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MIXED MEANS FOR CENTERED AND UNCENTERED AVERAGING OPERATORS OVER SPHERES AND RELATED RESULTS

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This paper is dedicated to Professor Josip Pečarić

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Abstract. Mixed-mean inequalities for integral power means over centered and uncentered spheres are proved. Therefrom we deduce the Hardy type inequalities for corresponding averaging operators. Moreover, we discuss estimates related to the spherical maximal functions.

1. Introduction

This paper is a continuation of series of papers [3, 4, 5] which deal with the problem of deriving mixed-mean inequalities for various averaging operators acting on functions defined on $\mathbb{R}^n$. The mixed-mean inequalities are of interest themselves, but they can also produce important inequalities, of which the most important are the Hardy type inequalities.

Throughout the paper we assume that all involved functions are non-negative.

M. Christ and L. Grafakos introduced in [1] the averaging operator particularly suitable for deriving mixed-mean inequalities

$$(T_\delta f)(\mathbf{x}) = \frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|}\int_{B(\mathbf{x},\delta|\mathbf{x}|)} f(\mathbf{y})d\mathbf{y},\ f \in L^1_{\text{loc}}(\mathbb{R}^n)),$$

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where $\delta > 0$, $B(\mathbf{x}, r)$ is the ball in $\mathbb{R}^n$ centered at $\mathbf{x} \in \mathbb{R}^n$ and of radius $r > 0$, $|\mathbf{x}|$ is the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$ and $|A|$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. In the same paper they proved the Hardy type inequality for the operator T_δ and henceforth deduced its operator norm on $L^p(\mathbb{R}^n)$. The basic tool in their proof was Young's inequality $\|f * K\|_p \le \|f\|_p \|K\|_1$ for the convolution on the group $\left(\mathbb{R}^+, \frac{dt}{t}\right)$. An interesting and important feature of this norm is that it is an lower bound for the Hardy-Littlewood (centered) maximal function

$$(M_c f)(\mathbf{x}) = \sup_{r>0} \frac{1}{|B(\mathbf{x}, r)|} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) d\mathbf{y}.$$

In [5] by proving the appropriate mixed-mean inequality, we derived the generalization of this result, in the sense that we obtained the operator norm on weighted L^p spaces (with power weights) of the operator

$$(T_{\delta,\alpha} f)(\mathbf{x}) = \frac{1}{|B(\mathbf{x}, \delta|\mathbf{x}|)|_\alpha} \int_{B(\mathbf{x}, \delta|\mathbf{x}|)} f(\mathbf{y}) |\mathbf{y}|^\alpha d\mathbf{y},$$

where $|A|_\alpha = \int_A |\mathbf{y}|^\alpha d\mathbf{y}$.

The second motivation for this paper is that maximal function can be defined for various collections $\mathcal{C}$ of sets, $\mathcal{C} = \{C \ : \ C \subset \mathbb{R}^n\}$, by

$$(M_{\mathcal{C}} f)(\mathbf{x}) = \sup_{C \in \mathcal{C}} \frac{1}{|C|} \int_C f(\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

This maximal function is closely related to one of the main problems in real-variable theory: For what collections $\mathcal{C}$

$$\lim_{\operatorname{diam}(C) \to 0} \frac{1}{|C|} \int_C f(\mathbf{x} - \mathbf{y}) d\mathbf{y} = f(\mathbf{x}) \ a.e.$$

holds for "all" f (see [14]).

In this paper we consider two collection of sets, a collection of centered spheres and a collection of uncentered spheres and analogous averaging operators

$$\left(\mathcal{S}^c_{\delta,\alpha} f\right)(\mathbf{x}) = \frac{1}{|S^{n-1}(\mathbf{x}, \delta|\mathbf{x}|)|_\alpha} \int_{S^{n-1}(\mathbf{x}, \delta|\mathbf{x}|)} f(\mathbf{y}) |\mathbf{y}|^\alpha ds(\mathbf{y}), \ \delta > 0,$$

$$\left(\mathcal{S}^{unc}_{\delta,\alpha} f\right)(\mathbf{x}) = \frac{1}{|S^{n-1}(\delta\mathbf{x}, |1-\delta||\mathbf{x}|)|_\alpha} \int_{S^{n-1}(\delta\mathbf{x}, |1-\delta||\mathbf{x}|)} f(\mathbf{y}) |\mathbf{y}|^\alpha ds(\mathbf{y}), \ \delta \in \mathbb{R}, \delta \neq 1,$$

defined for suitable f (say continuous with compact support), where $S^{n-1}(\mathbf{a}, r)$ is the sphere in $\mathbb{R}^n$ centered at $\mathbf{a} \in \mathbb{R}^n$ and of radius $r > 0$ and ds is the induced Lebesgue measure. Of course, in both cases the operator norms of these operators are lower bounds for operator norms of appropriate maximal functions defined by

$$(\mathcal{M}_c f)(\mathbf{x}) = \sup_{r>0} \frac{1}{|S^{n-1}(\mathbf{x}, r)|} \int_{S^{n-1}(\mathbf{x}, r)} f(\mathbf{y}) ds(\mathbf{y}),$$

$$(\mathcal{M}_{\text{unc}} f)(\mathbf{x}) = \sup_{\mathbf{a} \in \mathbb{R}^n, r>0, \mathbf{x} \in S^{n-1}(\mathbf{a}, r)} \frac{1}{|S^{n-1}(\mathbf{a}, r)|} \int_{S^{n-1}(\mathbf{a}, r)} f(\mathbf{y}) ds(\mathbf{y}).$$

The importance of these lower bounds can be seen by comparing the operator norm of an maximal function, when it is known, with the maximum (with respect to δ) of the operator norms of operators defined as $\mathcal{S}^{\mathrm{c}}_{\delta,\alpha}$ and $\mathcal{S}^{\mathrm{unc}}_{\delta,\alpha}$. For example, this can be done using results from [8] and calculating the norms of an operator defined analogously as $\mathcal{S}^{\mathrm{unc}}_{\delta,\alpha}$ but for balls instead of spheres.

