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A GENERALIZED BECKENBACH–DRESHER INEQUALITY AND RELATED RESULTS

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In honour of Professor Lars–Erik Persson on the occasion of his 65th birthday

Communicated by J. E. Pečarić

Abstract. After a short expose of the history of the Beckenbach–Dresher inequality, general result and the Aczél type inequality are given and super(sub)additivity of the function $G_{p,q,u}(f,g;A,B) := \frac{A^{\frac{u}{p}}(f^p)}{B^{\frac{u-1}{q}}(g^q)}$ is proved. Also, a difference which is inspired by one integral analogue of the Beckenbach–Dresher inequality is considered.

1. Preliminaries and history

Almost sixty years ago E.F Beckenbach [1] published an inequality, which has aroused interest until nowdays. He proved that for positive real numbers $x_i, y_i > 0$, $i = 1, \ldots, n$ and for $1 \le p \le 2$ the following inequality

$$\frac{\sum_{i=1}^{n}(x_i+y_i)^p}{\sum_{i=1}^{n}(x_i+y_i)^{p-1}} \le \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} x_i^{p-1}} + \frac{\sum_{i=1}^{n} y_i^p}{\sum_{i=1}^{n} y_i^{p-1}}$$

is valid. If $0 \le p \le 1$, then the inequality is reversed.

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Few years later M. Dresher [5] investigated moment spaces and stated that an integral analogue of the previous result holds. In fact he proved that if $p \geq 1 \geq q \geq 0$, and $f, g \geq 0$, then

$$\left(\frac{\displaystyle\int (f+g)^p d\varphi}{\displaystyle\int (f+g)^q d\varphi}\right)^{\frac{1}{p-q}} \leq \left(\frac{\displaystyle\int f^p d\varphi}{\displaystyle\int f^q d\varphi}\right)^{\frac{1}{p-q}} + \left(\frac{\displaystyle\int g^p d\varphi}{\displaystyle\int g^q d\varphi}\right)^{\frac{1}{p-q}}.$$

Some related results can be found in [3], [4] and [7]. In recent literature the above inequality is called the Beckenbach–Dresher inequality.

Three decades later some new, more general results appeared. Firstly, in [9], J. Pečarić and P.R. Beesack introduced two new moments: sums and integrals are substituted by isotonic linear functionals and functions, which appeared in the numerator are different than functions in the denominator. Namely, they gave the following theorem.

Theorem 1.1. *Let $A, B : L \to \mathbb{R}$ be two isotonic linear functionals and $f_i, u_i : E \to [0, \infty\rangle$, $(i = 1, \ldots, n)$, be functions such that f_i^p, u_i^q, $(\sum_{i=1}^n f_i)^p$, $(\sum_{i=1}^n u_i)^q \in L$, where $0 < q < 1 \leq p$ and $B(u_i^q) > 0$. Then*

$$\left[\frac{A\left((\displaystyle\sum_{i=1}^n f_i)^p\right)}{B\left((\displaystyle\sum_{i=1}^n u_i)^q\right)}\right]^{\frac{1}{p-q}} \leq \sum_{i=1}^n \left[\frac{A\left(f_i^p\right)}{B\left(u_i^q\right)}\right]^{\frac{1}{p-q}}.$$

In this text we denote by L a class of real-valued functions on non-empty set E with the properties: if $f, g \in L$, then $(af + bg) \in L$ for all $a, b \in \mathbb{R}$; and the function $\mathbf{1}$ belongs to L, where $\mathbf{1}(t) = 1$ for $t \in E$. A functional $A : L \to \mathbb{R}$ is called an isotonic linear functional if

A1. $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$, $a, b \in \mathbb{R}$;
A2. $f \in L$, $f(t) \geq 0$ on E implies $A(f) \geq 0$.

Almost in the same time J. Petree and L.E. Persson published another generalization of Beckenbach–Dresher inequality, [11]. They also include isotonic linear functionals, but with the same functions as arguments and they introduced a new, the third positive parameter u.

