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ON SOME CHARACTERIZATIONS OF CARLESON TYPE MEASURE IN THE UNIT BALL

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Abstract. The aim of this paper is to obtain some new characterizations of Carleson type measure for holomorphic Triebel–Lizorkin spaces and holomorphic Besov type spaces in the unit ball.

1. Introduction and notations

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball of $\mathbb{C}^n$ and S the unit sphere of $\mathbb{C}^n$. Let dv be the normalized Lebesgue measure on B and $d\sigma$ the normalized rotation invariant Lebesgue measure on S. We denote by $H(B)$ the class of all holomorphic functions on B. If $f \in H(B)$ and $f = \sum_k f_k$ is its homogeneous expansion, we denote the high radial derivative by $\mathcal{R}^m f = \sum_k k^m f_k$.

Let $0 < p < \infty$ and $\alpha > -1$. Recall that the weighted Bergman space A^p_α consists of those functions $f \in H(B)$ such that

$$\|f\|^p_{A^p_\alpha} = \int_B |f(z)|^p dv_\alpha(z) = C_\alpha \int_B |f(z)|^p (1 - |z|^2)^\alpha dv(z) < \infty,$$

where $C_\alpha = \Gamma(n + \alpha + 1)/(n!\Gamma(\alpha + 1))$. When $\alpha = 0$, we get the classical Bergman space which will be denoted by A^p. See [5, 10] for more details of weighted Bergman spaces.

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Let $0 < p, q < \infty$, $k > s$, $k, s \in \mathbb{R}$, $f \in H(B)$. Recall that $f \in F^{p,q}_s$, called the holomorphic Triebel–Lizorkin spaces, if

$$\|f\|^p_{F^{p,q}_s} = \int_S \left(\int_0^1 |(I + \mathcal{R})^k f(r\xi)|^q (1-r)^{(k-s)q-1} dr \right)^{p/q} d\sigma(\xi) < \infty.$$

The holomorphic Besov type spaces for the same values of parameters is defined as follows (see [7]).

$$B^{p,q}_s = \{f \in H(B) : \|f\|^q_{B^{p,q}_s} = \int_0^1 M^q_p((I + \mathcal{R})^k f, r)(1-r)^{q(k-s)-1} dr < \infty\},$$

where I is identity operator and

$$M^p_p(f, r) = \int_S |f(r\xi)|^p d\sigma(\xi) \qquad (0 < p < \infty,\ r \in (0, 1))$$

In the unit ball, these classes do not depend on k and include Hardy, Hardy–Sobolev, Bergman classes for particular values of parameters. They were considered by J.Ortega and J. Fàbrega in [7, 8].

Let $r > 0$ and $z \in B$, the Bergman metric ball at z is defined as

$$D(z, r) = \left\{ w \in B : \beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} < r \right\}.$$

Here the involution φ_z has the form

$$\varphi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z \rangle},$$

where $s_z = (1 - |z|^2)^{1/2}$, P_z is the orthogonal projection into the space spanned by $z \in B$, i.e., $P_z w = \frac{\langle w,z \rangle z}{|z|^2}$, $\quad P_0 w = 0$ and $Q_z = I - P_z$ (see, for example, [9] or [10]).

Various Carleson type embedding theorems in the unit ball are well known. In general, the formulation is the following. Let G be a region, μ be a finite positive Borel measure and $\mathbb{X}$ a Banach space of holomorphic functions in G. We say that μ is a Carleson measure for $\mathbb{X}$ if there exists a constant $C > 0$ such that for any $f \in \mathbb{X}$,

$$\int_G |f(z)|^p d\mu(z) \le C \|f\|^p_{\mathbb{X}} \qquad (0 < p < \infty).$$

For various Banach spaces in the unit ball, the characterizations of Carleson measure are known, see for example [1, 2, 3, 10].

In this paper, we completely describe some Carleson type measures in the unit ball for holomorphic Triebel–Lizorkin spaces and Besov type spaces. Note that such results in the unit disk were obtained in [4, 6].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $C^{-1} B \le A \le CB$.

2. On characterizations of Carleson type measure for holomorphic Triebel–Lizorkin type spaces and holomorphic Besov spaces in the unit ball

To state and prove our results in this section, let's collect some nice properties of the Bergman metric ball that will be used in this paper.

Lemma 2.1. *([10]) There exists a positive integer N such that for any $0 < r \leq 1$ we can find a sequence $\{a_k\}$ in B with the following properties:*
(1) $B = \cup_k D(a_k, r)$;
(2) The sets $D(a_k, r/4)$ are mutually disjoint;
(3) Each point $z \in B$ belongs to at most N of the sets $D(a_k, 2r)$.