Our results will be given in a priori forms, in the sense that we shall not go into details about existence and integrability of functions $\mathcal{S}^{\mathrm{c}}_{\delta,\alpha}f$ and $\mathcal{S}^{\mathrm{unc}}_{\delta,\alpha}f$. For further details in this matter see [13, 14]. In what follows we assume that all integrals exist on the respective domains of their definitions.

We shall frequently use the obvious identities

$$|B(r)|_\alpha = \frac{n}{n+\alpha} r^{n+\alpha}|B|,\ |B(\mathbf{x},\delta|\mathbf{x}|)|_\alpha = |\mathbf{x}|^{n+\alpha}\,|B(\mathbf{e},\delta)|_\alpha\,,$$

$$\left|S^{n-1}(r)\right|_\alpha = r^{n+\alpha-1}\left|S^{n-1}\right|,\ \left|S^{n-1}(\mathbf{x},\delta|\mathbf{x}|)\right|_\alpha = |\mathbf{x}|^{n+\alpha-1}\left|S^{n-1}(\mathbf{e},\delta)\right|_\alpha,$$

where $B = B(\mathbf{0},1)$ and $S^{n-1} = S^{n-1}(\mathbf{0},1)$ are the unit ball and the unit sphere respectively.

We shall also use the integral representation (see [15])

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(\theta)d\theta = \int_{SO(n)} f\left(\sigma\mathbf{e}\right)d\sigma, \tag{1.1}$$

where $d\sigma$ is the normalized Haar measure on the rotation group $SO(n)$ of $\mathbb{R}^n$ (which is left and right invariant due to the compactness of $SO(n)$, [10]), $d\theta$ is induced Lebesgue measure on unit sphere S^{n-1}, $\mathbf{e}\in\mathbb{R}^n$ is any unit vector. Note that we change notation of the surface measure ds in the case of unit sphere, in order to be in accordance with the standard notation of polar coordinates in integral over domains in $\mathbb{R}^n$.

2. Mixed-mean inequality

We begin with a technical lemma, which is especially useful in calculating the norms of the operators $\mathcal{S}^{\mathrm{c}}_{\delta,\alpha}$ and $\mathcal{S}^{\mathrm{unc}}_{\delta,\alpha}$. This lemma is a generalization of the calculus arc length formula.

Lemma 2.1. *Suppose that some hypersurface in $\mathbb{R}^n$ is given in polar coordinates with $y = u\phi = tF(\phi\cdot\theta)\phi$, where $t>0$, $\theta\in S^{n-1}$ are fixed, $\phi\in U$, U is an open subset of S^{n-1}, and $F:[-1,1]\to\mathbb{R}$ is an differentiable function. Then*

$$ds(\boldsymbol{y}) = t^{n-1}F^{n-2}(\phi\cdot\theta)\sqrt{F^2(\phi\cdot\theta)+F'^2(\phi\cdot\theta)\left(1-\phi\cdot\theta^2\right)}\,d\phi \tag{2.1}$$

Proof. Using rotational invariance of the induced Lebesgue measure on S^{n-1} it is enough to prove the case when $\theta = \mathbf{e}_n \equiv (0,\dots,0,1)$. In that case $\phi\cdot\theta = \cos\varphi_{n-1},\ \varphi_{n-1}\in[0,\pi]$ and the equation of the hypersurface is $\mathbf{y} = tF(\cos\varphi_{n-1})\,\phi$. The polar coordinates are used in the sense that $\phi = \left(\sin\varphi_{n-1}\bar{\phi},\cos\varphi_{n-1}\right)$, where $\bar{\phi}\in S^{n-2}$. To prove the formula we should calculate the Jacobian $\mathrm{J}\Phi$, where

$$\mathbf{y} = (y_1,\dots,y_n) = \Phi(\varphi_1,\dots,\varphi_{n-1}) = tF(\cos\varphi_{n-1})\left(\sin\varphi_{n-1}\bar{\phi},\cos\varphi_{n-1}\right).$$

Note that $\mathrm{J}\Phi$ is a $n\times(n-1)$ determinant. Using Pythagorean theorem for non-square determinants (see for example [7]) we have

$$(\mathrm{J}\Phi)^2 = \sum_{k=1}^{n}\left(\frac{\partial\left(y_1,\dots,\hat{y_k},\dots,y_n\right)}{\partial\left(\varphi_1,\dots,\varphi_{n-1}\right)}\right)^2,$$

where $\hat{y_k}$ denotes the missing variable. A straightforward calculation reveals

$$\frac{\partial\left(y_1,\ldots,,y_{n-1}\right)}{\partial\left(\varphi_1,\ldots,\varphi_{n-1}\right)} \tag{2.2}$$

$$= t^{n-1}F^{n-2}\left(-F'\sin^2\varphi_{n-1}+F\cos\varphi_{n-1}\right)\sin^{n-2}\varphi_{n-1}\begin{vmatrix} \frac{\partial\bar{\phi}_1}{\partial\varphi_1} & \cdots & \frac{\partial\bar{\phi}_1}{\partial\varphi_{n-2}} & \bar{\phi}_1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\bar{\phi}_{n-1}}{\partial\varphi_1} & \cdots & \frac{\partial\bar{\phi}_{n-1}}{\partial\varphi_{n-2}} & \bar{\phi}_{n-1} \end{vmatrix}$$

and for $k = 1,\ldots,n-1$,

$$\frac{\partial\left(y_1,\ldots,\hat{y_k},\ldots,y_n\right)}{\partial\left(\varphi_1,\ldots,\varphi_{n-1}\right)} \tag{2.3}$$

$$= -t^{n-1}F^{n-2}\left(F'\cos\varphi_{n-1}+F\right)\sin^{n-1}\varphi_{n-1}\frac{\partial\left(\bar{\phi}_1,\ldots,\hat{\bar{\phi}_k},\ldots,\bar{\phi}_{n-1}\right)}{\partial\left(\varphi_1,\ldots,\varphi_{n-2}\right)}.$$