Theorem 1.2. *Let $A, B : L \to \mathbb{R}$ be two isotonic linear functionals and $f_i, u_i : E \to [0, \infty\rangle$, $(i = 1, \ldots, n)$, be functions such that f_i^p, f_i^q, $(\sum_{i=1}^n f_i)^p$, $(\sum_{i=1}^n f_i)^q \in L$, $(i = 1, \ldots, n)$.*

If $u \geq 1$ and $q \leq 1 \leq p$ $(q \neq 0)$, then

$$\frac{A^{\frac{u}{p}}\left((\sum_{i=1}^{n} f_i)^p\right)}{B^{\frac{u-1}{q}}\left((\sum_{i=1}^{n} f_i)^q\right)} \leq \sum_{i=1}^{n} \frac{A^{\frac{u}{p}}(f_i^p)}{B^{\frac{u-1}{q}}(f_i^q)}. \tag{1.1}$$

If $0 < u \leq 1$, $p \leq 1$ and $q \leq 1$, $p, q \neq 0$, then the inequality is reversed.

While the result from [9] was proved using the functional version of the Minkowski and Hölder inequalities, above result is a consequence of the following more general theorem ([8], [11]).

Theorem 1.3. *Let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be an increasing function and let $g : D \to \mathbb{R}_+$ be superadditive.*

a) If F is convex and $f : D \to \mathbb{R}_+^n$ is subadditive, then

$$g(x+y)F\left(\frac{f(x+y)}{g(x+y)}\right) \leq g(x)F\left(\frac{f(x)}{g(x)}\right) + g(y)F\left(\frac{f(y)}{g(y)}\right).$$

b) If F is concave and f is superadditive, then inequality holds in the opposite direction.

Putting $f(z) = A^{1/p}(z^p)$, $g(z) = B^{1/q}(z^q)$, $F(z) = z^u$ we get inequality (1.1).

Finally, in [6] B. Guljaš, C.E.M. Pearce and J. Pečarić generalized the set of indices and gave another integral analogues, which can not be obtained from earlier functional versions given in [9] and [11]. Also, they considered a limiting case, when one or both parameters p and q tend to 0.

Theorem 1.4. *Let (X, Σ_X, μ), (Y, Σ_Y, ν) and (Y, Σ_Y, λ) be measure spaces. Let f, g be non-negative functions on $X \times Y$ such that f is integrable with respect to the measure $(\mu \times \nu)$ and g is integrable with respect to $(\mu \times \lambda)$.*

a) If

(i) $u \geq 1$ and $q \leq 1 \leq p$ $(q \neq 0)$, or

(ii) $u < 0$ and $p \leq 1 \leq q$ $(p \neq 0)$, and all terms exist, then

$$\frac{\left(\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y \left(\int_X g(x,y)d\mu(x)\right)^q d\lambda(y)\right)^{\frac{u-1}{q}}} \leq \int_X \frac{\left(\int_Y f^p(x,y)d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y g^q(x,y)d\lambda(y)\right)^{\frac{u-1}{q}}} d\mu(x). \tag{1.2}$$

If (iii) $0 < u \leq 1$, $p \geq 1$ and $q \leq 1$, $p, q \neq 0$, then the inequality is reversed.

b) If $u \geq 1$ and $p \leq 1$, then

$$\left(\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right)^{\frac{u}{p}} \exp\left(\frac{1-u}{\int_Y d\lambda}\int_Y \log\left(\int_X g(x,y)d\mu(x)\right)d\lambda(y)\right)$$
$$\leq \int_X \left(\int_Y f^p(x,y)d\nu(y)\right)^{\frac{u}{p}} \exp\left(\frac{1-u}{\int_Y d\lambda}\int_Y \log g(x,y)d\lambda(y)\right)d\mu(x).$$

It is worthy to say that the above inequalities describe some properties of different means such as a counter-harmonic mean, the Gini mean, the Stolarsky mean, etc. More about those means a reader can find in monograph [2].

2. Generalized Beckenbach–Dresher inequality and inequality of the Aczél type

The next theorem is obtained combining results of Theorems 1.1 and 1.2.

