lambda| = \sqrt{1+z^2}$ (where we mean the arithmetic value of the root). Hence, for all points of Γ

$$\lambda = \operatorname{sgn}(\lambda)\sqrt{1+z^2}, \tag{3.6}$$

where

$$\operatorname{sgn}(\lambda) = \begin{cases} 1, & \lambda > 0, \\ -1, & \lambda \leq 0 \end{cases}. \tag{3.7}$$

By the identity $z = \operatorname{sh} \operatorname{arcsh} z$, the equality (3.6) can be rewritten in the following form

$$\lambda = \operatorname{sgn}(\lambda)\sqrt{\operatorname{ch}^2(\operatorname{arcsh} z)} = \operatorname{sgn}(\lambda) \operatorname{ch}(\operatorname{arcsh} z),$$

in view of positivity of the hyperbolic cosine function. By this representation we can write

$$\begin{aligned} A = S + iT &= \int_L \int_L (\lambda + z) dE_\lambda dF_z = \int_L \int_L (\operatorname{sgn}(\lambda) \operatorname{ch}(\operatorname{arcsh} z) + z) dE_\lambda dF_z \\ &= \int_L \int_{L_+} e^{\operatorname{arcsh} z} dE_\lambda dF_z + \int_L \int_{L_-} (-e^{-\operatorname{arcsh} z}) dE_\lambda dF_z, \end{aligned} \tag{3.8}$$

where $L_+ = (0,\infty) \cap L,\ L_- = (-\infty, 0] \cap L.$

Define the following operator

$$V = \int_L \int_L \operatorname{sgn}(\lambda) \operatorname{arcsh} z dE_\lambda dF_z = \int_L \operatorname{sgn}(\lambda) dE_\lambda \int_L \operatorname{arcsh} z dF_z. \tag{3.9}$$

The operator V is bounded self-adjoint and J-imaginary (by this we mean $JV = -VJ$). In fact, since the operator S is J-real, then its resolution of unity E_λ commutes with J (see [37]). Therefore the operator

$$I := \int_L \operatorname{sgn}(\lambda) dE_\lambda, \tag{3.10}$$

is a bounded J-real self-adjoint involutory operator. On the other hand, since $\operatorname{arcsh} x$, $x \in \mathbb{R}$, is an odd function, we can approximate it uniformly on $[-L, L]$ by its Fejér trigonometric sums with the sine functions. Each sine function in the Fejér sums we can approximate by a part of its power expansion. Thus, elementary calculations show that for arbitrary $\varepsilon > 0$ there exists a polynomial

$$p_\varepsilon(x) = \sum_{k=0}^{n_\varepsilon} a_{\varepsilon,k} x^{2k+1}, \qquad a_{\varepsilon,k} \in \mathbb{R},\ n_\varepsilon \in \mathbb{Z}_+,$$

such that

$$|\operatorname{arcsh} x - p_\varepsilon(x)| < \varepsilon, \qquad \forall x \in [-L, L].$$

Consider operators

$$A_n = p_{\frac{1}{n}}(iT) = \int_L p_{\frac{1}{n}}(z) dF_z = i \sum_{k=0}^{n_\varepsilon} (-1)^k a_{\varepsilon,k} T^{2k+1}, \qquad n \in \mathbb{N}.$$

Since T is J-real, the operators A_n are J-imaginary. For arbitrary $x \in H$ we have

$$\|A_n x - \operatorname{arcsh}(iT)x\|^2 = \int_L |p_{\frac{1}{n}}(z) - \operatorname{arcsh}(z)|^2 d(F_z x, x) \leq \frac{\|x\|^2}{n^2}.$$

Therefore

$$A_n x \to \operatorname{arcsh}(iT)x, \qquad \forall x \in H.$$

Consequently, we can write

$$-A_n x = J A_n J x \to J \operatorname{arcsh}(iT) J x, \qquad \forall x \in H,$$

and

$$-A_n x \to -\operatorname{arcsh}(iT)x, \qquad \forall x \in H.$$

So, the operator $\operatorname{arcsh}(iT)$ is J-imaginary.
From relations (3.8),(3.9),(3.10) we conclude that

$$A = I e^V.$$

Set $K = -iV$, and we obtain the required representation (3.1).