Remark 2.2. If $\{a_k\}$ is a sequence from Lemma 2.1, according to the result on [10, p. 76], there exist positive constants C_1, C_2 such that

$$C_1 \int_B |f(z)|^p dv_\alpha(z) \leq \sum_{k=1}^{\infty} |f(a_k)|^p (1-|a_k|^2)^{n+1+\alpha} \leq C_2 \int_B |f(z)|^p dv_\alpha(z).$$

Such a sequence will be called a Bergman sampling sequence.

Lemma 2.3. *([10]) For each $r > 0$ there exists a positive constant C_r such that*

$$C_r^{-1} \leq \frac{1-|a|^2}{1-|z|^2} \leq C_r, \quad C_r^{-1} \leq \frac{1-|a|^2}{|1-\langle z, a\rangle|} \leq C_r,$$

for all a and z such that $\beta(a,z) < r$. Moreover, if r is bounded above, then we may choose C_r independent of r.

Lemma 2.4. *([10]) Suppose $r > 0$, $p > 0$ and $\alpha > -1$.Then there exists a constant $C > 0$ such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w)$$

for all $f \in H(B)$ and $z \in B$.

Now we are in a position to state and prove the main results of this paper.

Theorem 2.5. *Let μ be a positive Borel measure on B, $f \in H(B)$. Let $\{a_k\}$ be a Bergman sampling sequence. Assume that $s < 0$, $q < p$ or $s < 0, q = p, \tau \leq p$. Then*

$$\int_B |f(z)|^p d\mu(z) \leq C \|f\|^p_{F_s^{q,\tau}}$$

if and only if

$$\mu(D(a,r)) \leq C_2 (1-|a|^2)^{np/q - sp}, \ a \in B,$$

or

$$\mu(D(a_k,r)) \leq C_2 (1-|a_k|^2)^{np/q-sp}$$

for all $k \geq 1$ and some positive constants C_1 and C_2.

Proof. First we consider the case of $q < p, s < 0$. If the inequality

$$\Big(\int_B |f(z)|^p d\mu\Big)^{1/p} \le C\|f\|_{F_s^{q,\tau}},\ s < 0,$$

is true, then putting

$$f(z) = \Big(\frac{(1-|a|)^{n+\alpha+1}}{(1-\langle z,a\rangle)^{2(n+1+\alpha)}}\Big)^{1/p},\ z,\ a \in B,$$

in the above inequality(in the case of $k = 0$), where α is large enough, it holds

$$K = \Big(\int_B \frac{(1-|a|)^{n+\alpha+1}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} d\mu(z)\Big)^{1/p} \le C\|f\|_{F_s^{q,\tau}}.$$

On one hand, using [10, Theorem 1.12],

$$\begin{aligned}
\|f\|_{F_s^{\tau,q}} &= \Big(\int_S \Big(\int_0^1 \frac{(1-|a|^2)^{(n+\alpha+1)\tau/p}(1-|z|^2)^{-s\tau-1}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)\tau/p}} d|z|\Big)^{q/\tau} d\sigma(\xi)\Big)^{1/q} \\
&\le C(1-|a|^2)^{(n+\alpha+1)/p}\Big(\int_S \frac{d\sigma(\xi)}{|1-\langle \xi,a\rangle|^{(\frac{2\tau(n+1+\alpha)}{p}+\tau s)\frac{q}{\tau}}}\Big)^{1/q} \\
&\le \frac{C(1-|a|^2)^{(n+\alpha+1)/p}}{(1-|a|^2)^{2(n+1+\alpha)/p+s-n/q}} \le \frac{C}{(1-|a|^2)^{(n+1+\alpha)/p+s-n/q}}.
\end{aligned}$$

On the other hand,

$$K \ge \Big(\int_{D(a,r)} \frac{(1-|a|^2)^{n+1+\alpha} d\mu(z)}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}}\Big)^{1/p} \ge \frac{\mu^{1/p}(D(a,r))}{(1-|a|^2)^{(1+\alpha+n)/p}}.$$

Therefore

$$\mu(D(a,r)) \le C(1-|a|^2)^{np/q-sp},\ a \in B,$$

or $\mu(D(a_k,r)) \le C(1-|a_k|^2)^{np/q-sp}$ for Bergman sampling sequence $\{a_k\}$.

