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ON EDMUNDS–TRIEBEL SPACES

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This paper is dedicated to Professor Lars-Erik Persson on the occasion of his 65th birthday

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Abstract. We consider the Edmunds–Triebel logarithmic spaces $A_\theta(logA)_{b,q}$ produced by a Banach couple $\overline{A} = (A_0,\ A_1)$, as special cases of extrapolation spaces and get estimates of a measure of weak noncompactness of the unit balls of these spaces in terms of the measures of weak noncompactness of the unit balls of the spaces A_0 and A_1. We obtain also estimates of the n-th Jordan–von Neumann constant C^n_{NJ} and the n-th James constant J_n of the spaces $A_\theta(logA)_{b,q}$ in terms of the corresponding constants of the spaces A_0 and A_1.

1. Introduction and preliminaries

For two Banach spaces A_0 and A_1, such that A_0 is densely and continuously embedded in A_1, $0 < \theta < 1$, $1 < q < \infty$ and $b \in \mathbb{R} \setminus \{0\}$, Edmunds–Triebel [8] defined the logarithmic space $A_\theta(logA)_{b,q}$. Nikolova, Persson and Zachariades [16] proved that these spaces satisfy the (p, p') Clarkson inequality for suitable p, $1 \le p \le 2$, as well as some properties about the types and the cotypes of these spaces. Nikolova and Zachariades [15] proved that if one of A_0 and A_1 is uniformly convex, then the logarithmic space $A_\theta(\log A)_{b,q}$ is also uniformly convex and they gave an estimate of the moduli of convexity of $A_\theta(\log A)_{b,q}$ in terms of the moduli of convexity of A_0 and A_1.

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Kryczka, Prus and Szczepanik [13] defined a new measure of weak noncompactness $\gamma(M)$, for every M nonempty and bounded subset of a Banach space, as well as the measure of weak nonocompactness $\Gamma(T)$ for every bounded operator between two Banach spaces.
A logarithmic space can be considered as a special case of an extrapolation space [9]. Using this consideration of a logarithmic space, in this note we get an estimate of the measure of weak noncompactness of an operator T between two logarithmic spaces $A_\theta(\log A)_{b,q}$ and $B_\theta(\log B)_{b,q}$ in terms of the measures of weak noncompactness of the restrictions of T such that $T : A_0 \to B_0$ and $T : A_1 \to B_1$. As corollary of this result we get an estimate of the measure of weak noncompactness of the unit ball of the space $A_\theta(logA)_{b,q}$ in terms of the measures of weak noncompactness of the unit balls of the spaces A_0 and A_1. Also, using the consideration of a logarithmic space as extrapolation space, we get estimates of the n-th Jordan–von Neumann constant C^n_{NJ} and n-th James constant J_n of $A_\theta(logA)_{b,q}$ in terms of the corresponding constants of the spaces A_0 and A_1.
Jawerth and Milman [9] defined the Σ_q and Δ_q extrapolation methods for $1 < q < \infty$. According to their definition, a family $(A_i)_{i\in\mathbb{Z}}$ is called strongly compatible if there exist two Banach spaces Δ and Σ such that $\Delta \hookrightarrow A_i \hookrightarrow \Sigma$ (continuous embeddings) for every $i \in \mathbb{Z}$. The norms of the inclusion maps $\Delta \hookrightarrow A_i$ and $A_i \hookrightarrow \Sigma$ are denoted by $M_{\Delta(i)}$ and $M_{\Sigma(i)}$, respectively. Let $1 < q < \infty$. If $\sum\limits_{i\in\mathbb{Z}}(M_{\Sigma(i)})^q$ (resp. $\sum\limits_{i\in\mathbb{Z}}(M_{\Delta(i)})^q$) is finite, then the extrapolation spaces $\Sigma_q((A_i)_{i\in\mathbb{Z}})$ (resp. $\Delta_q((A_i)_{i\in\mathbb{Z}})$) are defined as follows:
The space $\Sigma_q((A_i)_{i\in\mathbb{Z}})$ is the space of all $\alpha \in \Sigma$ for which there exists $(\alpha_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} A_i$ such that $\sum\limits_{i\in\mathbb{Z}} \|\alpha(i)\|^q_{A_i} < \infty$ and $\sum\limits_{i\in\mathbb{Z}} \alpha_i$ is (absolutely) convergent to α in Σ. The norm in $\Sigma_q((A_i)_{i\in\mathbb{Z}})$ is defined by $\|\alpha\|_{\Sigma_q((A_i)_{i\in\mathbb{Z}})} = \inf \left(\sum\limits_{i\in\mathbb{Z}} \|\alpha_i\|^q_{A_i}\right)^{\frac{1}{q}}$, where the infimum is taken over all representations $(\alpha_i)_{i\in\mathbb{Z}}$ of α as above.
The space $\Delta_q((A_i)_{i\in\mathbb{Z}})$ is the space of all $\alpha \in \bigcap_{i\in\mathbb{Z}} A_i$ with $\sum\limits_{i\in\mathbb{Z}} \|\alpha\|^q_{A_i} < \infty$. The norm in $\Delta_q((A_i)_{i\in\mathbb{Z}})$ is defined by $\|\alpha\|_{\Delta_q((A_i)_{i\in\mathbb{Z}})} = \left(\sum\limits_{i\in\mathbb{Z}} \|\alpha\|^q_{A_i}\right)^{\frac{1}{q}}$.
For every strongly compatible family $(A_i)_{i\in\mathbb{Z}}$ we have $\Delta \hookrightarrow \Sigma_q((A_i)_{i\in\mathbb{Z}}) \hookrightarrow \Sigma$ and $\Delta \hookrightarrow \Delta_q((A_i)_{i\in\mathbb{Z}}) \hookrightarrow \Sigma$. Let $(A_i)_{i\in\mathbb{Z}}$ and $(B_i)_{i\in\mathbb{Z}}$ be two strongly compatible families with spaces Σ_α and Σ_β, respectively. We write $T : (A_i)_{i\in\mathbb{Z}} \to (B_i)_{i\in\mathbb{Z}}$ if $T : \Sigma_\alpha \to \Sigma_\beta$ is a linear operator such that $T(A_i) \subseteq B_i$ and $\|T|A_i\| \le 1$ for every $i \in \mathbb{Z}$. When $T : (A_i)_{i\in\mathbb{Z}} \to (B_i)_{i\in\mathbb{Z}}$, then