If the operator A is positive then

$$I = Ae^{-V} = (e^{-\frac{V}{2}})^* A e^{-\frac{V}{2}},$$

is positive, as well. Therefore I is a positive square root of E. By the uniqueness of such a root we conclude that $I = E$. □

By virtue of Lemma 3.1 we can prove a generalization of [4, Chapter XI, Theorem 1].

Theorem 3.2. *Let A be a bounded J-unitary operator in a Hilbert space H. The operator A admits the following representation:*

$$A = Re^{iK}, \tag{3.11}$$

where R is a J-real unitary operator in H, and K is a bounded J-real skew-self-adjoint operator in H.

Proof. Suppose that representation (3.11) is true. Then

$$A^* A = e^{iK} R^* R e^{iK} = e^{2iK}.$$

Now we shall drop the assumption of existence of representation (3.11) and notice that the operator $G := A^* A$ is positive self-adjoint and J-unitary. In fact, since the operator A is bounded by assumption and J-unitary, then A^* is also bounded and J-unitary. The product of bounded J-unitary operators is a bounded J-unitary operator. This is verified directly. By Lemma 3.1 we find a bounded J-real skew-self-adjoint operator K such that

$$G = e^{2iK}. \tag{3.12}$$

Set

$$R = Ae^{-iK}. \tag{3.13}$$

By (3.12) we can write

$$R^* R = e^{-iK} A^* A e^{-iK} = E.$$

Hence, the operator R is unitary. We can write

$$Je^{-iK}J = J(\cos(iK) - i\sin(iK))J = \cos(iK) + i\sin(iK),$$

since the operator iK is J-real and its resolution of unity commutes with J. Consequently, we have

$$Je^{-iK}J = e^{iK} = (e^{-iK})^{-1}, \tag{3.14}$$

and the operator e^{-iK} is J-unitary. By (3.13),(3.14) and using that the operator A is J-unitary we conclude that

$$R^{-1} = e^{iK}A^{-1} = Je^{-iK}JJA^*J = J(Ae^{-iK})^*J = \widetilde{(R^*)}.$$

Therefore the operator R is J-unitary. Then $R^{-1} = R^* = JR^*J$, and therefore R^* is a J-real operator. Using the matrix representations of the operators R^* and R in an arbitrary corresponding basis we conclude that the operator R is J-real. $\square$

The following lemma generalizes [4, Chapter XI, Lemma 2].

Lemma 3.3. *Let A be a J-self-adjoint and unitary operator in a Hilbert space H. The operator A admits the following representation:*

$$A = e^{iS}, \tag{3.15}$$

where S is a bounded J-real self-adjoint operator in H.

Proof. For the J-self-adjoint operator A $A^* = \widetilde{A}$, and we can write

$$A = S + iT,$$

where $S = \frac{1}{2}(A + \widetilde{A}) = \frac{1}{2}(A + A^*)$, $T = \frac{1}{2i}(A - \widetilde{A}) = \frac{1}{2i}(A - A^*)$. The operators S and T are J-real and self-adjoint. Since the operator A is unitary, we have

$$E = A^*A = (S - iT)(S + iT) = S^2 + T^2 + i(ST - TS).$$

It follows that the operators T and S commute and

$$S^2 + T^2 = E. \tag{3.16}$$

Since the operators S and T are commuting bounded self-adjoint operators, they admit the following spectral resolutions

$$S = \int_L \lambda dE_\lambda, \qquad T = \int_L z dF_z,$$

where $E_\lambda,\ F_z$ are commuting resolutions of unity of operators, and $L=(l_1,l_2]$, $l_1,l_2\in\mathbb{R}$, is a finite interval of the real line which contains the spectra of the operators S and T. Since the operators S and T are J-real, their resolutions of unity commute with J. By equality (3.16) and using the spectral resolutions we get

$$\int_L\int_L(\lambda^2+z^2-1)dE_\lambda dF_z=0,$$

where the integral means the limit in the norm of H of the corresponding Riemann-Stieltjes type sums. Thus, in all points of increase of the measure $dE_\lambda dF_z$ the following relation is true