Conversely, suppose that (4) or (5) holds. From [10, Theorem 2.25] we see that

$$\|f\|_{L^p(B,d\mu)} \le C\|f\|_{A^p_{-tp-1}},$$

where $t = s + n/p - n/q < 0$. Since (see [7])

$$\|f\|_{A^p_{-tp-1}} \le C\|f\|_{F_s^{q,\tau}} \qquad (s < 0)$$

we get the desired result.

For the case of $q = p$, we just use another embedding from [7]

$$F_s^{p,r} \subset F_s^{p,p},\ \text{when } r \le p, s < 0,$$

and repeat step by step arguments as the first case. □

Remark 2.6. Since $F_s^{p,q}$ include Hardy, weighted Bergman spaces, we get extensions of embedding theorems from [10, p. 59] and [10, p. 168].

Theorem 2.7. *Let μ be a positive Borel measure on B, $f \in H(B)$, $\{a_k\}$ a Bergman sampling sequence. Let $s < 0$, $q < p$ or $s < 0, q = p, \tau \leq p$. Then*

$$\int_B |f(z)|^p d\mu(z) \leq C\|f\|^p_{B_s^{q,\tau}}$$

if and only if

$$\mu(D(a,r)) \leq C_2(1-|a|^2)^{np/q-sp}, \ a \in B,$$

or $\mu(D(a_k,r)) \leq C_2(1-|a_k|^2)^{np/q-sp}$ for all $k \geq 1$ and some positive constants C_1 and C_2.

Proof. The proof can be done similarly as in the case of $F_s^{q,\tau}$ spaces and is based on embedding theorems from [7], hence we omit the details. □

Recall that a positive Borel measure μ on B is called a γ-Carleson measure if there exists a constant $C > 0$ such that (see [10])

$$\mu(Q_r(\zeta)) \leq Cr^\gamma$$

for all $\zeta \in S$ and $r > 0$, where $\gamma > 0$ and

$$Q_r(\zeta) = \{z \in B : |1 - \langle z, \zeta\rangle|^{1/2} < r\}.$$

In the following assertion we use one more time the properties of Bergman metric ball for characterization of Carleson type measure.

Theorem 2.8. *Let $g \in H(B)$, $\beta > -n-1$, $\gamma > 0$ and $q > 1$ such that $\beta q + n > 0$. Then*

$$\sup_{s\in(0,1),\xi\in S} \frac{1}{s^{2\gamma}} \int_{Q_s(\xi)} \frac{|g(z)|^{pq}(1-|z|)^{q(n+1+\beta)}}{1-|z|} dv(z) < \infty$$

if and only if

$$\sup_{s\in(0,1),\xi\in S} \frac{1}{s^{2\gamma}} \int_{Q_s(\xi)} \left(\frac{\int_{D(z,r)} |g(w)|^p dv(w)}{(1-|z|)^{-\beta}}\right)^q \frac{dv(z)}{1-|z|} < \infty.$$

Proof. By subharmonicity of g and Lemma 2.4

$$\begin{aligned} &\sup_{s\in(0,1),\xi\in S} \frac{1}{s^{2\gamma}} \int_{Q_s(\xi)} \frac{|g(z)|^{pq}(1-|z|)^{q(n+1+\beta)}}{1-|z|} dv(z) \\ \leq\ & C \sup_{s\in(0,1),\xi\in S} \frac{1}{s^{2\gamma}} \int_{Q_s(\xi)} \left(\frac{\int_{D(z,r)} |g(w)|^p dv(w)}{(1-|z|)^{-\beta}}\right)^q \frac{1}{1-|z|} dv(z) < \infty. \end{aligned}$$

Conversely, using Hölder inequality and Lemma 2.3 we have

$$\begin{aligned} &\frac{1}{s^{2\gamma}} \int_{Q_s(\xi)} \left(\frac{\int_{D(z,r)} |g(w)|^p dv(w)}{(1-|z|)^{-\beta}}\right)^q \frac{1}{1-|z|} dv(z) \\ \leq\ & C \frac{1}{s^{2\gamma}} \int_{Q_s(\xi)} \int_{D(z,r)} \frac{|g(w)|^{pq} dv(w)}{(1-|w|)^{-\beta q}} \frac{(1-|z|)^{(n+1)(q-1)}}{1-|z|} dv(z). \end{aligned}$$

