Banach J. Math. Anal. 4 (2010), no. 2, 1–10

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

HOMOMORPHISMS OF l^1-ALGEBRAS ON SIGNED POLYNOMIAL HYPERGROUPS

RUPERT LASSER[1] AND EVA PERREITER[1] *

Communicated by L. Székelyhidi

ABSTRACT. Let $\{R_n\}$ and $\{P_n\}$ be two polynomial systems which induce signed polynomial hypergroup structures on $\mathbb{N}_0$. We investigate when the Banach algebra $l^1(\mathbb{N}_0, h^R)$ can be continuously embedded into or is isomorphic to $l^1(\mathbb{N}_0, h^P)$. We find sufficient conditions on the connection coefficients c_{nk} given by $R_n = \sum_{k=0}^{n} c_{nk} P_k$, for the existence of such an embedding or isomorphism. Finally we apply these results to obtain amenability-properties of the l^1-algebras induced by Bernstein-Szegő and Jacobi polynomials.

1. INTRODUCTION AND PRELIMINARIES

The L^1-algebras of (signed) hypergroups show very distinctive properties compared with those of L^1-algebras of groups. The large number of l^1-algebras of (signed) polynomial hypergroups (which are determined by a sequence $\{R_n\}_{n\in\mathbb{N}_0}$ of orthogonal polynomials) show a broad diversity of properties. Thus, they are highly suitable to illustrate the difference between groups and hypergroups.

The purpose of this paper is to derive sufficient conditions for the existence of homomorphisms and isomorphisms between the l^1-algebras of two (signed) polynomial hypergroups. The results are applied to transfer amenability and related properties from one l^1-algebra to another.

Isomorphisms of hypergroups are studied in [4]. However, the main results there deal with isometric isomorphisms. Isometric isomorphisms between L^1-algebras are quite rare, which is not surprising since the basic translation operators are

Date: Received: 6 November 2009; Accepted: 21 November 2009.
* Corresponding author.
2000 *Mathematics Subject Classification.* Primary 43A62; Secondary 43A22, 43A20, 46H20.
Key words and phrases. Banach algebra homomorphism, hypergroup, amenability.

in general not unitary and the values of characters are in general not of modulus 1. Here we consider isomorphisms between L^1-algebras which are in general non-isometric.

During the past forty years there have been many results characterizing various amenability properties of Banach algebras, in particular L^1-algebras on groups, see the extensive list of references in [5]. Only very recently the investigations of the amenability of L^1-algebras of polynomial hypergroups have been started, see [12, 13].

Consider a sequence $\{R_n\}_{n\in\mathbb{N}_0}$ of real polynomials, $\deg R_n = n$, orthogonal with respect to a probability measure with compact and infinite support $\pi^R \in \mathcal{M}^1(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}} R_n R_m d\pi^R = \delta_{nm} h_n^{-1} \quad \text{for } m,n \in \mathbb{N}_0,$$

where $h_n^{-1} = \|R_n\|^2_{L^2(\mathbb{R},\pi^R)} > 0$. We assume that the normalization $R_n(1) = 1$, $n \in \mathbb{N}_0$, is possible. A recurrence relation of the following form is implied: $R_0 = 1,\ R_1(x) = \frac{1}{a_0}(x - b_0)$,

$$R_1 R_n = a_n R_{n+1} + b_n R_n + c_n R_{n-1} \quad \text{for all } n \in \mathbb{N}, \tag{1.1}$$

where $a_n,\ b_n,\ c_n \in \mathbb{R}$ and $a_n \neq 0,\ c_n \neq 0$. This three term recurrence can be extended to the product formula

$$R_m R_n = \sum_{k=|n-m|}^{n+m} g(m,n,k) R_k, \quad m,n \in \mathbb{N}_0,$$

where all $g(m,n,k)$ are real, $g(m,n,|n-m|) \neq 0,\ g(m,n,n+m) \neq 0$ and $\sum_{k=|n-m|}^{n+m} g(m,n,k) = 1$ because of our normalization.

If for the linearization coefficients holds $\sum_{k=|n-m|}^{n+m} |g(m,n,k)| \leq M$ for $n,m \in \mathbb{N}_0$ and some $M > 0$, then a commutative signed polynomial hypergroup structure is induced on $\mathbb{N}_0$: The involution is taken as the identity and the convolution $\omega :\ \mathcal{M}(\mathbb{N}_0) \times \mathcal{M}(\mathbb{N}_0) \to \mathcal{M}(\mathbb{N}_0)$ as

$$\varepsilon_n * \varepsilon_m = \sum_{k=|n-m|}^{n+m} g(m,n,k)\varepsilon_k \quad \text{for } n,m \in \mathbb{N}_0,$$

where ε_k is the point measure at $k \in \mathbb{N}_0$. Thus the neutral element is ε_0. If $g(m,n,k) \geq 0$ for $k = |n-m|, \ldots, n+m,\ n,m \in \mathbb{N}_0$, then a polynomial hypergroup structure is induced on $\mathbb{N}_0$. In this case $\sum_{k=|n-m|}^{n+m} |g(m,n,k)| = \sum_{k=|n-m|}^{n+m} g(m,n,k) = 1$ and $M = 1$. Signed hypergroups are for example treated in [16, 17] or [15] and summed up in [18]. More on the subject of (polynomial) hypergroups can be found in [10, 11] or [3].