$$\lambda^2+z^2-1=0. \tag{3.17}$$

The circle (3.17) in the plane $\mathbb{R}^2$ we denote by Γ. For all points of the circle Γ it is true $|z|=\sqrt{1-\lambda^2}$. Therefore for all points of Γ

$$z=\operatorname{sgn}(z)\sqrt{1-\lambda^2}, \tag{3.18}$$

where $\operatorname{sgn}(\cdot)$ is from (3.7). By the identity $\lambda=\cos\arccos\lambda,\ \lambda\in[-1,1]$, the equality (3.18) can be rewritten in the following form

$$z=\operatorname{sgn}(z)\sqrt{\sin^2(\arccos\lambda)}=\operatorname{sgn}(z)\sin(\arccos\lambda),$$

where we have used the positivity of the sine function on $[0,\pi]$. By this representation we can write

$$\begin{aligned} A=S+iT&=\int_L\int_L(\lambda+iz)dE_\lambda dF_z\\ &=\int_L\int_L(\cos\arccos\lambda+i\operatorname{sgn}(z)\sin(\arccos\lambda))dE_\lambda dF_z\\ &=\int_{L_+}\int_L e^{i\arccos\lambda}dE_\lambda dF_z+\int_{L_-}\int_L e^{-i\arccos\lambda}dE_\lambda dF_z, \end{aligned} \tag{3.19}$$

where $L_+=(0,\infty)\cap L,\ L_-=(-\infty,0]\cap L$. Define the following operator

$$S:=\int_L\int_L\operatorname{sgn}(z)\arccos\lambda dE_\lambda dF_z=\int_L\operatorname{sgn}(z)dF_z\int_L\arccos\lambda dE_\lambda.$$

It is obvious that S is a J-real self-adjoint operator. From relation (3.19) it follows that (3.15) is true. □

Using the last lemma we shall prove the following theorem which is a generalization of [4, Chapter XI, Theorem 2].

Theorem 3.4. *Let A be a unitary operator in a Hilbert space H. The operator A admits the following representation:*

$$A=Re^{iS}, \tag{3.20}$$

where R is a J-real unitary operator in H, and S is a bounded J-real self-adjoint operator in H.

Proof. Suppose that representation (3.20) is true. Then $A^* = e^{-iS}R^*$ and

$$\widetilde{A^*} = \widetilde{e^{-iS}}\widetilde{R^*} = J(\cos S - i\sin S)J\widetilde{R^*} = (\cos S + i\sin S)\widetilde{R^*} = e^{iS}R^*,$$

since S and R are J-real. Since R is unitary, we can write

$$\widetilde{A^*}A = e^{iS}R^*Re^{iS} = e^{2iS}.$$

Now we shall drop the assumption that representation (3.20) is true. Since the operator A is unitary, operators $A^{-1} = A^*$, JA^*J and $G := \widetilde{A^*}A$ are unitary, as well. The operator G is J-self-adjoint. In fact, by Proposition 2.8 we can write $G^* = A^*\widetilde{A} = J\widetilde{A^*}AJ = \widetilde{G}$. By Lemma 3.3 we can find a J-real self-adjoint operator S such that

$$G = e^{2iS}.$$

Set

$$R = Ae^{-iS}. \tag{3.21}$$

The operator R is unitary as a product of two unitary operators. We can write $\widetilde{R^*} = Je^{iS}A^*J = e^{-iS}\widetilde{A^*}$. Therefore

$$\widetilde{R^*}R = e^{-iS}\widetilde{A^*}Ae^{-iS} = e^{-iS}Ge^{-iS} = E.$$

Since the range of the unitary operator R is H, the latter equality implies that $\widetilde{R^*} = R^{-1}$. Thus, the operator R is J-unitary. Since the operator R is unitary and J-unitary, it is J-real. From (3.21) it follows the representation (3.20). $\square$

Let A be a linear bounded operator in a Hilbert space H and J be a conjugation in H. It is easy to check that operators $A^TA = JA^*JA$, $AA^T = AJA^*J$ are bounded J-self-adjoint operators. The operator A we shall call *J-normal* if $A^TA = AA^T$. It is clear that bounded J-self-adjoint, J-skew-self-adjoint and J-unitary operators are J-normal.
The following theorem is a generalization of [4, Chapter XI, Theorem 3]. We adapt the idea of the proof in the finite-dimensional case with a use of the Riesz calculus and the properties obtained in Section 2.1.