Since $D(z,r) \subset Q_\rho(\xi)$ by [10, Lemma 5.23] for some z, ξ such that $z = (1-\sigma\rho^2)\xi$, where $\xi \in S$, $\rho \in (0,1)$, $\sigma \in (0,1)$(depending on r but not on ρ), moreover $(1-|z|)^\gamma \asymp \rho^{2\gamma}$, by Lemma 2.3 we have

$$\begin{aligned}
&\int_{D(z,r)} \frac{|g(w)|^{pq} dv(w)}{(1-|w|)^{-\beta q}} \frac{(1-|z|)^{(n+1)(q-1)}}{1-|z|} \\
\leq\ & (1-|z|)^{-(n+1)} \int_{D(z,r)} \frac{|g(w)|^{pq}(1-|w|)^{q(n+1+\beta)}}{(1-|w|)} dv(w) \\
\leq\ & (1-|z|)^{-(n+1)} \rho^{2\gamma} \sup_{\rho\in(0,1),\xi\in S} \frac{1}{\rho^{2\gamma}} \int_{Q_\rho(\xi)} \frac{|g(w)|^{pq}(1-|w|)^{q(n+1+\beta)}}{(1-|w|)} dv(w) \\
\leq\ & C(1-|z|)^{-(n+1)+\gamma}.
\end{aligned}$$

It remains to note that

$$\frac{1}{s^{2\gamma}} \int_{Q_s(\xi)} (1-|z|)^{-(n+1)+\gamma} dv(z) \leq C$$

by [10, Lemma 5.23]. □

It is interesting that Bergman metric ball $D(a_k, r)$ can be used also in the study of embedding theorems in the unit ball of the type

$$\int_B |f(z)|^p d\mu(z) \leq C\|f\|_{\mathbb{Y}}^p, \qquad (0 < p < \infty). \tag{2.1}$$

We want to find sufficient conditions on measure μ such that (2.1) holds. Here $\mathbb{Y}$ is a holomorphic function space with finite quasinorm of the type (see [7])

$$\|A_{q_1}^\alpha(f)(\xi)\|_{L^{p_1}(S)} = \Big\| \Big(\int_{\Gamma_\sigma(\xi)} \frac{|f(z)|^{q_1} dv_\alpha(z)}{(1-|z|^2)^{n+1}} \Big)^{1/q_1} \Big\|_{L^{p_1}(S)}$$

or

$$\|A_\infty^\alpha(f)(\xi)\|_{L^{p_1}(S)} = \Big\| \sup_{z\in\Gamma_\sigma(\xi)} |f(z)|(1-|z|)^\alpha \Big\|_{L^{p_1}(S)}$$

or

$$\|C_{q_1}^\alpha(f)(\xi)\|_{L^{p_1}(S)} = \Big\| \sup_t \Big(\frac{1}{|I_{\xi,t}|} \int_{\widetilde{I}_{\xi,t}} \frac{|f(z)|^{q_1} dv_\alpha(z)}{(1-|z|^2)} \Big)^{1/q_1} \Big\|_{L^{p_1}(S)}.$$

Here $0 < q_1 < \infty$, $\alpha \geq 0$, $0 < p_1 \leq \infty$,

$$I_{\xi,t} = \{\eta \in S, |1-\langle \xi, \eta\rangle| < t\},\ \widetilde{I}_{\xi,t} = \{z \in B, |1-\langle \xi, z\rangle| < t\},\ t > 0,\ \xi \in S$$

and

$$\Gamma_\sigma(\xi) = \{z \in B : |1-\langle z, \xi\rangle| < \sigma(1-|z|)\}.$$

We give only one example connected with spaces defined with the help of $A_{q_1}^\alpha(f)$ functions. Note that similar arguments to get sufficient condition on measure can be used for spaces defined by $C_{q_1}^\alpha(f)(\xi)$ function. We have

$$\begin{aligned}
\int_B |f(z)|^p (1-|z|)^\alpha d\mu(z) &\leq \sum_{k=1}^\infty \max_{z\in D(a_k,r)} |f(z)|^p (1-|z|)^\alpha \mu(D(a_k, r)) \\
&\leq C \int_B |f(z)|^p g_1(z)(1-|z|)^\alpha dv(z),
\end{aligned}$$

by Lemmas 2.1 and 2.4, where

$$g_1(z) = \sum_{k=1}^{\infty} (1-|a_k|)^{-(n+1)} \mu(D(a_k, r)) \times \chi_{D(a_k,r)}(z).$$

It remains to use the estimate (see [7])

$$\int_B \frac{|f(z)||g(z)|}{1-|z|}(1-|z|)^{\alpha} dv(z) \leq C \int_S A^{\alpha}_{q'}(f)(\xi) d\xi \|C^{\alpha}_q(g)\|_{L^{\infty}(S)},$$

to get condition on μ which will be sufficient for estimate (2.1). Here $\frac{1}{q} + \frac{1}{q'} = 1,\ 0 < q' \leq \infty$.

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