Using $\sum_{k=1}^{n-1}\bar{\phi}_k^2 = 1$ and Pythagorean theorem we obtain

$$\begin{vmatrix} \frac{\partial\bar{\phi}_1}{\partial\varphi_1} & \cdots & \frac{\partial\bar{\phi}_1}{\partial\varphi_{n-2}} & \bar{\phi}_1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\bar{\phi}_{n-1}}{\partial\varphi_1} & \cdots & \frac{\partial\bar{\phi}_{n-1}}{\partial\varphi_{n-2}} & \bar{\phi}_{n-1} \end{vmatrix}^2 = \det\left[\begin{bmatrix} \frac{\partial\bar{\phi}_1}{\partial\varphi_1} & \cdots & \frac{\partial\bar{\phi}_1}{\partial\varphi_{n-2}} \\ \vdots & \vdots & \vdots \\ \frac{\partial\bar{\phi}_{n-1}}{\partial\varphi_1} & \cdots & \frac{\partial\bar{\phi}_{n-1}}{\partial\varphi_{n-2}} \end{bmatrix}\begin{bmatrix} \frac{\partial\bar{\phi}_1}{\partial\varphi_1} & \cdots & \frac{\partial\bar{\phi}_1}{\partial\varphi_{n-2}} \\ \vdots & \vdots & \vdots \\ \frac{\partial\bar{\phi}_{n-1}}{\partial\varphi_1} & \cdots & \frac{\partial\bar{\phi}_{n-1}}{\partial\varphi_{n-2}} \end{bmatrix}^T\right]$$

$$= \sum_{k=1}^{n-1}\left(\frac{\partial\left(\bar{\phi}_1,\ldots,\hat{\bar{\phi}_k},\ldots,\bar{\phi}_{n-1}\right)}{\partial\left(\varphi_1,\ldots,\varphi_{n-2}\right)}\right)^2 = \left(\mathrm{J}\bar{\phi}\right)^2. \tag{2.4}$$

Using (2.2), (2.3) and (2.4) we have

$$\begin{aligned}
(\mathrm{J}\Phi)^2 &= \sum_{k=1}^{n}\left(\frac{\partial\left(y_1,\ldots,\hat{y_k},\ldots,y_n\right)}{\partial\left(\varphi_1,\ldots,\varphi_{n-1}\right)}\right)^2 \\
&= t^{2(n-1)}F^{2(n-2)}\left(F'\cos\varphi_{n-1}+F\right)^2\sin^{2(n-1)}\varphi_{n-1}\left(\mathrm{J}\bar{\phi}\right)^2 \\
&+ t^{2(n-1)}F^{2(n-2)}\left(-F'\sin^2\varphi_{n-1}+F\cos\varphi_{n-1}\right)^2\sin^{2(n-2)}\varphi_{n-1}\left(\mathrm{J}\bar{\phi}\right)^2 \\
&= t^{2(n-1)}F^{2(n-2)}\sin^{2(n-2)}\varphi_{n-1} \\
&\left[\sin^2\varphi_{n-1}\left(F'\cos\varphi_{n-1}+F\right)^2+\left(-F'\sin^2\varphi_{n-1}+F\cos\varphi_{n-1}\right)^2\right]\left(\mathrm{J}\bar{\phi}\right)^2 \\
&= t^{2(n-1)}F^{2(n-2)}\sin^{2(n-2)}\varphi_{n-1}\left[F^2+F'^2\sin^2\varphi_{n-1}\right]\left(\mathrm{J}\bar{\phi}\right)^2.
\end{aligned} \tag{2.5}$$

Finally, using (2.5) follows

$$\begin{aligned}
ds(\mathbf{y}) &= \mathrm{J}\Phi\; d\varphi_1\cdots d\varphi_{n-1} \\
&= t^{n-1}F^{n-2}\sin^{n-2}\varphi_{n-1}\sqrt{F^2+F'^2\sin^2\varphi_{n-1}}\;\mathrm{J}\bar{\phi}\; d\varphi_1\cdots d\varphi_{n-1} \\
&= t^{n-1}F^{n-2}\sqrt{F^2+F'^2\sin^2\varphi_{n-1}}\;\sin^{n-2}\varphi_{n-1}\; d\varphi_{n-1}d\bar{\phi} \\
&= t^{n-1}F^{n-2}\sqrt{F^2+F'^2\left(1-\cos^2\varphi_{n-1}\right)}\; d\phi,
\end{aligned}$$

which, jointly with rotational invariance, gives (2.1). □

Our basic inequality reads as follows. When there is no danger of confusion, we write S instead of S^{n-1}.