Theorem 3.5. *Let A be a linear bounded operator in a Hilbert space H and $0 \notin \sigma(A)$. Let J be a conjugation in H. Suppose that the spectrum of the operator AA^T has an empty intersection with a radial ray $L_\varphi = \{z \in \mathbb{C} : \ z = xe^{i\varphi}, \ x \geq 0\}$ ($\varphi \in [0, 2\pi)$) in the complex plane. Then the operator A admits the following representation*

$$A = SU, \tag{3.22}$$

where S is a bounded J-self-adjoint operator in H, and U is a bounded J-unitary operator in H. Here

$$S = \sqrt{AA^T},$$

where the square root is understood according to the Riesz calculus. The operators U and S commute if and only if the operator A is J-normal. Moreover, the operator A admits the following representation

$$A = U_1S_1, \tag{3.23}$$

where U_1 is a bounded J-unitary operator in H, and $S_1 = \sqrt{A^T A}$ is a bounded J-self-adjoint operator in H. The operators U_1 and S_1 commute if and only if A is J-normal.

In particular, representations (3.22) and (3.23) are true for operators

$$A = E + K, \tag{3.24}$$

where K is a compact operator in H, $\|K\| < 1$.

Proof. We set

$$S = \sqrt{AA^T} = \int_\Gamma \sqrt{\lambda} R_\lambda(AA^T) d\lambda.$$

The contour Γ is constructed in the following way. Let $T_R = \{z \in \mathbb{C} : \ |z| = R\}$ be a circle which contains $\sigma(AA^T)$ inside, $R > 0$. Let $d > 0$ be the distance between the closed set $\sigma(AA^T)$ and the segment $[0, Re^{i\varphi}]$, where φ is from the statement of the Theorem. Consider parallel segments on the distance $\frac{d}{2}$ of the above segment. Join this segments by a half of a circle in a neighborhood of zero and complete the contour with a part of the big circle T_R. We have constructed the contour Γ which contains the spectrum of the operator AA^T inside and its intersection with the ray L_φ is empty. We choose an arbitrary analytic branch of the root in $\mathbb{C} \backslash L_\varphi$.

A bounded operator $B := AA^T$ is J-self-adjoint as it was noticed above. Consequently, its resolvent is also a J-self-adjoint operator. In fact, by virtue of Proposition 2.10 we can write

$$\begin{aligned} R^*_\lambda(B) &= ((B - \lambda E)^{-1})^* = (B^* - \overline{\lambda} E)^{-1} = (\widetilde{B} - \overline{\lambda} E)^{-1} \\ &= (J(B - \lambda E)J)^{-1} = J(B - \lambda E)^{-1} J = J R_\lambda(B) J, \quad \lambda \in \rho(B). \end{aligned}$$

The operator S is J-self-adjoint as a limit of J-self-adjoint integral sums. Moreover, there exists the inverse S^{-1} which is J-self-adjoint, as well. Set

$$U = S^{-1} A,$$

and notice that $U^{-1} = A^{-1} S$ (recall that $0 \notin \sigma(A)$). Then

$$U\widetilde{U^*} = S^{-1} A \widetilde{A^*} \widetilde{(S^{-1})^*} = S^{-1} S^2 S^{-1} = E.$$

Multiplying the latter equality from the left side by U^{-1} we get

$$\widetilde{U^*} = U^{-1}.$$

Thus, the operator U is J-unitary.

Suppose that the operators U and S in representation (3.22) commute. Then

$$AA^T = SU(\widetilde{U^*})(\widetilde{S^*}) = S^2,$$

$$A^T A = (\widetilde{U^*}) SSU = S^2.$$

Conversely, if operators A and A^T commute then using last relations (without the latter equality) we write:

$$S^2 = (\widetilde{U^*}) S^2 U = U^{-1} S^2 U,$$

$$US^2 = S^2 U.$$

Since U commutes with S^2, it commutes with an arbitrary function of this operator. In particular, U commutes with S.