$$T(\Sigma_q((A_i)_{i\in\mathbb{Z}})) \subseteq \Sigma_q((B_i)_{i\in\mathbb{Z}}), \quad T(\Delta_q((A_i)_{i\in\mathbb{Z}})) \subseteq \Delta_q((B_i)_{i\in\mathbb{Z}})$$

and the operators $T|\Sigma_q((A_i)_{i\in\mathbb{Z}})$ and $T|\Delta_q((A_i)_{i\in\mathbb{Z}})$ are bounded.
Let A_0 and A_1 be two Banach spaces such that A_0 is densely and continuously embedded in A_1, and $[A_0, A_1]_\eta$ be the complex interpolation space for $0 < \eta < 1$. For every $0 < \theta < 1,\ 1 < q < \infty$, and $b \in \mathbb{R} \setminus \{0\}$ the logarithmic space $A_\theta(logA)_{b,q}$ was defined in [8]. These spaces can be regarded as a special case of

extrapolation spaces Σ_q and Δ_q as follows:
For $b > 0$ the logarithmic space $A_\theta(logA)_{b,q}$ is the space $\Sigma_q((A_i)_{i\in\mathbb{Z}})$, where $A_i = \{0\}$ for $i < J$ and $A_i = 2^{ib}[A_0, A_1]_{\eta(i)}$ for $i \geq J$, where $J \in \mathbb{N}$ such that $\theta - 2^{-J} > 0$ and $\eta(i) = \theta - 2^{-i}$ for $i \geq J$.
For $b < 0$ the logarithmic space $A_\theta(logA)_{b,q}$ is the space $\Delta_q((Ai)_{i\in\mathbb{Z}})$, where $A_i = A_1$ with norm $\|a\| = 0$ for $i < J$, and $A_i = 2^{ib}[A_0, A_1]_{\theta(i)}$ for $i \geq J$, where $J \in \mathbb{N}$ such that $\theta + 2^{-J} < 1$ and $\theta(i) = \theta + 2^{-i}$ for $i \geq J$.
It is clear that different J define isomorphic spaces.
In [8] the following properties of the family of logarithmic spaces were proved.
(i) If $0 < \theta_0 < \theta < \theta_1 < 1,\ -\infty < b_0 < 0 < b_1 < 1$ and $1 < q < infty$, then

$$A_{\theta_0} \subset A_\theta(logA)_{b_1,q} \subset A_\theta(logA)_{b_0,q} \subset A_{\theta_1}.$$

(ii) If $0 < \theta < 1,\ -\infty < b_0 < 0 < b_1 < 1$ and $1 < q \leq \hat{q} < \infty$, then

$$A_\theta(logA)_{b_1,q} \subset A_\theta(logA)_{b_1,\hat{q}} \subset A_\theta \subset A_\theta(logA)_{b_0,q} \subset A_\theta(logA)_{b_0,\hat{q}}.$$

As it is noted in [8] the index q is comparatively not so important. Note also that if $0 < \theta < 1,\ -\infty < b_0 < b_1 < \infty$ and $1 < q,\ \hat{q} < \infty$, then

$$A_\theta(logA)_{b_1,q} \subset A_\theta(logA)_{b_0,\hat{q}}.$$

Many classical spaces are isomorphic to logarithmic spaces. For instance, if Ω is a bounded open subset of R^n with Lebesgue n-measure $\mu(\Omega) < \infty,\ 1 < p < \infty$ and $b \in \mathbb{R}$, then the usual Zygmund space $L_p(LogL)_b(\Omega)$ (i.e. the set of all measurable functions $f : \Omega \to C$ such that $\int_\Omega |f(x)|^p \log^{bp}(2 + |f(x)|)dx < \infty$) is isomorphic to the logarithmic space $A_\theta(logA)_{b,p}$, where $A_0 = L_\infty(\Omega),\ A_1 = L_1(\Omega)$ and $\theta = p^{-1}$(see [8]). This space was used in certain limiting situations in spectral theory in [8]. Sometimes it is more convenient (for instance if $1 < p \leq 2$) to take $A_0 = L_2(\Omega),\ A_1 = L_1\Omega),\ \theta = \frac{2-p}{p}$ and a slightly modified variant of $A_\theta(logA)_{b,p}$ to get $L_p(LogL)_b(\Omega)$. For instance, if $b < 0$, then $L_p(LogL)_b(\Omega)$ is the space $\Delta_p(A(i))$, where $A(i) = [A_0, A_1]_{\mu(i)},\ \mu(i) = \frac{2}{p} + 2^{-i+1} - 1 < 1,\ i \geq J$ such that $\frac{1}{p} + 2^{-J} < 1$, and $A(i) = A_1$ with norm $\|a\| = 0$ for $i < J$. In [8] also the related logarithmic Sobolev spaces $H^s_p(LogH)_b(\Omega)$ are considered, as well as the spaces $H^s_p(LogH)_{b,q}(\Omega)$, $H^s_p(\Lambda LogH)_{b,q}(\Omega)$ and $B^s_p(\Lambda LogB)_{b,q}(\Omega)$, where Λ is related to Laplacian and its iterates.
Let X be a Banach space and M_X the family of all nonempty bounded subsets of X. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X and $u_1, u_2 \in X$, then u_1 and u_2 are said to be a pair of successive convex combinations (scc) for $(x_n)_{n\in\mathbb{N}}$ if $u_1 \in conv\{x_1, \cdots, x_r\}$ and $u_2 \in conv\{x_{r+1}, x_{r+2}, \cdots\}$ for some integer $r \geq 1$. For every $M \in M_X$ the measure of weak noncompactness $\gamma(M)$ defined in [13] is given by

$$\gamma(M) = \sup\{csep(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in convM\},$$

where $csep(x_n)_{n\in\mathbb{N}} = \inf\{\|u_1 - u_2\| : u_1,\ u_2$ are scc for $(x_n)_{n\in\mathbb{N}}\}$.
The measure of weak noncompactness γ is related to the well-known James criterion:
A weakly closed $M \subset X$ is not weakly compact iff there exists $\delta > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in M, such that $dist(conv\{x_1, \cdots, x_r\}, conv\{x_{r+1}, x_{r+2}, \cdots\}) \geq \delta$ for every $r \in \mathbb{N}$.