For the signed polynomial hypergroup induced by $\{R_n\}_{n\in\mathbb{N}_0}$ we consider for $m \in \mathbb{N}_0$ the translations $T_m :\ l^{fin}(\mathbb{N}_0) \to l^{fin}(\mathbb{N}_0),\ T_m f(n) = \sum_{k=|n-m|}^{n+m} g(m,n,k) f(k)$, where $l^{fin}(\mathbb{N}_0)$ is the space of sequences with only finitely many non-zero entries. The Haar measure on $\mathbb{N}_0$ with respect to these translations is $(h_n)_{n\in\mathbb{N}_0}$, where

$h_0 = 1$. Let $l^p(\mathbb{N}_0, h) = \{(f(n))_{n\in\mathbb{N}_0} : \sum_{n=0}^{\infty} |f(n)|^p h_n < \infty\}$. The Plancherel isomorphism $\mathcal{F}^{-1} : L^2(\mathbb{R}, d\pi) \to l^2(\mathbb{N}_0, h)$ is given by $\check{f}(k) = \int_{\mathbb{R}} f R_k d\pi$. We define a norm on $l^1(\mathbb{N}_0, h)$ by $\|f\|_1 = M \cdot \sum_{n=0}^{\infty} |f(n)| h_n$. For $n \in \mathbb{N}_0$ we set $\varepsilon_n = \frac{\delta_n}{h_n}$, $\delta_n := (\delta_{nk})_{k\in\mathbb{N}_0}$, with norm $\|\varepsilon_n\|_1 = M \cdot \frac{1}{h_n} h_n = M$. Observe that $\hat{\varepsilon}_n = R_n|_D$. The convolution on $l^1(\mathbb{N}_0, h)$ given by $f * g(n) = \sum_{k=0}^{\infty} T_n f(k) g(k) h_k$ turns $l^1(\mathbb{N}_0, h)$ into a commutative Banach algebra with unit ε_0 (but note that $\|\varepsilon_0\|_1 = M$). Its structure space $\Delta(l^1(\mathbb{N}_0, h))$ can be identified with a compact set $D \subset \mathbb{C}$ characterized in the following two ways:

$$\begin{aligned} D &= \{z \in \mathbb{C} : \ |R_n(z)| \le C \ \text{ for all } n \in \mathbb{N}_0 \text{ and some } C > 0\} \\ &= \{z \in \mathbb{C} : \ |R_n(z)| \le M \ \text{ for all } n \in \mathbb{N}_0\}. \end{aligned}$$

Furthermore, $\operatorname{supp} \pi$ is a closed subset of D. The Gelfand transform $\mathcal{F} : l^1(\mathbb{N}_0, h) \to C(D)$, $\hat{v} = \sum_{k=0}^{\infty} v(k) R_k|_D \ h_k$, coincides with the inverse Plancherel isomorphism on $l^1(\mathbb{N}_0, h) \subset l^2(\mathbb{N}_0, h)$, which justifies using the same notation. The corresponding Wiener algebra is $A(D) := \mathcal{F}(l^1(\mathbb{N}_0, h))$ with norm $\|\hat{v}\| := \|v\|_1$. By [10] $l^1(\mathbb{N}_0, R)$ is semisimple.

To avoid confusion we frequently write D^R, h_n^R, ε_k^R etc. stressing the dependence on the polynomial system $\{R_n\}_{n\in\mathbb{N}_0}$ that induces the signed hypergroup.

2. Homomorphisms and Isomorphisms

The first aim of this section is to find a homomorphism between l^1-algebras on signed polynomial hypergroups whose connection coefficients fulfill certain requirements. Afterwards we will give conditions such that the constructed homomorphism is an isomorphism.

Lemma 2.1. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be polynomial sequences inducing signed polynomial hypergroups. A bounded linear map $S : l^1(\mathbb{N}_0, h^R) \to l^1(\mathbb{N}_0, h^P)$ is a homomorphism of Banach algebras if and only if*

(1) *$S\varepsilon_0 = \varepsilon_0^P$ and*
(2) *$S(\varepsilon_1 * \varepsilon_n) = S\varepsilon_1 *^P S\varepsilon_n$ for all $n \in \mathbb{N}_0$.*

Proof. If $S : l^1(\mathbb{N}_0, h^R) \to l^1(\mathbb{N}_0, h^P)$ is a homomorphism of Banach algebras, then (i) and (ii) are clearly valid. We show that these conditions are sufficient. From our two assumptions it immediately follows that $S(\varepsilon_0 * \varepsilon_n) = S\varepsilon_n = S\varepsilon_0 *^P S\varepsilon_n$ and $S(\varepsilon_1 * \varepsilon_n) = S\varepsilon_1 *^P S\varepsilon_n$ for all $n \in \mathbb{N}_0$. Let $k \ge 1$ and suppose as induction hypothesis that $S(\varepsilon_j * \varepsilon_n) = S\varepsilon_j *^P S\varepsilon_n$ for all $0 \le j \le k$ and $n \in \mathbb{N}_0$. First we obtain