Theorem 2.1. *(Generalized Beckenbach–Dresher inequality)*
Let $A, B : L \to \mathbb{R}$ be two isotonic linear functionals and $f_i, u_i : E \to [0, \infty\rangle$, $(i = 1, \dots, n)$, be functions such that f_i^p, u_i^q, $(\sum_{i=1}^n f_i)^p$, $(\sum_{i=1}^n u_i)^q \in L$ and $A(f_i^p), B(u_i^q)$, $A((\sum_{i=1}^n f_i)^p)$, $B((\sum_{i=1}^n u_i)^q)$ are positive for some real p, q. If either
(i) $u \geq 1$ and $q \leq 1 \leq p$ $(q \neq 0)$, or
(ii) $u < 0$ and $p \leq 1 \leq q$ $(p \neq 0)$, then

$$\frac{A^{\frac{u}{p}}\left((\sum_{i=1}^n f_i)^p\right)}{B^{\frac{u-1}{q}}\left((\sum_{i=1}^n u_i)^q\right)} \leq \sum_{i=1}^n \frac{A^{\frac{u}{p}}(f_i^p)}{B^{\frac{u-1}{q}}(u_i^q)}. \tag{2.1}$$

If $0 < u \leq 1$, $p \leq 1$ and $q \leq 1$, $p, q \neq 0$, then the inequality (2.1) is reversed.

Proof. The proof is based on the idea from [9]. Let $u \geq 1$ and $q \leq 1 \leq p$ $(q \neq 0)$. Then using the functional Minkowski inequality ([10, p.114]) we get

$$\begin{aligned} \frac{A^{\frac{u}{p}}\left((\sum_{i=1}^n f_i)^p\right)}{B^{\frac{u-1}{q}}\left((\sum_{i=1}^n u_i)^q\right)} &\leq \frac{\left[\sum_{i=1}^n A^{1/p}(f_i^p)\right]^u}{\left[\sum_{i=1}^n B^{1/q}(u_i^q)\right]^{u-1}} \\ &= \left[\sum_{i=1}^n A^{1/p}(f_i^p)\right]^u \left[\sum_{i=1}^n B^{1/q}(u_i^q)\right]^{1-u} \\ &\leq \sum_{i=1}^n A^{u/p}(f_i^p) B^{(1-u)/q}(u_i^q) = \sum_{i=1}^n \frac{A^{\frac{u}{p}}(f_i^p)}{B^{\frac{u-1}{q}}(u_i^q)}, \end{aligned}$$

where in the last inequality the functional Hölder inequality, ([10, p.113]), is used with conjugate exponents $u \geq 0$ and $1 - u \leq 0$. The other cases are proved similarly. □

Next theorem is the Aczél-type result for Generalized Beckenbach–Dresher inequality.

Theorem 2.2. *Let $A, B : L \to \mathbb{R}$ be two isotonic linear functionals, $f_{0,i}, u_{0,i} > 0$ and $f_i, u_i : E \to [0, \infty\rangle$, $(i = 1, \dots, n)$, be functions such that f_i^p, u_i^q, $(\sum_{i=1}^n f_i)^p$,*

$(\sum_{i=1}^n u_i)^q \in L$ and $f_{0,i}^p - A(f_i^p), u_{0,i}^q - B(u_i^q)$, $(\sum_{i=1}^n f_{0,i})^p - A((\sum_{i=1}^n f_i)^p)$, $(\sum_{i=1}^n u_{0,i})^q - B((\sum_{i=1}^n u_i)^q)$ are positive for some real p, q. If either

(i) $u \geq 1$, $(0 < p \leq 1)$ and $(q \leq 1$ or $q < 0)$ or

(ii) $u < 0$, $(0 < q \leq 1)$ and $(p \leq 1$ or $p < 0)$, then

$$\frac{\left((\sum_{i=1}^n f_{0,i})^p - A((\sum_{i=1}^n f_i)^p)\right)^{\frac{u}{p}}}{\left((\sum_{i=1}^n u_{0,i})^q - B((\sum_{i=1}^n u_i)^q)\right)^{\frac{u-1}{q}}} \leq \sum_{i=1}^n \frac{\left(f_{0,i}^p - A(f_i^p)\right)^{\frac{u}{p}}}{\left(u_{0,i}^q - B(u_i^q)\right)^{\frac{u-1}{q}}}.$$

If $0 < u \leq 1$, $(q \geq 1$ or $q < 0)$ and $(p \geq 1$ or $p < 0)$, then the inequality is reversed.

The proof is very similar to the previous proof but instead of the Minkowski inequality we use the Bellman inequality, ([10, p.125]).