From this criterion it is clear that $\gamma(M) = 0$ iff M is relatively compact.
The measure γ coincides with the function measuring the deviation from relative weak compactness based on the double-limit criterion, considered in [3]. Namely,

$$\gamma(M) = \sup\{\lim_{m\to\infty}\lim_{n\to\infty} f_m(x_n) - \lim_{n\to\infty}\lim_{m\to\infty} f_m(x_n) : (x_n)_{n\in\mathbb{N}} \subset M,\ (f_m)_{m\in\mathbb{N}} \subset B_{X^*} \text{ and the limits exist}\}.$$

So, $\gamma(M)$ is the worst distance between iterated limits for sequences in M and sequences in the dual unit ball B_{X^*}.
Another measure of weak noncompactness was introduced by de Blasi [4]. This measure is given by the formula

$$\omega(M) = \inf\{t > 0 : M \subset C + t\overline{B_X}, C \subset X \text{ is weakly compact}\}$$

for each $M \in M_X$. Hence, $\omega(M)$ is the worst distance from M to weakly compact sets of X. This measure was successfully applied to operator theory and to the theory of differential and integral equations. Logarithmically convex estimates for the measure of weak noncompactness ω have been established by Askoj and Maligranda [2], Cobos and Martinez [6, 7]. In [1] relations between the measures of weak noncompactness γ and ω were proved.
While the γ is a counterpart of separation measure of non compactness, de Blasi measure appears as a counterpart for the weak topology of Hausdorff measure of noncompactness. We have $\gamma(M) \leq 2\omega(M)$ in general, but γ is not equivalent to ω (see [1, 3, 12]). They coincide in c_0([13]), and if M is a nonempty bounded subset of $L_1(\mu)$, where μ is a finite measure, then $\gamma(M) = 2\omega(M)$.
For every bounded operator $T : E \to F$ the number $\Gamma(T) = \gamma(T(B_E))$ is called measure of weak noncompactness of the operator T. For weak topologies Gantmacher established that the operator $T : E \to F$ is weakly compact iff T^* is weakly compact. The quantitative result is $\gamma(T(B_E)) \leq \gamma(T^*(B_{F^*})) \leq 2\gamma(T(B_E))$ [1]. From [3, theorem 4] we obtain that there are no constants m and M, such that $m\omega(T(B_E)) \leq \omega(T^*(B_{F^*})) \leq M\omega(T(B_E))$ for any bounded operator $T : E \to F$.
For more details about the measure of weak noncompactness γ see [1, 12, 13].
The following result was proved in [12].

Theorem 1.1. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be two Banach couples, $0 < \theta < 1$ and $T : \overline{A} \to \overline{B}$. Then*

$$\Gamma_{[\theta]}(T) \leq \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^{\theta},$$

where $\Gamma_{[\theta]}(T)$ and $\Gamma_j(T), j = 0, 1,$ are the measures of weak noncompactness Γ of the operators $T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ and $T : A_j \to B_j,\ j = 0, 1,$ respectively.

If $(A_i)_{i\in\mathbb{Z}}$ and $(B_i)_{i\in\mathbb{Z}}$ are two strongly compatible families, $T : (A_i)_{i\in\mathbb{Z}} \to (B_i)_{i\in\mathbb{Z}}$ and $1 < q < \infty$, then we denote by $\Gamma_{\Sigma_q}(T)$ (resp. $\Gamma_{\Delta_q}(T)$) the measures of weak noncompactness Γ of the operator $T : \Sigma_q((A_i)_{i\in\mathbb{Z}}) \to \Sigma_q((B_i)_{i\in\mathbb{Z}})$ (resp. $T : \Delta_q((A_i)_{i\in\mathbb{Z}}) \to \Delta_q((B_i)_{i\in\mathbb{Z}})$). Then, we can write Theorem 4.1 in [11] as follows.

Theorem 1.2. *Let $(A_i)_{i\in\mathbb{Z}}$ and $(B_i)_{i\in\mathbb{Z}}$ be two strongly compatible families, $T : (A_i)_{i\in\mathbb{Z}} \to (B_i)_{i\in\mathbb{Z}}$ and $1 < q < \infty$. Then*

(i) $\Gamma_{\Sigma_q}(T) \leq \sup\{\Gamma(T : A_i \to B_i) : i \in \mathbb{Z}\}$, *and*
(ii) $\Gamma_{\Delta_q}(T) \leq \sup\{\Gamma(T : A_i \to B_i) : i \in \mathbb{Z}\}$.

Let X be a Banach space and $n = 2, 3, \cdots$
(i) The *n-th James* non-square *constant* $J_n(X)$ of X is defined by

$$J_n(X) = \sup\left\{\min_{\theta_i=\pm 1} \|\sum_{i=1}^{n} \theta_i x_i\| : x_1, \cdots, x_n \in B_X\right\}.$$

Note that $J_2(X)$ is just the James constant $J(X)$. Moreover, we note that $1 \leq J_n(X) \leq n$; if $\dim X = \infty$, then $J_n(X) \geq n^{1/2}$; $J_n(\ell^1) = J_n(\ell^1_m) = n$ for $m \geq n$. It is clear that X is uniformly non-ℓ^1_n if and only if $J_n(X) < n$ and X is B-convex if and only if $J_n(X) < n$ for some $n \geq 2$. For more information see [14].
(ii) The *n-th Jordan-von Neumann constant* $C^{(n)}_{NJ}(X)$ of X is defined [10] by

$$C^{(n)}_{NJ}(X) = \sup\left\{\frac{\sum\limits_{\theta_j=\pm 1} \|\sum\limits_{j=1}^{n} \theta_j x_j\|^2}{2^n \sum\limits_{j=1}^{n} \|x_j\|^2} : x_1, \cdots, x_n \in X,\ \sum_{j=1}^{n} \|x_j\| \neq 0\right\}.$$