$$\begin{aligned} S(\varepsilon_1 * \varepsilon_k * \varepsilon_n) &= a_n S(\varepsilon_k * \varepsilon_{n+1}) + b_n S(\varepsilon_k * \varepsilon_n) + c_n S(\varepsilon_k * \varepsilon_{n-1}) \\ &= a_n S\varepsilon_k *^P S\varepsilon_{n+1} + b_n S\varepsilon_k *^P S\varepsilon_n + c_n S\varepsilon_k *^P S\varepsilon_{n-1} \\ &= S\varepsilon_k *^P S(\varepsilon_1 * \varepsilon_n) = S\varepsilon_1 *^P S\varepsilon_k *^P S\varepsilon_n \\ &= S(\varepsilon_1 * \varepsilon_k) *^P S\varepsilon_n \end{aligned}$$

and furthermore

$$S\varepsilon_{k+1} *^P S\varepsilon_n = \frac{1}{a_k} \cdot S(\varepsilon_1 * \varepsilon_k - b_k\varepsilon_k - c_k\varepsilon_{k-1}) *^P S\varepsilon_n$$
$$= \frac{1}{a_k} \cdot S\left((\varepsilon_1 * \varepsilon_k - b_k\varepsilon_k - c_k\varepsilon_{k-1}) * \varepsilon_n\right) = S(\varepsilon_{k+1} * \varepsilon_n).$$

Hence we have shown that $S(\varepsilon_k * \varepsilon_n) = S\varepsilon_k *^P S\varepsilon_n$ for all $k, n \in \mathbb{N}_0$. Since S is assumed to be bounded, for $v, w \in l^1(\mathbb{N}_0, h^R)$, $v = \sum_{k=0}^\infty v_k\varepsilon_k$, $w = \sum_{n=0}^\infty w_n\varepsilon_n$, it follows that

$$S(v * w) = \sum_{k=0}^\infty \sum_{n=0}^\infty v_k w_n S(\varepsilon_k * \varepsilon_n) = \sum_{k=0}^\infty \sum_{n=0}^\infty v_k w_n S\varepsilon_k *^P S\varepsilon_n = Sv *^P Sw.$$

□

Given a family of orthogonal polynomials $\{P_k\}_{k\in\mathbb{N}_0}$ and a polynomial R_n of degree n we consider the linear combination $R_n = \sum_{k=0}^n c_{nk}P_k$. In all the following we define $c_{nk} = 0$ for $k > n$ which enables us to write $R_n = \sum_{k=0}^\infty c_{nk}P_k$.

The following theorem is more general than [9, Theorem 3.1], since we do not impose conditions on the dual objects.

Theorem 2.2. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be families of orthogonal polynomials inducing signed polynomial hypergroups such that $R_n = \sum_{k=0}^n c_{nk}P_k$. If there is $C > 0$ such that*

$$\sum_{k=0}^n |c_{nk}| \le C \quad \text{for all } n \in \mathbb{N}_0, \tag{2.1}$$

then the linear operator $S : l^1(\mathbb{N}_0, h^R) \to l^1(\mathbb{N}_0, h^P)$ determined by $S\varepsilon_n := \sum_{k=0}^n c_{nk}\varepsilon_k^P$ is a continuous homomorphism of Banach algebras with dense range. Furthermore, $D^P \subseteq D^R$ and $\widehat{Sf} = \hat{f}|_{D^P}$ for all $f \in l^1(\mathbb{N}_0, h^R)$.

Proof. Let $v = \sum_{n=0}^N v_n\varepsilon_n \in l^1(\mathbb{N}_0, h^R)$; then $\|v\|_1 = M^R \sum_{n=0}^N |v_n|$. By

$$\|Sv\| = \left\|\sum_{n=0}^N \sum_{k=0}^n v_n c_{nk}\varepsilon_k^P\right\| \le M^P \sum_{n=0}^N \sum_{k=0}^n |v_n||c_{nk}| \le \frac{M^P}{M^R}\|v\|_1 \cdot C$$

the linear map S is bounded on a dense subset of $l^1(\mathbb{N}_0, h^R)$ and can thus be uniquely extended to a bounded linear operator on $l^1(\mathbb{N}_0, h^R)$. Condition (2.1) implies $D^P \subseteq D^R$ for the following reason: If $z \in \mathbb{C}$ such that $|P_k(z)| \le M^P$ for all $k \in \mathbb{N}_0$, we obtain $|R_n(z)| = |\sum_{k=0}^n c_{nk}P_k(z)| \le \sum_{k=0}^n |c_{nk}| \cdot M^P \le C \cdot M^P$ for all $n \in \mathbb{N}_0$. (i) in Lemma 2.1 is fulfilled since $R_0 = 1 = P_0$, which means $c_{00} = 1$ and $S\varepsilon_0 = \varepsilon_0^P$. Now consider (ii) in Lemma 2.1. For $n \in \mathbb{N}_0$ we observe that $\widehat{S\varepsilon_n} = \sum_{k=0}^n c_{nk}\hat{\varepsilon}_k^P = \sum_{k=0}^n c_{nk}P_k|_{D^P} = R_n|_{D^P}$. Therefore