Now we shall establish the possibility of resolution (3.23) for the operator A. By virtue of Proposition 2.10 for an arbitrary linear bounded operator D in H we can write

$$JR_\lambda^*(D)J = J(D^* - \overline{\lambda}E)^{-1}J = (JD^*J - \lambda E)^{-1} = R_\lambda(D^T), \quad \lambda \in \rho(D).$$

Therefore

$$\rho(D) = \rho(D^T)$$

holds for an arbitrary linear bounded operator D in H. Applying this equality to the operator A we conclude that $0 \notin \sigma(A^T)$. Choose an arbitrary $\lambda \in L_\varphi$. Notice that

$$A^T A - \lambda E = A^{-1}(AA^T - \lambda E)A.$$

Therefore $(A^T A - \lambda E)^{-1}$ exists and

$$(A^T A - \lambda E)^{-1} = A^{-1}(AA^T - \lambda E)^{-1}A$$

is defined on the whole H and bounded. Thus, the ray L_φ does not intersect with the spectrum of the operator $A^T A$. Applying the proven part of the Theorem to the operator A^T we get a resolution $A^T = SU$, where $S = \sqrt{A^T A}$ is a bounded J-self-adjoint operator and U is a bounded J-unitary operator. By virtue of Proposition 2.8 we can write

$$A = \widetilde{U^*}\widetilde{S^*} = U^{-1}S.$$

It remains to notice that U^{-1} is a bounded J-unitary operator.

Let the operator A has the form (3.24). In this case $0 \notin \sigma(A)$ and we can write

$$AA^T = (E + K)J(E + K^*)J = E + C,$$

where $C := K + JK^*J + KJK^*J$. Notice that the operator C is compact as a sum of compact operators. It is not hard to see that the operator $J(E + K^*)^{-1}J(E + K)^{-1}$ is the inverse to the operator AA^T. Therefore $0 \notin \sigma(AA^T)$. Since the spectrum of the compact operator C is discrete with a unique point of concentration 0, one can find the ray which is required in the statement of the Theorem. $\square$

There is an essential difference between the properties of the J-form $[\cdot, \cdot]_J$ and the properties of the indefinite metric in the Krein spaces. This difference does not allow to apply methods from [24] to obtain or to study the J-polar decomposition in our case. In particular, the J-form can take arbitrary complex values. Some elementary properties of the J-form are illustrated by the result below. This result shows that the null set $H_{J;0}$ (which was defined above as $H_{J;0} = \{x \in H : [x, x]_J = 0\}$) is not a subspace (we can not even say that it is a linear set). Consider an arbitrary Hilbert space H. Let J be a conjugation in H and $\mathcal{F} = \{f_k\}_{k\in\mathbb{Z}_+}$ be a corresponding orthonormal basis in H. Set

$$H_R := \{x \in H : (x, f_k) \in \mathbb{R}, \ k \in \mathbb{Z}_+\}.$$

Notice that for an arbitrary element $x \in H$ we can write the following resolution:

$$x = x_R + ix_I, \qquad x_R, x_I \in H_R. \tag{3.25}$$

Namely, if $x = \sum_{k=0}^{\infty} x_k f_k$, we set $x_R := \sum_{k=0}^{\infty} \operatorname{Re} x_k f_k$, $x_I := \sum_{k=0}^{\infty} \operatorname{Im} x_k f_k$. It is easy to see that representation (3.25) is unique.

Define the following vectors:

$$f_{k,l}^{+} := \frac{1}{\sqrt{2}}(f_k + if_l), \quad f_{k,l}^{-} := \frac{1}{\sqrt{2}}(f_k - if_l), \qquad k, l \in \mathbb{Z}_+.$$

Theorem 3.6. *Let H be a Hilbert space and J be a conjugation in H. Let $\mathcal{F} = \{f_k\}_{k=0}^{\infty}$ be a corresponding orthonormal basis in H. The set $H_{J;0}$ has the following properties:*