This constant has been studied also in [17, 14]. Note that $C^{(n)}_{NJ}(X) = (K^n_{2,2})^2(X)$, where $K^n_{2,2}$ is n-th Khintchin constant; $1 \leq C^{(n)}_{NJ}(X) \leq n, n \geq 2$; $C^{(n)}_{NJ}(X) = 1$ for some (resp.any) $n \geq 2$ iff X is Hilbert space; $C^{(n)}_{NJ}(X) < n$ iff X is uniformly non-ℓ^1_n.

2. The main results

Using Theorems 1.1 and 1.2 we can prove the following result concerning an estimate of the measure of weak noncompactness of an operator between logarithmic spaces.

Theorem 2.1. *Let A_0, A_1, B_0, B_1 be Banach spaces such that A_0 is densely and continuously embedded into A_1 and B_0 is densely and continuously embedded into B_1, $0 < \theta < 1$, $1 < q < \infty$ and $b \in \mathbb{R} \setminus \{0\}$. Let also $T : A_1 \to B_1$ be a bounded operator such that $T(A_0) \subseteq B_0$ and $\Gamma_j(T)$, $j = 0, 1$, be the measures of weak noncompactness Γ of the operators $T : A_j \to B_j$, $j = 0, 1$.*
(i) If $\Gamma_0(T) = 0$, or $\Gamma_1(T) = 0$, then $\Gamma(T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}) = 0$.
(ii) If $\Gamma_0(T)\Gamma_1(T) \neq 0$, then
a) for $b < 0$

$$\Gamma(T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}) \leq \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^\theta \max\left(1, \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{2^{-J}}\right),$$

and
b) for $b > 0$

$$\Gamma(T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}) \leq \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^\theta \max\left(1, \left(\frac{\Gamma_0(T)}{\Gamma_1(T)}\right)^{2^{-J}}\right),$$

where J is the integer from the definitions of $A_\theta(logA)_{b,q}$.

Proof. (i) If $\Gamma_0(T) = 0$, or $\Gamma_1(T) = 0$, from Theorems 1.1 and 1.2 we obtain that $\Gamma(T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}) = 0$.
(ii) Let $\Gamma_0(T)\Gamma_1(T) \neq 0,\ b < 0,\ J \in \mathbb{N}$, such that $\theta + 2^{-J} < 1$, and $\theta(i) = \theta + 2^{-i}$ for $i \geq J$. We put $A(i) = [A_0, A_1]_{\theta(i)},\ B(i) = [B_0, B_1]_{\theta(i)}$ and $C = \sup_{i\in\mathbb{Z}}\{\Gamma(T : A(i) \to B(i))\}$. By Theorems 1.1 and 1.2 we get $C \leq \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{\theta(J)} \Gamma_0(T)$. If $\frac{\Gamma_1(T)}{\Gamma_0(T)} \leq 1$, then $C \leq \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{\theta} \Gamma_0(T) = \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^\theta$. If $\frac{\Gamma_1(T)}{\Gamma_0(T)} \geq 1$, then $C \leq \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{\theta+2^{-J}} \Gamma_0(T) = \Gamma_0(T)^{(1-\theta)}\Gamma_1(T)^\theta \left(\frac{\Gamma_1(T)}{\Gamma_0(T)}\right)^{2^{-J}}$.
So, the result follows from Theorem 1.2.
The proof for the case $b > 0$ is analogue. □

From Theorem 2.1 we obtain the following Corollaries.

Corollary 2.2. *Let A_0 and A_1 be two Banach spaces such that A_0 is densely and continuously embedded in A_1, $0 < \theta < 1$, $1 < q < \infty$ and $b \in \mathbb{R} \setminus \{0\}$.*
(i)$\gamma(B_{A_\theta(logA)_{b,q}}) \leq \gamma(B_{A_0})^{(1-\theta)}\gamma(B_{A_1})^\theta$
(ii) If A_0 or A_1 is reflexive, then the space $A_\theta(logA)_{b,q}$ is also reflexive.

Corollary 2.3. *Let $A_0,\ A_1,\ B_0,\ B_1,\ T,\ \theta, b$ and q be as in theorem 3.1. If one of the operators $T : A_j \to B_j,\ j = 0, 1,$ is weakly compact, then the operator $T : A_\theta(logA)_{b,q} \to B_\theta(logB)_{b,q}$ is also weakly compact.*

In order to estimate the n-th Jordan–von Neumann constant of a logarithmic space we prove two Lemmas concerning estimations of the n-th Jordan - von Neumann constants of interpolation and extrapolation spaces.

Lemma 2.4. *If (A_0, A_1) is a couple of Banach spaces and $0 < \theta < 1$, then $C^{(n)}_{NJ}([A_0, A_1]_\theta) \leq C^{(n)}_{NJ}(A_0)^{1-\theta}C^{(n)}_{NJ}(A_1)^\theta$.*