$$S\widehat{(\varepsilon_1 * \varepsilon_n)} = a_n\widehat{S\varepsilon_{n+1}} + b_n\widehat{S\varepsilon_n} + c_n\widehat{S\varepsilon_{n-1}} = a_nR_{n+1}|_{D^P} + b_nR_n|_{D^P} + c_nR_{n-1}|_{D^P}$$
$$= (R_1 \cdot R_n)|_{D^P} = R_1|_{D^P} \cdot R_n|_{D^P} = \widehat{S\varepsilon_1} \cdot \widehat{S\varepsilon_n}$$

and $S(\varepsilon_1 * \varepsilon_n) = S\varepsilon_1 *^P S\varepsilon_n$ for all $n \in \mathbb{N}_0$, since $l^1(\mathbb{N}_0, h^P)$ is semisimple. Thus S is a continuous homomorphism of Banach algebras. S has dense range since the polynomials are dense in $A(D^P)$. Finally, for $f = \sum_{k=0}^{\infty} f_k \varepsilon_k \in l^1(\mathbb{N}_0, h^R)$ it follows that $\widehat{Sf} = \lim_{N\to\infty} \widehat{S\left(\sum_{k=0}^{N} f_k \varepsilon_k\right)} = \lim_{N\to\infty} \sum_{k=0}^{N} f_k R_n|_{D^P} = \hat{f}|_{D^P}$, where the limits are w.r.t. $\| \, \|_{D^P,\infty}$. □

Remark 2.3. (i) Condition (2.1) is in particular fulfilled when the connection coefficients are nonnegative; in this case our normalization yields that $\sum_{k=0}^{n} |c_{nk}| = \sum_{k=0}^{n} c_{nk} = 1, \ n \in \mathbb{N}_0$. The non-negativity of connection coefficients has for example been studied by Askey and Gasper in [2], Szwarc in [20], Trench in [21] or Wilson in [23].
(ii) If supp $\pi^R \subset D^P$, then S is injective. In fact, in this case it follows from $\widehat{Sf} = \hat{f}|_{D^P} = 0$ that $\hat{f}|_{\text{supp } \pi^R} = 0$. This is only possible if $f = 0$.

A Banach algebra A is called regular, if for every closed subset V of $\Delta(A)$ and $\alpha \in \Delta(A) \setminus V$ there is $a \in A$ with $\hat{a}|_V = 0$ and $\hat{a}(\alpha) \neq 0$. Note that $l^1(\mathbb{N}_0, h)$ is regular whenever the Haar measure h is of polynomial growth, see [22]. If $l^1(\mathbb{N}_0, h)$ is regular, then supp $\pi = D$.

Proposition 2.4. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be families of orthogonal polynomials inducing signed polynomial hypergroups such that* (2.1) *is fulfilled. Suppose that $l^1(\mathbb{N}_0, h^R)$ is regular. Then S in Theorem 2.2 is injective if and only if $D^P = D^R$. Furthermore $l^1(\mathbb{N}_0, h^P)$ is also regular.*

Proof. Suppose that $D^P \subset D^R$ but $D^P \neq D^R$. D^P is closed in D^R. For all $\alpha \in D^R \setminus D^P$ there is $f \in l^1(\mathbb{N}_0, h^R)$ such that $\hat{f}|_{D^P} = 0$ and $\hat{f}(\alpha) \neq 0$. Since $\widehat{Sf} = \hat{f}|_{D^P}$, this means that S is not injective. On the other hand, S is obviously injective for $D^P = D^R$. Now take a closed subset $V \subset D^P$ and $\alpha \in D^P \setminus V$. There is $f \in l^1(\mathbb{N}_0, h^R)$ such that $\hat{f}|_V = 0$ and $\hat{f}(\alpha) \neq 0$. The same is true for $\hat{f}|_{D^P}$, so $l^1(\mathbb{N}_0, h^P)$ is also regular. □

Let us for a moment consider the semigroup $\mathbb{N}_0$. The induced convolution on $l^1(\mathbb{N}_0, 1)$, where 1 denotes the constant sequence with entries one, is determined by $\varepsilon_n * \varepsilon_m = \varepsilon_{n+m}$ for all $n, m \in \mathbb{N}_0$. The structure space $\Delta(l^1(\mathbb{N}_0, 1))$ can be identified with the closed unit disc $\mathbb{D} \subset \mathbb{C}$. The Gelfand transform reads $\mathcal{F}: \ l^1(\mathbb{N}_0, 1) \to C(\mathbb{D}), \ \hat{v}(x) = \sum_{k=0}^{\infty} v(k) x^k|_{\mathbb{D}}$ and thus maps $l^1(\mathbb{N}_0, 1)$ onto the space of absolutely convergent Taylor series on $\mathbb{D}$ [5, Example 2.1.13(v)]. In this sense one can say that the semigroup $\mathbb{N}_0$ is induced by the family of polynomials $\{x^n\}_{n\in\mathbb{N}_0}$. The analogue of (1.1) reads $x^1 \cdot x^n = x^{n+1}$, i.e. $a_n = 1$ and $b_n = c_n = 0$ for all $n \in \mathbb{N}_0$.