1. *The set $H_{J;0}$ is closed;*
2. *$x \in H_{J;0} \Rightarrow Jx \in H_{J;0}$, $\alpha x \in H_{J;0}$, $\alpha \in \mathbb{C}$;*
3. *$x, y \in H_{J;0}: \ x \perp_J y \Rightarrow \alpha x + \beta y \in H_{J;0}$, $\alpha, \beta \in \mathbb{C}$;*
4. *$H_{J;0} = \{x \in H: \ x = x_R + ix_I,\ x_R, x_I \in H_R,\ \|x_R\| = \|x_I\|,\ (x_R, x_I) = 0\}$;*
5. *The set $H_{J;0}$ has no inner points;*
6. *$\operatorname{span} H_{J;0} = H$;*
7. *The set $\{f_{2k,2k+1}^{+}, f_{2k,2k+1}^{-}\}_{k \in \mathbb{Z}_+}$ is an orthonormal basis in H whose elements belong to $H_{J;0}$.*

Proof. The 1-st statement follows from continuity of the operator J and from continuity of the scalar product in H.
The second and third statements follows from the linearity of the J-form and from the properties of the conjugation J.
The 4-th statement is directly verified.
Suppose that the set $H_{J;0}$ has an inner point x_0 such that

$$x \in H,\ \|x - x_0\| < \varepsilon \ \Rightarrow \ x \in H_{J;0}, \tag{3.26}$$

for a number $\varepsilon > 0$. Let us write for x_0 the resolution (3.25):

$$x_0 = x_{0,R} + ix_{0,I}, \qquad x_{0,R}, x_{0,I} \in H_R.$$

Suppose first that $x_{0,I} \neq 0$. Set

$$x_\varepsilon := x_0 + i\frac{\varepsilon}{2\|x_{0,I}\|}x_{0,I} = x_{0,R} + ix_{0,I}\left(1 + \frac{\varepsilon}{2\|x_{0,I}\|}\right).$$

Notice that $\|x_\varepsilon - x_0\| = \frac{\varepsilon}{2} < \varepsilon$, and by (3.26) we obtain that $x_\varepsilon \in H_{J;0}$. Applying the fourth statement with the points x_0 and x_ε, we get

$$\|x_{0,R}\| = \|x_{0,I}\|, \tag{3.27}$$

and

$$\|x_{0,R}\| = \|x_{0,I}\left(1 + \frac{\varepsilon}{2\|x_{0,I}\|}\right)\| = \|x_{0,I}\| + \frac{\varepsilon}{2} > \|x_{0,I}\|,$$

respectively. The contradiction proves statement 5 for the case $x_{0,I} \neq 0$.

If $x_{0,I} = 0$ then the fourth statement implies that relation (3.27) is true and therefore $x_0 = 0$. If zero is an inner point of the set $H_{J;0}$ then the second statement implies $H_{J;0} = H$. It is a nonsense, since, for example, the elements of the basis $\mathcal{F}$ do not belong to the set $H_{J;0}$.

Let us prove the seventh statement. Using orthonormality of the elements f_k, $k \in \mathbb{Z}_+$, we directly check that elements of the set $\{f^+_{2k,2k+1}, f^-_{2k,2k+1}\}_{k\in\mathbb{Z}_+}$, are orthonormal. Notice that

$$f_{2k} = \frac{1}{\sqrt{2}}(f^+_{2k,2k+1} + f^-_{2k,2k+1}), \quad f_{2k+1} = \frac{1}{\sqrt{2}i}(f^+_{2k,2k+1} - f^-_{2k,2k+1}), \qquad k \in \mathbb{Z}_+.$$

Therefore $\mathrm{span}\{f^+_{2k,2k+1}, f^-_{2k,2k+1}\}_{k\in\mathbb{Z}_+} = H$ and the set $\{f^+_{2k,2k+1}, f^-_{2k,2k+1}\}_{k\in\mathbb{Z}_+}$ is an orthonormal basis in H. It remains to notice that

$$[f^\pm_{2k,2k+1}, f^\pm_{2k,2k+1}]_J = \frac{1}{2}[f_{2k} \pm i f_{2k+1}, f_{2k} \pm i f_{2k+1}]_J = 0,$$

and therefore $f^\pm_{2k,2k+1} \in H_{J;0}$, $k \in \mathbb{Z}_+$.