Proof. Let $T : [\ell^2_n(A_0) \bigoplus \ell^2_n(A_1)]_1 \to [\ell^2_{2^n}(A_0) \bigoplus \ell^2_{2^n}(A_1)]_1$ be defined by
$T\left((x_1, \cdots, x_n), (y_1, \cdots, y_n)\right) = \left((\sum_{i=1}^n \theta_i x_i)_{(\theta_i)_{i=1}^n\in\{-1,1\}^n}, (\sum_{i=1}^n \theta_i y_i)_{(\theta_i)_{i=1}^n\in\{-1,1\}^n}\right)$.
Then $T(\ell^2_n(A_i)) \subseteqq \ell^2_{2^n}(A_i)$ and $\|T|\ell^2_n(A_i)\| = \sqrt{2^n C^{(n)}_{NJ}(A_i)}$ for $i = 0, 1$.
Since $[\ell^2_n(A_0), \ell^2_n(A_1)]_\theta = \ell^2_n([A_0, A_1]_\theta)$ and $[\ell^2_{2^n}(A_0), \ell^2_{2^n}(A_1)]_\theta = \ell^2_{2^n}([A_0, A_1]_\theta)$ we obtain that

$$T(\ell^2_n([A_0, A_1]_\theta)) \subseteqq \ell^2_{2^n}([A_0, A_1]_\theta)$$

and

$$\|T|\ell^2_n([A_0, A_1]_\theta)\| \leq \sqrt{2^n C^{(n)}_{NJ}(A_0)^{1-\theta}C^{(n)}_{NJ}(A_1)^\theta}.$$

But $\|T|\ell^2_n([A_0, A_1]_\theta)\| = \sqrt{2^n C^{(n)}_{NJ}([A_0, A_1]_\theta)}$. Thus

$$C^{(n)}_{NJ}([A_0, A_1]_\theta) \leq C^{(n)}_{NJ}(A_0)^{1-\theta}C^{(n)}_{NJ}(A_1)^\theta.$$

□

Lemma 2.5. *Let $(A_i)_{i\in\mathbb{Z}}$ be a strongly compatible family of Banach spaces, $1 < q < \infty$ and $\Sigma_q((A_i)_{i\in\mathbb{Z}}),\ \Delta_q((A_i)_{i\in\mathbb{Z}})$ be the extrapolation spaces. Then for any $n \geq 2$*

(i) $C^{(n)}_{NJ}(\Sigma_q((A_i)_{i\in\mathbb{Z}})) \leq n^{2/t-1}\sup_{i\in\mathbb{Z}} C^{(n)}_{NJ}(A_i)^{2/t'}$, *and*
(ii) $C^{(n)}_{NJ}(\Delta_q((A_i)_{i\in\mathbb{Z}})) \leq n^{2/t-1}\sup_{i\in\mathbb{Z}} C^{(n)}_{NJ}(A_i)^{2/t'}$,
where $t = \min\{q, q'\}$.

Proof. We put $C_i = C^{(n)}_{NJ}(A_i)$ for $i \in \mathbb{Z}$ and $C = \sup_{i\in\mathbb{Z}} C_i$. From [10] we obtain $C^{(n)}_{NJ}(\ell_q((A_i))) \leq n^{2/t-1}C^{2/t'}$.
(i) Let $n \geq 2$, $\alpha_1, \cdots, \alpha_n \in \Sigma_q((A_i)_{i\in\mathbb{Z}})$ with $\sum_{j=1}^n \|\alpha_j\| \neq 0$, and $\varepsilon > 0$. For every $j = 1, \cdots, n$ there exists a representation $(\alpha_j(i))_{i\in\mathbb{Z}}$ of α_j such that

$$\left(\sum_{i\in\mathbb{Z}} \|\alpha_j(i)\|^q_{A_i}\right)^{\frac{1}{q}} - \|\alpha_j\|_{\Sigma_q} < \varepsilon.$$

Then

$$\begin{aligned}
\sum_{\theta_j\in\{-1,1\}} \left\|\sum_{j=1}^n \theta_j\alpha_j\right\|^2_{\Sigma_q} &\leq \sum_{\theta_j\in\{-1,1\}} \sum_{i\in\mathbb{Z}} \left(\left\|\sum_{j=1}^n \theta_j\alpha_j(i)\right\|^q_{A_i}\right)^{\frac{2}{q}} \\
&= \sum_{\theta_j\in\{-1,1\}} \left\|\sum_{j=1}^n \theta_j\alpha_j\right\|^2_{\ell_q(A_i)} \\
&\leq 2^n n^{2/t-1} C^{2/t'} \sum_{j=1}^n \|\alpha_j\|^2_{l_q(A_i)} \\
&\leq 2^n n^{2/t-1} C^{2/t'} \sum_{j=1}^n (\|\alpha_j\|_{\Sigma_q} + \varepsilon)^2.
\end{aligned}$$

So, we get

$$\sum_{\theta_j\in\{-1,1\}} \left\|\sum_{j=1}^n \theta_j\alpha_j\right\|^2_{\Sigma_q} \leq 2^n n^{2/t-1} C^{2/t'} \sum_{j=1}^n \|\alpha_j\|^2_{\Sigma_q}.$$

(ii) The proof of (ii) is similar to the above. □

Using Lemmas 2.4 and 2.5 we obtain an estimate of the n-th Jordan–von Neumann constant of a logarithmic space produced from the couple (A_0, A_1) in terms of the n-th Jordan–von Neumann constants of the spaces A_0 and A_1.

Theorem 2.6. *Let A_0 and A_1 be two Banach spaces such that A_0 is densely and continuously embedded in A_1, $0 < \theta < 1$, $b \in \mathbb{R}\setminus\{0\}$, $1 < q < \infty$, $t = \min\{q, q'\}$ and J be the integer from the definition of the logarithmic space $A = A_\theta(logA)_{b,q}$.*
(i)If $b < 0$, then

$$C^n_{NJ}(A) \leq n^{2/t-1} C^n_{NJ}(A_0)^{\frac{2(1-\theta)}{t'}} C^n_{NJ}(A_1)^{\frac{2\theta}{t'}} \max\left\{1, \left(\frac{C^n_{NJ}(A_1)}{C^n_{NJ}(A_0)}\right)^{\frac{2^{-J+1}}{t'}}\right\}$$

(ii) If $b > 0$, then

$$C^n_{NJ}(A) \leq n^{2/t-1} C^n_{NJ}(A_0)^{\frac{2(1-\theta)}{t'}} C^n_{NJ}(A_1)^{\frac{2\theta}{t'}} \max\left\{1, \left(\frac{C^n_{NJ}(A_0)}{C^n_{NJ}(A_1)}\right)^{\frac{2^{-J+1}}{t'}}\right\}$$

.