Theorem 2.2. *Let $r, s, b, \delta, \alpha_1, \alpha_2 \in \mathbb{R}$ be such that $r \le s$, $r, s \ne 0$, $b > 0$, $\delta > 0$, $\alpha_2 > -n$ and $\alpha_1 > -n+1$ in the case $\delta = 1$. If f is a non-negative function on $B((1+\delta)b)$ (f positive in the case $r < 0$) and $\boldsymbol{b} = b\ \boldsymbol{e}$, $|\boldsymbol{e}| = 1$, then the inequality*

$$\left[\frac{1}{|B(b)|_{\alpha_2}} \int_{B(b)} \left(\frac{1}{|S(\boldsymbol{x}, \delta|\boldsymbol{x}|)|_{\alpha_1}} \int_{S(\boldsymbol{x},\delta|\boldsymbol{x}|)} f^r(\ \boldsymbol{y})|\boldsymbol{y}|^{\alpha_1} ds(\boldsymbol{y})\right)^{\frac{s}{r}} |\boldsymbol{x}|^{\alpha_2} d\boldsymbol{x}\right]^{\frac{1}{s}}$$
$$\le \left[\frac{1}{|S(\boldsymbol{b}, \delta b)|_{\alpha_1}} \int_{S(\boldsymbol{b},\delta b)} \left(\frac{1}{|B(|\ \boldsymbol{x}|)|_{\alpha_2}} \int_{B(|\boldsymbol{x}|)} f^s(\ \boldsymbol{y})|\boldsymbol{y}|^{\alpha_2} d\boldsymbol{y}\right)^{\frac{r}{s}} |\boldsymbol{x}|^{\alpha_1} ds(\boldsymbol{x})\right]^{\frac{1}{r}}. \tag{2.6}$$

holds. Inequality (2.6) is sharp and equality holds for functions of the form $f(\boldsymbol{x}) = C|\boldsymbol{x}|^\lambda$, $C > 0$. In the case $r \ge s$ the sign of inequality in (2.6) is reversed.

Proof. To transform the LHS of inequality (2.6) we use the polar coordinates, so let $\mathbf{x} = t\theta$ and $\mathbf{y} = u\phi$, $t, u \ge 0$, $\theta, \phi \in S^{n-1}$. The relation $|\mathbf{y} - \mathbf{x}| = \delta|\mathbf{x}|$ is now equivalent to expression $u = t\left(\phi \cdot \theta \pm \sqrt{\phi \cdot \theta^2 + \delta^2 - 1}\right)$, where $\phi \cdot \theta$ denotes the inner product in $\mathbb{R}^n$. In the case $0 < \delta < 1$, we have $\phi \cdot \theta \ge \sqrt{1 - \delta^2}$ and we must decompose the sphere into two parts, $S_+^{n-1}(\mathbf{x}, \delta|\mathbf{x}|)$ and $S_-^{n-1}(\mathbf{x}, \delta|\mathbf{x}|)$. In the case $\delta \ge 1$, the minus case has no geometrical meaning.

We continue by considering the case $0 < \delta < 1$. In the case $\delta \ge 1$ the proof follows the same lines. It is obvious that it is enough to prove (2.6) in the case $r = 1 < s = p$ and $b = 1$. In order to simplify the formulas we introduce the following notations $\phi\theta$ for inner product, $F_{1,2}(\phi\theta) = \phi\theta \pm \sqrt{\phi\theta^2 + \delta^2 - 1}$, $H_{1,2}(\phi\theta) = F_{1,2}^{n-2}(\phi\theta) \cdot \sqrt{F_{1,2}^2(\phi\theta) + F_{1,2}'^2(\phi\theta)\left(1 - \phi\theta^2\right)}$ and $I(\phi\theta)$ for the condition $\phi\theta \ge \sqrt{1 - \delta^2}$. Using above transformations and triangle inequality we obtain

$$\left[\frac{1}{|B|_{\alpha_2}} \int_B \left(\frac{1}{|S(\mathbf{x}; \delta|\mathbf{x}|)|_{\alpha_1}} \int_{S(\mathbf{x};\delta|\mathbf{x}|)} f(\mathbf{y})|\mathbf{y}|^{\alpha_1} ds(\mathbf{y})\right)^p |\mathbf{x}|^{\alpha_2} d\mathbf{x}\right]^{1/p}$$
$$\le \frac{1}{|B|_{\alpha_2}^{1/p} |S(\delta)|_{\alpha_1}}$$
$$\cdot \left[\left(\int_{t=0}^1 \int_\theta \left(\int_{I(\phi\theta)} f\left(tF_1(\phi\theta)\phi\right) F_1^{\alpha_1}(\phi\theta) H_1(\phi\theta) d\phi\right)^p t^{\alpha_2+n-1} dt d\theta\right)^{1/p}\right.$$
$$\left. + \left(\int_{t=0}^1 \int_\theta \left(\int_{I(\phi\theta)} f\left(tF_2(\phi\theta)\phi\right) F_2^{\alpha_1}(\phi\theta) H_2(\phi\theta) d\phi\right)^p t^{\alpha_2+n-1} dt d\theta\right)^{1/p}\right]. \tag{2.7}$$

Using integral equality (1.1) and rotational invariance of the induced Lebesgue measure on S^{n-1}, we transform the first term in square brackets in (2.7) as follows