Corollary 2.5. *Let $\{P_n\}_{n\in\mathbb{N}_0}$ be a symmetric family of orthogonal polynomials inducing a polynomial hypergroup, i.e. $b_n = 0$ for all $n \in \mathbb{N}_0$ in* (1.1). *Let furthermore $l^1(\mathbb{N}_0, 1)$ carry the convolution structure of the semigroup $\mathbb{N}_0$. Then $S: \ l^1(\mathbb{N}_0, 1) \to l^1(\mathbb{N}_0, h^P)$ determined by $\widehat{Sf} = \hat{f}|_{D^P}$ is a continuous homomorphism of Banach algebras with dense range.*

Proof. Lemma 2.1 and Theorem 2.2 also hold true if one replaces $\{R_n\}_{n\in\mathbb{N}_0}$ by $\{x^n\}_{n\in\mathbb{N}_0}$, $l^1(\mathbb{N}_0, h^R)$ by $l^1(\mathbb{N}_0, 1)$ and D^R by $\mathbb{D}$; the proofs are exactly the same ones. The connection coefficients in $x^n = \sum_{k=0}^{n} c_{nk} P_k$ are all nonnegative. Indeed,

$x = P_1(x)$ and using the recurrence

$$\begin{aligned} c_{n+1,0} &= c_{n1}c_1^P,\ c_{n+1,1} = c_{n2}c_2^P + c_{n0}, \\ c_{n+1,k} &= c_{n,k+1}c_{k+1}^P + c_{n,k-1}a_{k-1}^P, \quad 2 \le k \le n-1, \\ c_{n+1,n} &= c_{n,n-1}a_{n-1}^P,\ c_{n+1,n+1} = c_{nn}a_n^P, \end{aligned}$$

the non-negativity of c_{nk} follows by induction. An application of Theorem 2.2 concludes the proof. □

Next we want to obtain sufficient conditions for S of Theorem 2.2 to be an isomorphism. Let us at first consider two families of orthogonal polynomials $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ inducing signed polynomial hypergroups with $d\pi^P = f d\pi^R$ for some $f \in L^2(D^R, d\pi^R)$. Since both measures are probability measures, $f \ge 0$ π^R-a.e. For the representation $R_n = \sum_{k=0}^n c_{nk}P_k$ one gets for all $k, n \in \mathbb{N}_0$ that

$$\frac{c_{nk}}{h_k^P} = (\mathcal{F}^P)^{-1}R_n(k) = \int_{\mathbb{R}} P_k R_n d\pi^P = \int_{\mathbb{R}} f P_k R_n d\pi^R = (\mathcal{F}^R)^{-1}(f \cdot P_k)(n).$$

In particular we obtain for $k = 0$ that the Plancherel transform $\check{f} \in l^2(\mathbb{N}_0, h^R)$ reads $\check{f}(n) = c_{n0},\ n \in \mathbb{N}_0$, and thus $\sum_{n=0}^\infty |c_{n0}|^2 h_n^R < \infty$. The converse of this observation is also true in the following sense.

Proposition 2.6. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be families of orthogonal polynomials inducing signed polynomial hypergroups such that $R_n = \sum_{k=0}^n c_{nk}P_k$. Suppose that $\sum_{n=0}^\infty |c_{n0}|^2 h_n^R < \infty$ and define $f \in l^2(\mathbb{N}_0, h^R)$ by $f(n) = c_{n0},\ n \in \mathbb{N}_0$. Then*

1. *$(\mathcal{F}^R)^{-1}(\hat{f} \cdot P_k)(n) = \frac{c_{nk}}{h_k^P} = (\mathcal{F}^P)^{-1}R_n(k)$ for all $k, n \in \mathbb{N}_0$.*
2. *$d\pi^P = \hat{f} d\pi^R$. In particular, $\hat{f} \ge 0$ π^R-a.e. and supp $\pi^P \subset$ supp π^R.*

Proof. Note that $\hat{f} \in L^2(D^R, d\pi^R) \subset L^1(D^R, d\pi^R)$. In order to prove the first statement we have to show that $\frac{c_{nk}}{h_k^P} = \int_{D^P} P_k R_n d\pi^P = \int_{D^R} \hat{f} P_k R_n d\pi^R$ for all $k, n \in \mathbb{N}_0$. Fix $n \in \mathbb{N}_0$. For $k = 0$ the equality holds true by definition of f. For $k \in \mathbb{N}_0$ we know that

$$\begin{aligned} \int_{D^R} \hat{f} R_k R_n d\pi^R &= \sum_{j=|k-n|}^{|k+n|} g^R(k,n,j) \int_{D^R} \hat{f} P_0 R_j d\pi^R = \sum_{j=|k-n|}^{|k+n|} g^R(k,n,j) c_{j0} \\ &= \sum_{j=|k-n|}^{|k+n|} g^R(k,n,j) \int_{D^P} P_0 R_j d\pi^P = \int_{D^P} R_k R_n d\pi^P. \end{aligned} \tag{2.2}$$

Writing $P_m = \sum_{k=0}^m d_{mk} R_k$ it follows that for all $m \in \mathbb{N}_0$,

$$\begin{aligned} \int_{D^R} \hat{f} P_m R_n d\pi^R &= \sum_{k=0}^m d_{mk} \int_{D^R} \hat{f} R_k R_n d\pi^R = \sum_{k=0}^m d_{mk} \int_{D^P} R_k R_n d\pi^P \\ &= \int_{D^P} P_m R_n d\pi^P = \frac{c_{nm}}{h_m^P}. \end{aligned}$$