The proof is similar to the one of Theorem 2.1, using the Lemmas 2.4 and 2.5. In the following we estimate the n-James constant of an interpolation space $[A_0, A_1]_\theta$ in terms of the n-th James constants of the spaces A_0 and A_1.

Theorem 2.7. *Let (A_0, A_1) be a couple of Banach spaces, $0 < \theta < 1$ and $A_\theta = [A_0, A_1]_\theta$. Then*

$$\frac{J_n(A_\theta)}{n} \leq \left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^n}} \left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^n}}$$

Proof. We put $\beta_0 = J_n(A_0),\ \beta_1 = J_n(A_1)$. Let $0 < q < 1$. We will prove that for any $x_1, \dots, x_n \in B_{A_\theta}$ there exist $\varepsilon_1, \dots, \varepsilon_n = \pm 1$ such that

$$\|\sum_{k=1}^n \varepsilon_k x_k\| \leq B_n = \beta_0^{\frac{(1-\theta)q}{2^n}} \beta_1^{\frac{\theta q}{2^n}} n^{1-\frac{q}{2^n}}.$$

Then considering $q \to 1$, we will get the assertion of the theorem. Consider first the case $\beta_0 < n$ and $\beta_1 < n$. Let $\varepsilon > 0$, be such that $\beta_0 + \varepsilon < n$ and $\beta_1 + \varepsilon < n$. By contradiction, let there exist $x_1^\theta, \dots, x_n^\theta \in B_{A_\theta}$ such that $\| \sum_{k=1}^n \varepsilon_k x_k^\theta \| > B_n$ for every $\varepsilon_1, \cdots, \varepsilon_n = \pm 1$. As in the proof of Casini and Vignati [5], for fixed $\eta > 0$ and $k = 1, \dots, n$ we note that there exist functions $f_k \in F(\bar{A})$ such that $f_k(\theta) = \frac{x_k^\theta}{1+\eta} = x_k'$ and

$$\|f_k\| = \max_{j=0,1}(\sup_{t\in\mathbb{R}} \|f_k(j+it)\|_{A_j} \leq 1.$$

For $j = 0, 1$ and every choice of $\varepsilon_k = \pm 1$ we define

$$E^j_{\varepsilon_1,\dots,\varepsilon_n} = \{t \in \mathbb{R} : \|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(j+it)\|_{A_j} < \frac{\beta_j + \varepsilon}{n}\}.$$

From the inequality

$$\begin{aligned} \log\left\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k x_k'\right\|_\theta \quad &\leq \quad \int_{-\infty}^{\infty} \log\left\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it)\right\|_{A_0} \mu_0(\theta, t)\, dt \\ &+ \int_{-\infty}^{\infty} \log\left\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(1+it)\right\|_{A_1} \mu_1(\theta, t)\, dt, \end{aligned}$$

where $\mu_j(\theta,t),\ j=0,1$ give the Poisson kernel for the strip, we obtain

$$\begin{aligned}\log\frac{\frac{B_n}{n}}{1+\eta} &< \int_{E^0_{\varepsilon_1,\dots,\varepsilon_n}} \log\left\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it)\right\|_{A_0} \mu_0(\theta,t)\,dt \\ &+\int_{E^{0,c}_{\varepsilon_1,\dots,\varepsilon_n}} \log\left\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it)\right\|_{A_0} \mu_0(\theta,t)\,dt \\ &+\int_{E^1_{\varepsilon_1,\dots,\varepsilon_n}} \log\left\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it)\right\|_{A_1} \mu_1(\theta,t)\,dt \\ &+\int_{E^{1,c}_{\varepsilon_1,\dots,\varepsilon_n}} \log\left\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it)\right\|_{A_1} \mu_1(\theta,t)\,dt.\end{aligned}$$

Since we have $\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(j+it)\|_{A_j} \le 1,\ j=0,1$ for every $t\in\mathbb{R}$, we get

$$\log\frac{\frac{B_n}{n}}{1+\eta} < (1-\theta)|E^0_{\varepsilon_1,\dots,\varepsilon_n}|\frac{\beta_0+\varepsilon}{n} + \theta|E^1_{\varepsilon_1,\dots,\varepsilon_n}|\frac{\beta_0+\varepsilon}{n}$$

Since η is an arbitrary positive number we have

$$\frac{B_n}{n} \le \left(\frac{\beta_0+\varepsilon}{n}\right)^{(1-\theta)|E^0|}\left(\frac{\beta_1+\varepsilon}{n}\right)^{\theta|E^1|},$$

where $E^j = E^j_{\varepsilon_1,\dots,\varepsilon_n}$. Replacing B_n we get

$$\left(\frac{\beta_0+\varepsilon}{n}\right)^{(1-\theta)(\frac{q}{2^n}-|E^0|)}\left(\frac{\beta_1+\varepsilon}{n}\right)^{\theta(\frac{q}{2^n}-|E^1|)} \le 1.$$

At least one of the multipliers should be ≤ 1, let for instance this be the first one. Then since $\beta_0+\varepsilon\le n$ we get $|E^0|\le\frac{q}{2^n}$. Then $|\bigcup E^0_{\varepsilon_1,\dots,\varepsilon_n}| \le \frac{q2^n}{2^n} = q($ the union is taken over all permutation of signs). This means that

$$\left(\bigcup E^0_{\varepsilon_1,\dots,\varepsilon_n}\right)^c \ne \emptyset,$$

i.e. there exist t_θ such that for every choice of signs $\varepsilon_1,\dots,\varepsilon_n$ we have

$$\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(it_\theta)\|_{A_0} \ge \frac{\beta_0+\varepsilon}{n}.$$

This leads us to the inequality

$$\max_{f_k}\min_{\varepsilon_k=\pm1}\|\sum_{k=1}^n \varepsilon_k f_k(it_\theta)\|_{A_0} \ge \beta_0+\varepsilon$$

which gives a contradiction.
If one of $\beta_0,\ \beta_1$ is equal to n, then the proof goes similarly. If for example $\beta_1=n$ we consider only the sets $E^0_{\varepsilon_1,\dots,\varepsilon_n}$, and we use that $\log\|\frac{1}{n}\sum_{k=1}^n \varepsilon_k f_k(1+it)\|_{A_j}\le 0$ for every $t\in\mathbb{R}$.
If both $\beta_0=n, \beta_1=n$ the result is obvious. The proof is complete. □