Let p, q, u be real numbers and let $G_{p,q,u}(f, g; A, B)$ be a mapping defined as

$$G_{p,q,u}(f, g; A, B) := \frac{A^{\frac{u}{p}}(f^p)}{B^{\frac{u-1}{q}}(g^q)},$$

where A and B are isotonic linear functionals on L and f and g are positive functions, $f^p, g^q \in L$. Theorem 2.1 can be read as the following: the mapping $G_{p,q,u}(f, g; A, B)$ is super(sub)additive in arguments f and g for certain choices of parameters p, q and u. But we have superadditivity in the other two arguments. Namely, the following theorem holds.

Theorem 2.3. *If p, q and u satisfy*

(a) $u \geq 1$, $(p < 0$ or $p \geq 1)$ and $0 < q \leq 1$

or

(b) $u < 0$, $0 < p \leq 1$ and $(q < 0$ or $q \geq 1)$, then

$$G_{p,q,u}(f, g; A_1 + A_2, B_1 + B_2) \leq G_{p,q,u}(f, g; A_1, B_1) + G_{p,q,u}(f, g; A_2, B_2),$$

where A_1, A_2, B_1, B_2 are isotonic linear functionals and f and g are positive functions such that the above terms exist.

If (c) $0 < u \leq 1$, $0 < p \leq 1$ and $0 < q \leq 1$, then the inequality is reversed.

Proof. Let us suppose that $u \geq 1$, $(p < 0$ or $p \geq 1)$ and $0 < q \leq 1$. Using subadditivity of the function $x^{1/p}$ for $p < 0$ or $p \geq 1$ and superadditivity of the function $x^{1/q}$ for $0 < q \leq 1$, and using the Hölder inequality we have the following

$$\begin{aligned} G_{p,q,u}(f, g; A_1 + A_2, B_1 + B_2) &= \left((A_1 + A_2)^{\frac{u}{p}}(f^p)\right)\left((B_1 + B_2)^{\frac{1-u}{q}}(g^q)\right) \\ &\leq \left(A_1^{\frac{1}{p}}(f^p) + A_2^{\frac{1}{p}}(f^p)\right)^u \left(B_1^{\frac{1}{q}}(g^q) + B_2^{\frac{1}{q}}(g^q)\right)^{1-u} \\ &\leq A_1^{\frac{u}{p}}(f^p) B_1^{\frac{1-u}{q}}(g^q) + A_2^{\frac{u}{p}}(f^p) B_2^{\frac{1-u}{q}}(g^q) \\ &= G_{p,q,u}(f, g; A_1, B_1) + G_{p,q,u}(f, g; A_2, B_2). \end{aligned}$$

The other cases are proved on the similar way. □

As a simple consequence of the previous theorem we have the following corollary.

Corollary 2.4. *Let w_1 and w_2 be non-negative functions, A and B be isotonic linear functionals on L and f and g be positive functions such that $w_1 f^p$, $w_2 f^p$, $w_1 g^q$, $w_2 g^q \in L$. Let us denote $G(w) = \frac{A^{\frac{u}{p}}(wf^p)}{B^{\frac{u-1}{q}}(wg^q)}$.*

If p, q, u satisfy (a) or (b) of the previous theorem, then

$$G(w_1 + w_2) \leq G(w_1) + G(w_2). \tag{2.2}$$

If p, q, u satisfy (c), then the inequality (2.2) is reversed and if $w_1 \leq w_2$, then

$$G(w_1) \leq G(w_2).$$

Proof. Putting $A_i(f) = A(w_i f)$ and $B_i(g) = B(w_i g)$, $i = 1, 2$ in Theorem 2.3 we obtain the statement of the corollary. □

3. Integral Beckenbach–Dresher difference

Let (X, Σ_X, μ), (Y, Σ_Y, ν) and (Y, Σ_Y, λ) be measure spaces. Let f, g be non-negative functions on $X \times Y$ such that f is integrable with respect to the measure $(\mu \times \nu)$ and g is integrable with respect to $(\mu \times \lambda)$.

An integral Beckenbach–Dresher difference $BD_1(\mu)$ is defined as

$$BD_1(\mu) = \int_X \frac{\left(\int_Y f^p(x,y)d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y g^q(x,y)d\lambda(y)\right)^{\frac{u-1}{q}}} d\mu(x) - \frac{\left(\int_Y \left(\int_X f(x,y)d\mu(x)\right)^p d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y \left(\int_X g(x,y)d\mu(x)\right)^q d\lambda(y)\right)^{\frac{u-1}{q}}},$$

where we suppose that all terms exist.