$$
\begin{aligned}
&|S|^{-\frac{1}{p}}\left(\int_{t=0}^{1}\int_{\theta}\left(\int_{I(\phi\theta)} f\left(tF_1\left(\phi\theta\right)\phi\right)F_1^{\alpha_1}(\phi\theta)H_1(\phi\theta)d\phi\right)^p t^{\alpha_2+n-1}dtd\theta\right)^{1/p}\\
&=\left(\int_{t=0}^{1}\int_{\sigma}\left(\int_{I(\phi\sigma\mathbf{e})} f\left(tF_1\left(\phi\sigma\ \mathbf{e}\right)\phi\right)F_1^{\alpha_1}(\phi\sigma\ \mathbf{e})H_1(\phi\sigma\mathbf{e})d\phi\right)^p t^{\alpha_2+n-1}dtd\sigma\right)^{1/p}\\
&=\Bigg(\int_{t=0}^{1}\int_{\sigma}\left(\int_{I(\sigma^{-1}\phi\ \mathbf{e})} f\left(tF_1\left(\sigma^{-1}\phi\ \mathbf{e}\right)\phi\right)F_1^{\alpha_1}(\sigma^{-1}\phi\ \mathbf{e})H_1(\sigma^{-1}\phi\mathbf{e})d\phi\right)^p\\
&\qquad\qquad t^{\alpha_2+n-1}dtd\sigma\Bigg)^{1/p}\\
&=\left(\int_{t=0}^{1}\int_{\sigma}\left(\int_{I(\phi\ \mathbf{e})} f\left(tF_1\left(\phi\ \mathbf{e}\right)\sigma\phi\right)F_1^{\alpha_1}(\phi\mathbf{e})H_1(\phi\ \mathbf{e})d\phi\right)^p t^{\alpha_2+n-1}dtd\sigma\right)^{1/p}\\
&\le\int_{I(\phi\cdot\ \mathbf{e})}\left(\int_{t=0}^{1}\int_{\sigma} f^p\left(tF_1\left(\phi\ \mathbf{e}\right)\sigma\phi\right)t^{\alpha_2+n-1}dtd\sigma\right)^{\frac{1}{p}}F_1^{\alpha_1}(\phi\ \mathbf{e})H_1(\phi\mathbf{e})d\phi\\
&=\int_{I(\phi\mathbf{e})}\left(\frac{|B|_{\alpha_2}}{|B\left(F_1(\phi\ \mathbf{e})\right)|_{\alpha_2}}\int_0^{F_1(\phi\ \mathbf{e})}\int_{\sigma} f^p\left(t\sigma\phi\right)t^{\alpha_2+n-1}d\sigma dt\right)^{\frac{1}{p}}F_1^{\alpha_1}(\phi\mathbf{e})H_1\left(\phi\mathbf{e}\right)d\phi\\
&=\frac{|B|_{\alpha_2}^{\frac{1}{p}}}{|S|^{\frac{1}{p}}}\int_{I(\phi\mathbf{e})}\left(\frac{1}{|B\left(F_1(\phi\ \mathbf{e})\right)|_{\alpha_2}}\int_0^{F_1(\phi\ \mathbf{e})}\int_{\theta} f^p\left(t\theta\right)t^{\alpha_2+n-1}d\theta dt\right)^{\frac{1}{p}}\\
&\qquad\qquad F_1^{\alpha_1}(\phi\mathbf{e})H_1\left(\phi\mathbf{e}\right)d\phi\\
&=\frac{|B|_{\alpha_2}^{\frac{1}{p}}}{|S|^{\frac{1}{p}}}\int_{S_+(\ \mathbf{e};\delta)}\left(\frac{1}{|B\left(|\mathbf{y}|\right)|_{\alpha_2}}\int_{B(|\mathbf{y}|)} f^p\left(\mathbf{x}\right)|\mathbf{x}|^{\alpha_2}d\mathbf{x}\right)^{\frac{1}{p}}|\mathbf{y}|^{\alpha_1}ds(\mathbf{y}),
\end{aligned}\tag{2.8}
$$

where integral Minkowski inequality is used, and integral equality (1.1) again.

Analogous arguing for the second term in (2.7) gives

$$
\begin{aligned}
&\left(\int_{t=0}^{1}\int_{\theta}\left(\int_{I(\phi\theta)} f\left(tF_2\left(\phi\theta\right)\phi\right)F_2^{\alpha_1}(\phi\theta)H_2(\phi\theta)d\phi\right)^p t^{\alpha_2+n-1}dtd\theta\right)^{1/p}\\
&\quad\le|B|_{\alpha_2}^{\frac{1}{p}}\int_{S_-(\ \mathbf{e};\delta)}\left(\frac{1}{|B\left(|\mathbf{y}|\right)|_{\alpha_2}}\int_{B(|\mathbf{y}|)} f^p\left(\mathbf{x}\right)|\mathbf{x}|^{\alpha_2}d\mathbf{x}\right)^{\frac{1}{p}}|\mathbf{y}|^{\alpha_1}ds(\mathbf{y}).
\end{aligned}\tag{2.9}
$$

Using (2.7), (2.8) and (2.9), inequality (2.6) follows.

Finally, it is straightforward to check that both sides of inequality (2.6), rewritten for the function $f(\mathbf{x})=|\mathbf{x}|^{\lambda}$, are equal to

$$
b^{\lambda}M_r\left(|\mathbf{y}|^{\lambda};S^{n-1}(\ \mathbf{e};\delta);\alpha_1\right)M_s\left(|\mathbf{x}|^{\lambda};B;\alpha_2\right),
$$

which gives the sharpness of the inequality. □

Mixed-mean inequality for uncentered case is given in the following theorem.

Theorem 2.3. *Let $r, s, b, \delta, \alpha_1, \alpha_2 \in \mathbb{R}$ be such that $r \le s$, $r, s \ne 0$, $b > 0$, $\delta \ne 1$, $\alpha_2 > -n$ and $\alpha_1 > -n+1$ in the case $\delta = 1/2$. If f is a non-negative function on $B\left((|\delta| + |1-\delta|)b\right)$ (f positive in the case $r < 0$) and $\boldsymbol{b} = b\boldsymbol{e}$, $|\boldsymbol{e}| = 1$, then the inequality*