By using the same technique of the proof of Theorem 2.1 we also have the following result:

Proposition 2.8. *Let $n \geq 2$, (A_0, A_1) be a couple of Banach spaces and $0 < \theta < 1$.*
(i) If $\theta_j = \theta + 2^{-j}$, $J \in \mathbb{N}$ such that $\theta + 2^{-J} < 1$ and $A_{\theta_j} = [A_0, A_1]_{\theta_j}$, then

$$\sup_{j \geq J} \frac{J_n(A_{\theta_j})}{n} \leq \left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^n}} \left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^n}} \max\left\{1, \left(\frac{J_n(A_1)}{J_n(A_0)}\right)^{2^{-J-n}}\right\}.$$

(ii) If $\eta_j = \theta - 2^{-j}$, $J \in \mathbb{N}$ such that $\theta - 2^{-J} > 0$ and $A_{\eta_j} = [A_0, A_1]_{\eta_j}$, then

$$\sup_{j \geq J} \frac{J_n(A_{\eta_j})}{n} \leq \left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^n}} \left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^n}} \max\left\{1, \left(\frac{J_n(A_0)}{J_n(A_1)}\right)^{2^{-J-n}}\right\}.$$

Using Lemma 2.5, Proposition 2.8 and [14, Theorem 4] we obtain an estimate of the n-th James constant of a logarithmic space produced from the couple $\overline{A} = (A_0, A_1)$ in terms of the n-th James constants of the spaces A_0 and A_1.

Theorem 2.9. *Let $n \geq 2$, A_0 and A_1 be two Banach spaces such that A_0 is densely and continuously embedded in A_1, $b \in \mathbb{R} \setminus \{0\}$, $0 < \theta < 1$, $1 < q < \infty$, $t = \min\{q, q'\}$, $A = A_\theta(logA)_{b,q}$ and J be the integer from the definitions of $A = A_\theta(logA)_{b,q}$.*
(i) If $b < 0$, then

$$\frac{J_n(A)}{n} \leq$$
$$\left\{1 - \frac{1}{2^{n-1}n}\left[1 - \left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^{n-1}}} \left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^{n-1}}} \max\left\{1, \left(\frac{J_n(A_1)}{J_n(A_0)}\right)^{2^{-J-n+1}}\right\}\right]\right\}^{1/t'},$$

(ii) If $b > 0$, then

$$\frac{J_n(A)}{n} \leq$$
$$\left\{1 - \frac{1}{2^{n-1}n}\left[1 - \left(\frac{J_n(A_0)}{n}\right)^{\frac{1-\theta}{2^{n-1}}} \left(\frac{J_n(A_1)}{n}\right)^{\frac{\theta}{2^{n-1}}} \max\left\{1, \left(\frac{J_n(A_0)}{J_n(A_1)}\right)^{2^{-J-n+1}}\right\}\right]\right\}^{1/t'}.$$

Proof. From [14, Theorem 4] we get

$$K^n_{2,2}(A) \leq 2^{\frac{1-n}{2}}\left[2^{n-1}(n-1) + c_n\right]^{1/2},$$

where $c_n = ([J^s_n(A) - n + 1]_+)^2 + 2^{n-1} - 1 \leq (J_n(A) - n + 1)^2 + 2^{n-1} - 1$.
Since $J_n(A) \leq n$ we have $J_n(A) - n + 1 \leq \frac{J_n(A)}{n}$ and we get

$$K^n_{2,2}(A) \leq 2^{\frac{1-n}{2}}\left[2^{n-1}n - 1 + \left(\frac{J_n(A)}{n}\right)^2\right]^{1/2}.$$

Since $(K^n_{2,2}(A))^2 = C^n_{NJ}(A)$, we get from [14, Theorem 4] the inequality

$$\frac{J^2_n(A)}{n} \leq C^n_{NJ}(A) \leq n - 2^{1-n} + \frac{J^2_n(A)}{2^{n-1}n^2}.$$

When $b < 0$ from the above and Lemma 3.5 we obtain that

$$\frac{J_n^2(A)}{n} \le C_{NJ}^{(n)}(\Delta_q((A_{\theta_j})_{j\in\mathbb{Z}})) \le n^{2/t-1} \sup_{j\ge J}(C_{NJ}^{(n)}(A_{\theta_j}))^{2/t'} \le$$

$$\le n^{2/t-1}\left[n - 2^{1-n} + \sup_{j\ge J}\frac{J_n^2(A_{\theta_j})}{2^{n-1}n^2}\right]^{2/t'}.$$

We put $\beta_0 = J_n(A_0),\ \beta_1 = J_n(A_1)$. Then, from Proposition 2.8 we have

$$\frac{J_n^2(A)}{n} \le n^{2/t-1}\left[n - 2^{1-n} + 2^{1-n}\left(\frac{\beta_0}{n}\right)^{\frac{1-\theta}{2^{n-1}}}\left(\frac{\beta_1}{n}\right)^{\frac{\theta}{2^{n-1}}}\right]^{2/t'},$$

and since $\frac{1}{t}+\frac{1}{t'}=1$ we obtain

$$\frac{J_n^2(A)}{n^2} \le \left\{1 - \frac{1}{2^{n-1}n}\left[1 - \left(\frac{\beta_0}{n}\right)^{\frac{1-\theta}{2^{n-1}}}\left(\frac{\beta_1}{n}\right)^{\frac{\theta}{2^{n-1}}}\right]\right\}^{2/t'}.$$

The proof for the case $b > 0$ is similar so we omit the details. □

Corollary 2.10. *Let A_0, A_1 θ, q and b be as in Theorem 2.9. If one of the spaces A_0 and A_1 is uniformly non- ℓ_n^1, then the logarithmic space $A_\theta(logA)_{b,q}$ is uniformly non- ℓ_n^1.*

Proof: A space X is uniformly non- ℓ_n^1 iff $J_n(X) < n$. So, $J_n(A_0) < n$ or $J_n(A_1) < n$. Therefore, from Theorem 3.9 we obtain $\frac{J_n(A_\theta(logA)_{b,q})}{n} < 1$. Thus the space $A_\theta(logA)_{b,q}$ is uniformly non- ℓ_n^1.