The sixth statement follows from the seventh statement. □

It is of interest to study some specific types of J-symmetric and J-skew-symmetric operators in more details. For example, there is an extensive literature on the Jacobi matrices (see [1]) and their generalizations (e.g. [38]) which are related to symmetric operators. Recently, we solved the direct and inverse spectral problems for $(2N+1)$-diagonal complex symmetric and skew-symmetric matrices [39],[40], [41]). These matrices are related to J-symmetric and J-skew-symmetric operators and can be further studied.

Self-adjoint operators with a simple spectrum are the operators which lead to the functional model of an arbitrary self-adjoint operator. So, it is of interest to study J-self-adjoint operators with a simple spectrum. What properties do their matrices have? This may lead to a functional model for J-self-adjoint operators and then for J-normal operators. Thus, our results on matrix representations and on the J-polar decomposition can be viewed as a first step in this program, as well.

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References

1. N.I. Akhiezer, *Classical moment problem and some questions of analysis related to it* , Fizmatlit., Moscow, 1961 (Russian).
2. N.I. Akhiezer and I.M. Glazman, *Theory of linear operators in a Hilbert space. Vol.1*, Izdat-vo pri KhGU izd. obyed. Vysha shkola, Kharkov, 1977 (Russian).
3. Sh. Asadi and I.E. Lutsenko, *Skew-unitary transformations of operator bundles*, Vestnik Kharkovskogo universiteta, Mehanika i matematika **37** No.83 (1972), 21–27 (Russian).
4. F.R. Gantmacher, *Theory of matrices*, Nauka, Moscow, 1967 (Russian).
5. S.R. Garcia, *The eigenstructure of complex symmetric operators*, Operator Theory: Advances and Applications **179** (2008), 169–184.
6. S.R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), 1285-1315.
7. S.R. Garcia and M. Putinar, *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc. **359** (2007), 3913–3931.
8. S. Garcia and W.T. Ross, *A non-linear extremal problem on the Hardy space*, Computational Methods and Function Theory **9** (2009), No.2, 485–524.
9. E. Prodan, S.R. Garcia and M. Putinar, *Norm estimates of complex symmetric operators applied to quantum systems*, J. Phys. A: Math. and Gen. **39** (2006), 389–400.