$$\left[\frac{1}{|B(b)|_{\alpha_2}} \int_{B(b)} \left(\frac{1}{|S(\delta \boldsymbol{x}, |1-\delta||\boldsymbol{x}|)|_{\alpha_1}} \int_{S(\delta \boldsymbol{x}, |1-\delta||\boldsymbol{x}|)} f^r(\boldsymbol{y}) |\boldsymbol{y}|^{\alpha_1} ds(\boldsymbol{y}) \right)^{\frac{s}{r}} |\boldsymbol{x}|^{\alpha_2} d\boldsymbol{x} \right]^{\frac{1}{s}}$$
$$\le \left[\frac{1}{|S(\delta \boldsymbol{b}, |1-\delta|b)|_{\alpha_1}} \int_{S(\boldsymbol{b}, \delta b)} \left(\frac{1}{|B(|\boldsymbol{x}|)|_{\alpha_2}} \int_{B(|\boldsymbol{x}|)} f^s(\boldsymbol{y}) |\boldsymbol{y}|^{\alpha_2} d\boldsymbol{y} \right)^{\frac{r}{s}} |\boldsymbol{x}|^{\alpha_1} ds(\boldsymbol{x}) \right]^{\frac{1}{r}}. \tag{2.10}$$

holds. Inequality (2.6) is sharp and equality holds for functions of the form $f(\boldsymbol{x}) = C|\boldsymbol{x}|^{\lambda}$, $C > 0$. In the case $r \ge s$ the sign of inequality in (2.6) is reversed.

Proof. The proof is analogous to the proof of Theorem 2.2. In transforming inequality (2.10) in polar coordinates using $\mathbf{x} = t\theta$, $\mathbf{y} = u\phi$, the relation $\mathbf{y} \in S\left(\delta\mathbf{x}, |1-\delta||\mathbf{x}|\right)$ is equivalent to equation

$$u^2 - 2ut\delta\phi\cdot\theta + (2\delta - 1)t^2 = 0.$$

Three cases should be considered. For $0 \le \delta \le 1/2$ the equation of the sphere is given by

$$u = t\delta \left(\phi\cdot\theta + \sqrt{(\phi\cdot\theta)^2 + \frac{1-2\delta}{\delta^2}} \right),$$

for $\delta < 0$

$$u = t|\delta| \left(\sqrt{(\phi\cdot\theta)^2 + \frac{1-2\delta}{\delta^2}} - \phi\cdot\theta \right)$$

and for $\delta > 1/2, \delta \ne 1$ we must decompose the sphere into two parts given by

$$u = t\delta \left(\phi\cdot\theta \pm \sqrt{(\phi\cdot\theta)^2 + \frac{1-2\delta}{\delta^2}} \right),$$

with the condition $\phi\cdot\theta \ge \frac{\sqrt{1-2\delta}}{\delta}$.

The rest of the proof is as in the proof of Theorem 2.2. □

3. Hardy and Carleman type inequalities

The mixed means can be used in proving various integral inequalities, such as the Hardy and the Carleman inequality (for the classical theory see [2, 9, 11, 12] and for the multidimensional case see for example [6]). Analogously to the procedure given in [3, 4, 5], we apply the mixed mean inequality (2.6) to deduce the Hardy-type inequalities for the operators $\mathcal{S}^{\mathrm{c}}_{\delta,\alpha}$ and $\mathcal{S}^{\mathrm{unc}}_{\delta,\alpha}$ defined in the Introduction.

Theorem 3.1. *Let $p > 1$, $0 < b \le \infty$, α_1, $\alpha_2 \in \mathbb{R}$, $\delta > 0$ be such that $\alpha_2 > -n$ and $\alpha_1 > -n+1$ in the case $\delta = 1$. If f is a nonnegative function on $B((1+\delta)b)$ and $|\boldsymbol{e}| = 1$, then*

$$\left[\int_{B(b)} \left(\frac{1}{|S(\boldsymbol{x}, \delta|\boldsymbol{x}|)|_{\alpha_1}} \int_{S(\boldsymbol{x},\delta|\boldsymbol{x}|)} f(\boldsymbol{y})|\boldsymbol{y}|^{\alpha_1} ds(\boldsymbol{y})\right)^p |\boldsymbol{x}|^{\alpha_2} d\boldsymbol{x}\right]^{\frac{1}{p}} \le C(n,p;\delta;\alpha_1;\alpha_2) \left(\int_{B((1+\delta)b)} f^p(\boldsymbol{y})|\boldsymbol{y}|^{\alpha_2} d\boldsymbol{y}\right)^{\frac{1}{p}}, \tag{3.1}$$

where

$$C(n,p;\delta;\alpha_1;\alpha_2) = \frac{1}{|S(\boldsymbol{e},\delta)|_{\alpha_1}} \int_{S(\boldsymbol{e},\delta)} |\boldsymbol{x}|^{-\frac{n+\alpha_2}{p}} |\boldsymbol{x}|^{\alpha_1} ds(\boldsymbol{x}),$$

is the best possible constant.

Proof. Let $0 < b < \infty$. Inequality (2.6) for $r = 1$ and $s = p$ and obvious estimation $\int_{B(|\mathbf{x}|)} f^p(\mathbf{y})|\mathbf{y}|^{\alpha_2} d\mathbf{y} \le \int_{B((1+\delta)b)} f^p(\mathbf{y})|\mathbf{y}|^{\alpha_2} d\mathbf{y}$, which holds for every $\mathbf{x} \in S(\mathbf{b}, \delta b)$, implies the inequality

$$\left[\int_{B(b)} \left(\frac{1}{|S(\mathbf{x}, \delta|\mathbf{x}|)|_{\alpha_1}} \int_{S(\mathbf{x},\delta|\mathbf{x}|)} f(\mathbf{y})|\mathbf{y}|^{\alpha_1} ds(\mathbf{y})\right)^p |\mathbf{x}|^{\alpha_2} d\mathbf{x}\right]^{\frac{1}{p}} \le \frac{|B(b)|_{\alpha_2}^{1/p}}{|S(\mathbf{b},\delta b)|_{\alpha_1}} \int_{S(\mathbf{b},\delta b)} |B(|\mathbf{x}|)|_{\alpha_2}^{-1/p} |\mathbf{x}|^{\alpha_1} ds(\mathbf{x}) \left(\int_{B((1+\delta)b)} f^p(\mathbf{y})|\mathbf{y}|^{\alpha_2} d\mathbf{y}\right)^{\frac{1}{p}}.$$

Using $|B(b)|_{\alpha_2} = b^{n+\alpha_2}|B(1)|_{\alpha_2}$, $|S(\mathbf{b},\delta b)|_{\alpha_1} = b^{n+\alpha_1-1}|S(\mathbf{e},\delta)|_{\alpha_1}$, and simple substitution $\mathbf{x}' = \mathbf{x}/b$ and radiallity of the involved function we obtain (3.1). Since the constant $C(n,p;\delta;\alpha_1;\alpha_2)$ is independent of b, inequality (3.1) obviously holds for $b = \infty$ as well.