Corollary 2.11. *Let A_0, A_1 θ, q and b be as in Theorem 2.9. If one of the spaces A_0 and A_1 is B- convex, then the logarithmic space $A_\theta(logA)_{b,q}$ is B-convex.*

About the classical James constant $J(X)$, using [18], we get a sharper and simpler estimate of $J(A_\theta(logA)_{b,q})$.

Theorem 2.12. *Let A_0, A_1 θ, q, b, t, A and J be as in Theorem 2.9.*
(i) If $b < 0$, then

$$\frac{J(A_\theta(logA)_{b,q})}{2} \le \left(\frac{J(A_0)}{2}\right)^{\frac{1-\theta}{4t'}}\left(\frac{J(A_1)}{2}\right)^{\frac{\theta}{4t'}} \max\left\{1, \left(\frac{J(A_1)}{J(A_0)}\right)^{\frac{2^{-J-2}}{t'}}\right\}$$

(ii) If $b > 0$, then

$$\frac{J(A_\theta(logA)_{b,q})}{2} \le \left(\frac{J(A_0)}{2}\right)^{\frac{1-\theta}{4t'}}\left(\frac{J(A_1)}{2}\right)^{\frac{\theta}{4t'}} \max\left\{1, \left(\frac{J(A_0)}{J(A_1)}\right)^{\frac{2^{-J-2}}{t'}}\right\}.$$

Proof. Let $b < 0$. For any Banach space X we have $C_{NJ}(X) \le J(X)$ (see [18]). So, by using Proposition 2.8 for $n = 2$ we obtain that

$$\begin{aligned} \frac{J^2(A)}{4} &\le \frac{1}{2} C_{NJ}(A) \le \frac{1}{2} 2^{2/t-1} \sup_{j\ge J} (C_{NJ}(A_{\theta_j}))^{2/t'} \\ &\le \frac{1}{2} 2^{2/t-1} \sup_{j\ge J} (J(A_{\theta_j}))^{2/t'} = \sup_{j\ge J} \left(\frac{J(A_{\theta_j})}{2}\right)^{2/t'} \\ &\le \left(\frac{J(A_0)}{2}\right)^{\frac{1-\theta}{2t'}} \left(\frac{J(A_1)}{2}\right)^{\frac{\theta}{2t'}} \max\left\{1, \left(\frac{J(A_1)}{J(A_0)}\right)^{\frac{2^{-J-1}}{t'}}\right\}. \end{aligned}$$

The proof for the case $b > 0$ goes in the same way so we omit the details. ☐

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References

1. C. Angosto and B. Cascales, *Measures of weak noncompactness in Banach spaces*, Topology and Appl. **156** (2008), 1412–1421.
2. A.G. Askoj and L. Maligranda, *Real interpolation and measure of weak noncompactness*, Math. Nachr. **175** (1995), 5–12.
3. K. Astala and H. O. Tylli, *Seminorms related to weak compactness and to Tauberian operators*, Math. Proc. Cambridge Philos. Soc. **107** (1990), 36–375.
4. F.S. Blasi, *On a property of the unit shpere in a Banach space*, Colloq. Math. **65** (1992), 333–343.
5. E. Casini and M. Vignati, *The uniform non-squarness for the complex intrpolation spaces*, J. Math. Anal. Appl. **164** (1992), 518–523.
6. F. Cobos and A. Martinez, *Remarks on interpolation properties of the measures of weak non-compactness and ideal variations*, Math. Nachr. **208** (1999), 93–100.
7. F. Cobos and A. Martinez, *Extreme estimates for interpolated operators by the real method*, J. London Math. Soc. **60** (2000), 860–870.
8. D. Edmunds and H. Triebel, *Logarithmic spaces and related trace problems*, Funct. Approx. Comment. Math. **26** (1998), 189–204 .
9. B. Jawerth and M. Milman, *Extrapolation theory with applications*, Mem. Amer. Math. Soc. **89** (1991), no 440.
10. M. Kato, Y. Takahashi and K. Hashimoto, *On n-th von Neumann-Jordan constants for Banach spaces*, Bull. Kyushu Inst. Tech. Pure Appl. Math. **45** (1998), 25–33.
11. A. Kryczka, *Weak noncompactness in Banach sequence spaces and its extrapolation properties*, Math. Inequal. Appl. **11** (2008), 297–306.
12. A. Kryczka and S. Prus, *Measure of weak noncompactness under complex interpolation*, Studia Math. **147** (2001), 89–102.
13. A. Kryczka, S. Prus and M. Szczepanik, *Measure of weak noncompactness and real interpolation of operators*, Bull. Austral. Math. Soc. **62** (2000), 389–401.
14. L. Maligranda, L. Nikolova, L. E. Persson and T. Zachariades, *On n-th James and Khintchin constants of Banach spaces*, Math. Inequal. Appl. **11** (2008), 1–22.
15. L. Nikolova and T. Zachariades, *The uniform convexity of the Edmunds–Triebel logarithmic spaces*, J. Math. Anal. Appl., **283** (2003), 549–556.
16. L. Nikolova, L.E. Persson and T. Zachariades, *On Clarkson's Inequality, Type and Cotype for the Edmunds–Triebel Logarithmic Spaces*, Arch. Math. **80** (2003), 165–176.
17. L. Nikolova, L.E. Persson and T. Zachariades, *A study of some constants for Banach spaces*, C. R. Acad. Bulg. Sci. **57**(2) (2004), 5–8.

18. Y. Takahashi and M. Kato, *A simple inequality for the von Neumann-Jordan and James constants of a Banach space*, J. Math. Anal. Appl. **359** (2009), 602–609.

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