10. I.M. Glazman, *On an analog of the theory of extensions of Hermitian operators and a non-symmetric one-dimensional boundary problem on a semi-axis*, DAN SSSR **115** No.2 (1957), 214–216 (Russian).
11. I.M. Glazman, *Direct methods of qualititative spectral analysis of singular differential operators*, Gos. izdat. fiz.-mat. liter., Moscow, 1963 (Russian).
12. I.T. Gohberg and M.G. Krein, *Theory of Volterra operators in a Hilbert space and its applications*, Nauka, Moscow, 1967 (Russian).
13. T.B. Kalinina, *On the theory of extensions of K-symmetric operators*, Funkcionalniy analiz (Ulyanovsk) **7** (1976), 78–85 (Russian).
14. T.B. Kalinina, *On one perturbation of an operator in a Hilbert space with an antiunitary transformation*, Funkcionalniy analiz (Ulyanovsk) **7** (1979), 87–90 (Russian).
15. T.B. Kalinina, *On extensions of an operator in a Hilbert space with a skew-unitary transformation*, Funkcionalniy analiz (Ulyanovsk) **17** (1981), 68–75 (Russian).
16. T.B. Kalinina, *One extension of an operator in a Hilbert space with a skew-unitary transformation*, Funkcionalniy analiz (Ulyanovsk) **18** (1982), 63–71 (Russian).
17. T.B. Kalinina, *Generalized resolvents of an operator which is skew-symmetric with respect to an antilinear transformation of a Hilbert space*, Funkcionalniy analiz (Ulyanovsk) **20** (1983), 60–72 (Russian).
18. L.A. Kamerina, *Self-adjoint extensions of a symmetric operator in a space with an involution*, Funkcionalniy analiz (Ulyanovsk) **2** (1974), 16–26 (Russian).
19. L.A. Kamerina, *Unitary equivalence of operators of the class K_y*, Funkcionalniy analiz (Ulyanovsk) **5** (1975), 72–78 (Russian).
20. L.A. Kamerina, *A symmetric operator in the orthogonal sum of two Hilbert spaces with an involution*, Funkcionalniy analiz (Ulyanovsk) **7** (1976), 86–94 (Russian).
21. L.A. Kamerina, *Quasi-unitary equivalence of operators in spaces with an involution*, Funkcionalniy analiz (Ulyanovsk) **27** (1987), 72–78 (Russian).
22. I. Knowles, *On the boundary conditions characterizing J-selfadjoint extensions of J-symmetric operators*, J. Diff. Equations **40** (1981), 193–216.
23. A.N. Kochubey, *On extensions of J-symmetric operators*, Funkcionalniy analiz (Ulyanovsk) **10** (1978), 93–97 (Russian).
24. M.G. Krein and Yu.L. Shmulyan, *J-polar representation of plus-operators*, Matem. issledovaniya (Kishinev) Vol.1 **2** (1966), 172–210 (Russian).
25. V.P. Li, *On the theory of J-symmetric operators*, Funkcionalniy analiz (Ulyanovsk) **3** (1974), 84–91 (Russian).
26. V.P. Li, *A J-adjoint operator in a Hilbert space with an involution J*, Funkcionalniy analiz (Ulyanovsk) **3** (1974), 77–83 (Russian).
27. V.P. Li, *Quasi-dissipative and quasi-acummulative extensions of a J-symmetric operator*, Funkcionalniy analiz (Ulyanovsk) **4** (1975), 62–69 (Russian).
28. A.D. Makarova, *Extensions of J-symmetric operators with a non-dense domain*, Funkcionalniy analiz (Ulyanovsk) **8** (1977), 102–112 (Russian).
29. A.D. Makarova, *On the generalized resolvent of a J-symmetric operator*, Funkcionalniy analiz (Ulyanovsk) **14** (1980), 99–104 (Russian).
30. A.D. Makarova, *Extensions and generalized resolvents of an operator which commutes with an involution*, Funkcionalniy analiz (Ulyanovsk) **20** (1983), 83–92 (Russian).
31. B.G. Mironov, *On the theory of J-symmetric operators with a non-dense domain*, Funkcionalniy analiz (Ulyanovsk) **27** (1987), 128–133 (Russian).
32. B.G. Mironov, *On extensions and regular points of one extension of a J-symmetric operator*, Funkcionalniy analiz (Ulyanovsk) **29** (1989), 103–111 (Russian).
33. U.V. Riss, *Extension of the Hilbert space by J-unitary transformations*, Helv. Phys. Acta **71** (1998), 288–313.
34. D. Race, *The theory of J-selfadjoint extensions of J-symmetric operators*, J. Diff. Equations **57** (1985), 258–274.

35. L.M. Rayh, *On an extension of J-symmetricc operator with a non-dense domain*, Matem. zametki **17** N.5 (1975), 737–743.
36. L.M. Rayh and E.R. Tsekanovskii, *Biinvolutive self-adjoint biextensions of J-symmetric operators*, Teoriya funkciy, funcionalniy analiz i ih prilozheniya (Kharkov) **23** (1975), 79–93.
37. M.H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, AMS Colloquium Publications, Vol. 15, Providence, Rhode Island, 1932.
38. S.M. Zagorodnyuk, *On generalized Jacobi matrices and orthogonal polynomials*, New York J. Math. **9** (2003), 117–136.
39. S.M. Zagorodnyuk, *Direct and inverse spectral problems for (2N+1)-diagonal, complex, symmetric non-Hermitian matrices*, Serdica Math. J. **30** (2004), 471–482.
40. S.M. Zagorodnyuk, *The direct and inverse spectral problems for (2N+1)-diagonal complex transposition-antisymmetric matrices*, Methods Funct. Anal. Topology **14** (2008), 124–131.
41. S.M. Zagorodnyuk, *Integral representations for spectral functions of some non-selfadjoint Jacobi matrices*, Methods Funct. Anal. Topology **15** (2009), 91–100.

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