In the usual manner, for the best possibility of inequality (3.1) consider the functions $f_\epsilon(\mathbf{x}) = |\mathbf{x}|^{-(\alpha_2+n)/p+\epsilon}$. It is straightforward to check that the quotient of the integral expressions on the left side and the right side of inequality (3.1), in this particular choice of functions, tends to the constant $C(n,p;\delta;\alpha_1;\alpha_2)$ as ϵ tends to 0. □

Theorem 3.2. *Let $0 \ne p < 1$, $0 < b \le \infty$, α_1, $\alpha_2 \in \mathbb{R}$, $\delta > 0$ be such that $\alpha_2 > -n$ and $\alpha_1 > -n+1$ in the case $\delta = 1$. If f is a nonnegative function on $B((1+\delta)b)$, then*

$$\int_{B(b)} \left(\frac{1}{|S(\boldsymbol{x}, \delta|\boldsymbol{x}|)|_{\alpha_1}} \int_{S(\boldsymbol{x},\delta|\boldsymbol{x}|)} f^p(\boldsymbol{y})|\boldsymbol{y}|^{\alpha_1} ds(\boldsymbol{y})\right)^{\frac{1}{p}} |\boldsymbol{x}|^{\alpha_2} d\boldsymbol{x} \le C_1(n,p;\delta;\alpha_1;\alpha_2) \int_{B((1+\delta)b)} f(\boldsymbol{y})|\boldsymbol{y}|^{\alpha_2} d\boldsymbol{y},$$

where

$$C_1(n,p;\delta;\alpha_1;\alpha_2) = \left(\frac{1}{|S(\boldsymbol{e},\delta)|_{\alpha_1}} \int_{S(\boldsymbol{e},\delta)} |\boldsymbol{x}|^{-p(n+\alpha_2)} |\boldsymbol{x}|^{\alpha_1} ds(\boldsymbol{x}) \right)^{\frac{1}{p}}, \tag{3.2}$$

is the best possible constant.

Proof. The proof is analogous to the proof of Theorem 3.1 taking in Theorem 2.6 $r = p < s = 1$. For the best possibility of the constant $C_1(n,p;\delta;\alpha_1;\alpha_2)$, arguing is the same as in Theorem 3.1 using functions $f_\epsilon(\mathbf{y}) = |\mathbf{y}|^{-p(n+\alpha_2)+\epsilon}$. □

Finally, we give the related Carleman type inequality for geometric mean.

Theorem 3.3. *Let $0 < b \le \infty$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $\delta > 0$ be such that $\alpha_2 > -n$ and $\alpha_1 > -n+1$ in the case $\delta = 1$. If f is a positive function on $B((1+\delta)b)$, then*

$$\int_{B(b)} \exp\left[\frac{1}{|S(\boldsymbol{x},\delta|\boldsymbol{x}|)|_{\alpha_1}} \int_{S(\boldsymbol{x},\delta|\boldsymbol{x}|)} |\boldsymbol{y}|^{\alpha_1} \log f(\boldsymbol{y})\, ds(\boldsymbol{y}) \right] |\boldsymbol{x}|^{\alpha_2} d\boldsymbol{x}$$
$$\le C_2(n;\delta;\alpha_1,\alpha_2) \int_{B((1+\delta)b)} f(\boldsymbol{y}) |\boldsymbol{y}|^{\alpha_2}\, d\boldsymbol{y}, \tag{3.3}$$

where

$$C_2(n;\delta;\alpha_1,\alpha_2) = \exp\left[\frac{\alpha_2+n}{|S(\boldsymbol{e},\delta)|_{\alpha_1}} \int_{S(\boldsymbol{e},\delta)} |\boldsymbol{x}|^{\alpha_1} \log \frac{1}{|\boldsymbol{x}|}\, ds(\boldsymbol{x}) \right]$$

is the best possible constant.

Proof. Inequality (3.3) follows from (3.2) by taking the limiting procedure $\lim_{p\to 0}$.

We give here the proof that the constant $C_2(n;\delta;\alpha_1,\alpha_2)$ is the best possible one. To do that consider the functions $f_\epsilon(\mathbf{y}) = |\mathbf{y}|^{-n-\alpha_2+\epsilon}$, $\epsilon > 0$. The integral on the right hand side of inequality (3.3), for this choice of functions, is equal to $|S|(1+\delta)^\epsilon b^\epsilon/\epsilon$. The integral on the left hand side of inequality (3.2), for this choice of functions, using substitution $\mathbf{y} \mapsto \mathbf{y}/|\mathbf{x}|$ and obvious transformations gives