Theorem 3.1. *If*

(i) $u \geq 1$ and $q \leq 1 \leq p$ $(q \neq 0)$, or

(ii) $u < 0$ and $p \leq 1 \leq q$ $(p \neq 0)$, and all terms exist, then

$$BD_1(\mu_1 + \mu_2) \geq BD_1(\mu_1) + BD_1(\mu_2) \tag{3.1}$$

and if $\mu_2 - \mu_1$ is a measure, then

$$BD_1(\mu_1) \leq BD_1(\mu_2). \tag{3.2}$$

Also, if M and m are real numbers such that $M \geq m \geq 0$ and $\mu_1 - m\mu_2$ and $M\mu_2 - \mu_1$ are measures, then

$$M \cdot BD_1(\mu_2) \geq BD_1(\mu_1) \geq m \cdot BD_1(\mu_2). \tag{3.3}$$

If (iii) $0 < u \leq 1$, $p \leq 1$ and $q \leq 1$, $p, q \neq 0$, then the inequalities (3.1), (3.2) and (3.3) are reversed.

Proof. Let us suppose that (i) or (ii) is valid. Then we have

$$BD_1(\mu_1 + \mu_2) - BD_1(\mu_1) - BD_1(\mu_2) =$$

$$= \frac{\left(\int_Y \left(\int_X f(x,y)d\mu_1(x)\right)^p d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y \left(\int_X g(x,y)d\mu_1(x)\right)^q d\lambda(y)\right)^{\frac{u-1}{q}}} + \frac{\left(\int_Y \left(\int_X f(x,y)d\mu_2(x)\right)^p d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y \left(\int_X g(x,y)d\mu_2(x)\right)^q d\lambda(y)\right)^{\frac{u-1}{q}}}$$

$$- \frac{\left(\int_Y \left(\int_X f(x,y)d\mu_1(x) + \int_X f(x,y)d\mu_2(x)\right)^p d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y \left(\int_X g(x,y)d\mu_1(x) + \int_X g(x,y)d\mu_2(x)\right)^q d\lambda(y)\right)^{\frac{u-1}{q}}}$$

$$\geq 0,$$

where in the last inequality we use (1.2) from Theorem 1.4, when the measure ν is discrete.

Using the result of Theorem 1.4 that if (i) or (ii) are satisfied, then $BD_1(\mu) \geq 0$, we have

$$\begin{aligned} BD_1(\mu_2) &= BD_1(\mu_1 + (\mu_2 - \mu_1)) \\ &\geq BD_1(\mu_1) + BD_1(\mu_2 - \mu_1) \geq BD_1(\mu_1). \end{aligned}$$

□

For fixed measures μ, ν, λ and functions f and g we define a difference BD_2 on the following way

$$BD_2(A) = \int_A \frac{\left(\int_Y f^p(x,y)d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y g^q(x,y)d\lambda(y)\right)^{\frac{u-1}{q}}} d\mu(x) - \frac{\left(\int_Y \left(\int_A f(x,y)d\mu(x)\right)^p d\nu(y)\right)^{\frac{u}{p}}}{\left(\int_Y \left(\int_A g(x,y)d\mu(x)\right)^q d\lambda(y)\right)^{\frac{u-1}{q}}},$$

where A is a subset of X.

For BD_2 the following result holds.

Theorem 3.2. *If (i) or (ii) from Theorem 3.1 is valid and if $A_1, A_2 \subseteq X$, $A_1 \cap A_2 = \emptyset$, then*

$$BD_2(A_1 \cup A_2) \geq BD_2(A_1) + BD_2(A_2).$$

If $A_1 \subseteq A_2$, then

$$BD_2(A_1) \leq BD_2(A_2).$$

Especially, if S_k is a subset of X with k elements and if $S_m \supset S_{m-1} \supset \ldots \supset S_2$, then we have

$$BD_2(S_m) \geq BD_2(S_{m-1}) \geq \ldots \geq BD_2(S_2) \geq 0$$

and

$$BD_2(S_m) \geq \max\{BD_2(S_2) : S_2 \textit{ is any subset of } S_m \textit{ with 2 elements}\}.$$

If (iii) from Theorem 3.1 is valid, then the above inequalities are reversed with max → min.

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