For the second statement consider the compact set supp $\pi^R \cup$ supp $\pi^P \subset \mathbb{R}$. From the case $k=0$ in (2.2) we obtain that the bounded Borel measures $d\pi^P$ and $\hat{f} d\pi^R$ coincide on the dense subset span $(\{R_n\})$ of $C(\text{supp } \pi^R \cup \text{ supp } \pi^P)$. This means they have to be equal. □

Theorem 2.7. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ and $\{P_n\}_{n\in\mathbb{N}_0}$ be families of orthogonal polynomials inducing signed polynomial hypergroups such that $R_n = \sum_{k=0}^n c_{nk} P_k$. Suppose that*

(1) $\sum_{k=0}^n |c_{nk}| \leq C$ *for all* $n \in \mathbb{N}_0$.
(2) $\sum_{n=k}^\infty |c_{nk}| \frac{h_n^R}{h_k^P} \leq \tilde{C}$ *for all* $k \in \mathbb{N}_0$.
(3) *By Proposition 2.6 we know that* $d\pi^P = \hat{f} d\pi^R$, *where* $f \in l^1(\mathbb{N}_0, h^R) \subset l^2(\mathbb{N}_0, h^R)$ *is defined by* $f(n) = c_{n0}$, $n \in \mathbb{N}_0$, *and* $\hat{f} \geq 0$ *is continuous. Suppose that* $\hat{f} > 0$ *on supp* π^R.

Then $S: \ l^1(\mathbb{N}_0, h^R) \to l^1(\mathbb{N}_0, h^P)$, $S\varepsilon_n = \sum_{k=0}^n c_{nk}\varepsilon_k^P$, *in Theorem 2.2 is an isomorphism of Banach algebras. Furthermore,* $D^P = D^R$ *and* $\widehat{S(g)} = \hat{g}$ *for all* $g \in l^1(\mathbb{N}_0, h^R)$.

Proof. Applying Theorem 2.2 it suffices to show that for $P_k = \sum_{n=0}^k d_{kn} R_n$ holds $\sum_{n=0}^k |d_{kn}| = \frac{1}{M^R} \|(\mathcal{F}^R)^{-1} P_k\|_1 \leq \text{const.}$ for all $n \in \mathbb{N}_0$. Since $\hat{f} > 0$, f is invertible in $l^1(\mathbb{N}_0, h^R)$ and $(\|(\mathcal{F}^R)^{-1} P_k\|_1)_{k\in N_0}$ is bounded if and only if $(\|(\mathcal{F}^R)^{-1}(\hat{f} \cdot P_k)\|_1)_{k\in N_0}$ is bounded. By Proposition 2.6 and our second assumption we obtain

$$\frac{1}{M^R} \|(\mathcal{F}^R)^{-1}(\hat{f} \cdot P_k)\|_1 = \sum_{n=0}^\infty |(\mathcal{F}^R)^{-1}(\hat{f} \cdot P_k)(n)| h_n^R = \sum_{n=k}^\infty |c_{nk}| \frac{h_n^R}{h_k^P} \leq \tilde{C}$$

for all $k \in \mathbb{N}_0$. □

In the case of hypergroups the isomorphism above is isometric if and only if $\hat{f} = 1$, i.e. in the trivial case. In fact, it follows from [4] that an isometric isomorphism $l^1(\mathbb{N}_0, h^R) \to l^1(\mathbb{N}_0, h^P)$ maps point measures onto point measures. For our isomorphism this is only possible if $R_n = P_n$ for all $n \in \mathbb{N}_0$.

3. Application to Amenability-properties

Now we apply the constructed homomorphism to transfer amenability and related properties from one l^1-algebra to another. These properties are usually hard to verify directly, whereas the approach via inheritance under homomorphisms is a practicable alternative.

Let A be a commutative Banach algebra. A is called amenable if every bounded derivation from A into a dual Banach-A-bimodule is inner. A is weakly amenable if every bounded derivation from A into a symmetric Banach-A-bimodule is inner. Finally, let $\alpha \in \Delta(A)$ be a character. A is α-amenable if every bounded derivation from A into a Banach-A-bimodule X^α such that $a \cdot x = \langle \alpha, a\rangle \cdot x$, $a \in A$, $x \in X^\alpha$, is inner. Thus amenability of A implies weak and α-amenability.

Proposition 3.1. *Let A and B be Banach algebras, and let $\theta: \ A \to B$ be a continuous homomorphism with $\overline{\theta(A)} = B$.*

(1) *Suppose that A is amenable. Then B is amenable.*
(2) *Suppose that A is commutative and weakly amenable. Then B is weakly amenable.*
(3) *Suppose that A is commutative and let $\alpha \in \Delta(B)$. Suppose further that A is $\theta^*\alpha$-amenable. Then B is α-amenable.*

(i) and (ii) are well known; see for example [5, Proposition 2.8.64]. A proof of (iii) can be found in [8, Proposition 3.5].

We can now use the homeomorphism of Theorem 2.2 to apply Proposition 3.1. In [12] $l^1(\mathbb{N}_0, h^T)$ has been shown to be amenable, where $\{T_n\}_{n\in\mathbb{N}_0}$ are the Chebyshev polynomials of the first kind; up to now this is the only example of an amenable $l^1(\mathbb{N}_0, h^R)$. There is an abundance of polynomial sequences $\{R_n\}_{n\in\mathbb{N}_0}$ such that the connection coefficients $R_n = \sum_{k=0}^{n} c_{nk}T_k,\ n \in \mathbb{N}_0$, are non-negative, see [14], i.e. the assumption of Theorem 2.2 and the first condition of Theorem 2.7 are fulfilled. In the following example we will make use of the isomorphism of Theorem 2.7 and subsequently apply Proposition 3.1.

Example 1. Bernstein-Szegő polynomials: Given a polynomial $H:\ \mathbb{C} \to \mathbb{C}$, $H(z) = \sum_{k=0}^{r} \alpha_k z^k$, $r \geq 1$, with real coefficients $\alpha_k,\ 0 \leq k \leq r$, such that H has no zeros for $|z| \leq 1$ and $H(0) > 0$, define $\rho:\ [-1,1] \to \mathbb{R},\ \rho(\cos t) := |H(e^{it})|^2$. ρ is strictly positive in $[-1,1]$. The Bernstein-Szegő polynomials $\{B_n^\rho\}_{n\in\mathbb{N}_0}$ are defined as the ones orthogonal with respect to the probability measure π^ρ on $[-1,1]$, where $d\pi^\rho := c_\rho \cdot \rho(x)^{-1}(1-x^2)^{-\frac{1}{2}}dx = c_\rho \cdot \rho(x)^{-1} d\pi^T$. It is stated in [19, Chapter 2.6], or more explicitly in [7], that

$$B_n^\rho = \sum_{k=0}^{r} \alpha_k T_{n-k} \quad \text{for } n \geq r,$$

where $\{T_n\}_{n\in\mathbb{N}_0}$ are the Chebyshev polynomials of the first kind. Because of our normalization we can suppose $\sum_{k=0}^{r} \alpha_k = 1$. Using this representation a straightforward (but tedious) calculation shows that for every admissible ρ the Bernstein-Szegő polynomials $\{B_n^\rho\}_{n\in\mathbb{N}_0}$ induce a signed polynomial hypergroup. Furthermore, $h_n = \text{const.}$ for $n \geq r$ [7]. Thus, all requirements of Theorem 2.7 are fulfilled.

Corollary 3.2. *For every admissible ρ, $l^1(\mathbb{N}_0, h^{B^\rho})$ is isomorphic to $l^1(\mathbb{N}_0, h^T)$. In particular, $l^1(\mathbb{N}_0, h^{B^\rho})$ is amenable.*

Example 2. Jacobi polynomials: For $(\alpha,\beta) \in J' := \{(\gamma,\delta) \in \mathbb{R}^2 :\ \gamma \geq \delta > -1\}$, the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}_0}$ are orthogonal on $[-1,1]$ with respect to the probability measure $d\omega^{(\alpha,\beta)} = C^{(\alpha,\beta)} \cdot (1-x)^\alpha(1+x)^\beta dx$. For $(\alpha,\beta) \in J := \{(\alpha,\beta) \in J' :\ \alpha+\beta+1 \geq 0\}$, the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}_0}$ induce a polynomial hypergroup, see [10].

In [12] $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ has been shown to be not weakly amenable, where $\{P_n^{(\alpha,\alpha)}\}$ are Ultraspherical polynomials and $\alpha \geq 0$. Via Theorem 2.2 we can transfer this property to a large region of parameters of Jacobi polynomials.

Corollary 3.3. *The Banach algebra* $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta)}})$ *is not weakly amenable whenever* $\alpha+\beta \geq 0$ *or* $-\alpha+\beta+1 \leq 0$.

Proof. In [12] it has been shown that $l^1(\mathbb{N}_0, h^{P^{(\alpha,\alpha)}})$ is not weakly amenable for $\alpha \geq 0$. By [1, Theorem 7.1] we get that in $P_n^{(\alpha,\beta)} = \sum_{k=0}^n c_{nk} P_k^{(0,0)}$ all connection coefficients are positive whenever $\alpha+\beta \geq 0$, $(\alpha,\beta) \in J$. Thus by $\sum_{k=0}^n |c_{nk}| = \sum_{k=0}^n c_{nk} = 1$, $n \in \mathbb{N}_0$, we can apply Theorem 2.2 and obtain that $l^1(\mathbb{N}_0, P^{(\alpha,\beta)})$ is not weakly amenable whenever $\alpha+\beta \geq 0$, $(\alpha,\beta) \in J$. Next we consider $(\alpha,\beta) \in J$ with $-\alpha+\beta+1 \leq 0$. Then $(\alpha,\beta+1) \in J$ and with $\beta > -1$ it follows that $\alpha+\beta+1 \geq 2(\beta+1) > 0$, so $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta+1)}})$ is not weakly amenable as seen in the first part of this proof. By [1, (7.32) and following] we see that in $P_n^{(\alpha,\beta)} = \sum_{k=0}^n c_{nk} P_k^{(\alpha,\beta+1)}$ again all connection coefficients are positive. Another application of Theorem 2.2 completes the proof. ☐

Note that $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta)}})$, $\alpha+\beta \geq 0$ or $-\alpha+\beta+1 \leq 0$, is not weakly amenable because $l^1(\mathbb{N}_0, h^{P^{(0,0)}})$ is not weakly amenable. Also because of Theorem 2.2 the non x-amenability for $x \in (-1,1)$ of $l^1(\mathbb{N}_0, h^{P^{(0,0)}})$ is inherited by $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta)}})$. Nevertheless, $l^1(\mathbb{N}_0, h^{P^{(0,0)}})$ is (-1)-amenable, whereas $l^1(\mathbb{N}_0, h^{P^{(\alpha,\beta)}})$ lacks this property whenever $\alpha \neq \beta$, see [6, Example 4.6].

Example 3. Associated Legendre polynomials: The associated Legendre polynomials $\{P_n^\nu\}_{n\in\mathbb{N}_0}$ are orthogonal w.r.t. $d\pi^\nu = |{}_2F_1(\frac{1}{2}, \nu; \nu+\frac{1}{2}; \exp(2i \arccos x))|^{-2} dx$ on $[-1,1]$. They define polynomial hypergroups whenever $\nu \geq 0$. For $\nu = 0$ we obtain the classical Legendre polynomials. The connection coefficients in

$$P_n^\nu = \sum_{k=0}^n c_{nk} P_n^0$$

are non-negative, see [9].

Corollary 3.4. *The Banach algebra* $l^1(\mathbb{N}_0, h^{P^\nu})$ *is not weakly amenable for all* $\nu \geq 0$.

Although $l^1(\mathbb{N}_0, h^{P^\nu})$ is not weakly amenable, it is point-amenable for all $\nu \geq 0$ which has been shown in [13]. So they share this property with the classical Legendre polynomials.

References

1. R. Askey, *Orthogonal polynomials and special functions*, CBMS-NSF Regional Conf. Ser. in Appl. Math., SIAM, Philadelphia, 1975.
2. R. Askey and G. Gasper, *Jacobi polynomial expansions of Jacobi polynomials with non-negative coefficients*, Proc. Camb. Philos. Soc. **70** (1971), 243–255.
3. W.R. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, de Gruyter, Berlin, 1995.
4. W.R. Bloom and M.E. Walter, *Isomorphisms of hypergroups*, J. Aust. Math. Soc., Ser. A **52** (1992), 383–400.
5. H.G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society, Oxford, 2000.
6. F. Filbir, R. Lasser and R. Szwarc, *Reiter's condition P_1 and approximate identities for polynomial hypergroups*, Monatsh. Math. **143** (2004), 189–203.

7. V. Hösel and R. Lasser, *Approximation with Bernstein-Szegő polynomials*, Numer. Funct. Anal. Optim. **27** (2006), 377–389.
8. E. Kaniuth, A.-T. Lau and J. Pym, *On φ-amenability of Banach algebras*, Math. Proc. Camb. Philos. Soc. **144** (2008), 85–96.
9. R. Lasser, *Fourier-Stieltjes transforms on hypergroups*, Analysis (Munich) **2** (1982), 281–303.
10. ______, *Orthogonal polynomials and hypergroups*, Rend. Mat. Appl., VII. Ser. **3** (1983), 185–209.
11. ______, *Orthogonal polynomials and hypergroups II: The symmetric case*, Trans. Amer. Math. Soc. **341** (1994), 749–770.
12. ______, *Amenability and weak amenability of l^1-algebras of polynomial hypergroups*, Studia Math. **182** (2007), 183–196.
13. ______, *Point derivations on the L^1-algebra of polynomial hypergroups*, Colloq. Math. **116** (2009), 15–30.
14. R. Lasser and M. Rösler, *A note on property (T) of orthogonal polynomials*, Arch. Math. (Basel) **60** (1993), 459–463.
15. A.W. Parr, *Signed Hypergroups*, Ph.D. thesis, Graduate School of the University of Oregon, Department of Mathematics, 1997.
16. M. Rösler, *Convolution algebras which are not necessarily positivity-preserving*, Applications of hypergroups and related measure algebras, 299–318, Connett, William C. (ed.) et al., American Mathematical Society, Providence, 1995.
17. ______, *On the dual of a commutative signed hypergroup*, Manuscripta Math. **88** (1995), 147–163.
18. K.A. Ross, *Signed hypergroups - a survey*, Applications of hypergroups and related measure algebras, 319–329, Connett, William C. (ed.) et al., American Mathematical Society, Providence, 1995.
19. G. Szegő, *Orthogonal polynomials*, American Mathematical Society, Providence, 1975.
20. R. Szwarc, *Connection coefficients of orthogonal polynomials*, Canad. Math. Bull. **35** (1992), 548–556.
21. W.F. Trench, *Nonnegative and alternating expansions of one set of orthogonal polynomials in terms of another*, SIAM J. Math. Anal. **4** (1973), 111–115.
22. M. Vogel, *Spectral synthesis on algebras of orthogonal polynomial series*, Math. Z. **194** (1987), 99–116.
23. M.W. Wilson, *Nonnegative expansions of polynomials*, Proc. Amer. Math. Soc. **24** (1970), 100–102.

[1] Institute of Biomathematics and Biometry, Helmholtz Zentrum München, 85764 Neuherberg, Germany.

E-mail address: lasser@helmholtz-muenchen.de

E-mail address: eva.perreiter@helmholtz-muenchen.de