$$\int_{B(b)} \exp\left[-\frac{n+\alpha_2-\epsilon}{|S(\mathbf{x},\delta|\mathbf{x}|)|_{\alpha_1}} \int_{S(\mathbf{x},\delta|\mathbf{x}|)} |\mathbf{y}|^{\alpha_1} \log|\mathbf{y}|\, ds(\mathbf{y}) \right] |\mathbf{x}|^{\alpha_2} d\mathbf{x}$$
$$= \int_{B(b)} \exp\left[-\frac{n+\alpha_2-\epsilon}{|S(\mathbf{e},\delta)|_{\alpha_1}} \int_{S(\mathbf{e},\delta)} |\mathbf{y}|^{\alpha_1} (\log|\mathbf{y}| + \log|\mathbf{x}|)\, ds(\mathbf{y}) \right] |\mathbf{x}|^{\alpha_2} d\mathbf{x}$$
$$= \exp\left[-\frac{n+\alpha_2-\epsilon}{|S(\mathbf{e},\delta)|_{\alpha_1}} \int_{S(\mathbf{e},\delta)} |\mathbf{y}|^{\alpha_1} \log|\mathbf{y}| ds(\mathbf{y}) \right] \int_{B(b)} |\mathbf{x}|^{-n+\epsilon} d\mathbf{x}$$
$$= \exp\left[-\frac{n+\alpha_2-\epsilon}{|S(\mathbf{e},\delta)|_{\alpha_1}} \int_{S(\mathbf{e},\delta)} |\mathbf{y}|^{\alpha_1} \log|\mathbf{y}| ds(\mathbf{y}) \right] |S| \frac{b^\epsilon}{\epsilon},$$

which gives that the quotient of the integrals on the left hand side and on the right hand side of the inequality (3.2), for this particular choice of functions, tends to $C_2(n;\delta;\alpha_1,\alpha_2)$ as ϵ tends to 0. □

Keeping in mind Theorem 2.3, it is obvious what are the uncentered versions of Theorems 3.1, 3.2 and 3.3, so we give just the forms of the constants in analogous inequalities

$$C^{unc}(n,p;\delta;\alpha_1;\alpha_2) = \frac{1}{|S(\delta\mathbf{e},|1-\delta|)|_{\alpha_1}} \int_{S(\delta\mathbf{e},|1-\delta|)} |\mathbf{x}|^{-\frac{n+\alpha_2}{p}} |\mathbf{x}|^{\alpha_1} ds(\mathbf{x}), \quad p>1,$$

$$C_1^{unc}(n,p;\delta;\alpha_1;\alpha_2) = \left(\frac{1}{|S(\delta\mathbf{e},|1-\delta|)|_{\alpha_1}} \int_{S(\delta\mathbf{e},|1-\delta|)} |\mathbf{x}|^{-p(n+\alpha_2)} |\mathbf{x}|^{\alpha_1} ds(\mathbf{x})\right)^{\frac{1}{p}}, \quad 0\neq p<1,$$

$$C_2^{unc}(n;\delta;\alpha_1,\alpha_2) = \exp\left[\frac{\alpha_2+n}{|S(\delta\ \mathbf{e},|1-\delta|)|_{\alpha_1}} \int_{S(\delta\ \mathbf{e},|1-\delta|)} |\mathbf{x}|^{\alpha_1} \log\frac{1}{|\mathbf{x}|}\ ds(\mathbf{x})\right], \quad p=0.$$

4. Concluding remarks

We give several remarks on the constants $C(n,p;\delta)=C(n,p;\delta;0,0)$, $C_2(n;\delta)=C_2(n;\delta;0,0)$, $C^{unc}(n,p;\delta)=C^{unc}(n,p;\delta;0,0)$, $C_2^{unc}(n;\delta)=C_2^{unc}(n;\delta;0,0)$.

Using Lemma 2.1 for $\mathbf{x}=\mathbf{e}_n$, $\mathbf{y}=u\phi$ and using $d\phi=\sin^{n-2}\varphi_{n-1}d\varphi_{n-1}d\bar{\phi}$, $\bar{\phi}\in S^{n-2}$, $\phi\cdot\mathbf{e}_n=\cos\varphi_{n-1}$, $|S^{n-1}|=\frac{2\pi^{n/2}}{\Gamma(n/2)}$, we easily get

$$C(n,p;\delta) = \frac{1}{\delta^{n-2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} \left(t+\sqrt{t^2+\delta^2-1}\right)^{\frac{n}{p'}-1} \frac{(1-t^2)^{\frac{n-3}{2}}dt}{\sqrt{t^2+\delta^2-1}},\ \delta\geq 1,$$

$$C^{unc}(n,p;\delta) = \frac{|\delta|^{\frac{n}{p'}-2}}{(1-\delta)^{n-2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^{1} \left(t+\sqrt{t^2+\frac{1-2\delta}{\delta^2}}\right)^{\frac{n}{p'}-1} \frac{(1-t^2)^{\frac{n-3}{2}}dt}{\sqrt{t^2+\frac{1-2\delta}{\delta^2}}},\ \delta\leq\frac{1}{2},$$

$$C_2(n;\delta) = \exp\left[-\frac{n}{\delta^{n-2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \cdot\int_{-1}^{1} \log\left(t+\sqrt{t^2+\delta^2-1}\right)\left(t+\sqrt{t^2+\delta^2-1}\right)^{n-1} \frac{(1-t^2)^{\frac{n-3}{2}}dt}{\sqrt{t^2+\delta^2-1}}\right],\ \delta\geq 1,$$

$$\begin{aligned}
&C_2^{unc}(n;\delta)\\
&\quad = \exp\left[-\frac{n|\delta|^{n-2}}{(1-\delta)^{n-2}}\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\right.\\
&\quad\left.\cdot\int_{-1}^{1}\log\left[|\delta|\left(t+\sqrt{t^2+\frac{1-2\delta}{\delta^2}}\right)\right]\left(t+\sqrt{t^2+\frac{1-2\delta}{\delta^2}}\right)^{n-1}\frac{(1-t^2)^{\frac{n-3}{2}}dt}{\sqrt{t^2+\frac{1-2\delta}{\delta^2}}}\right],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \delta\le\frac{1}{2}.
\end{aligned}$$

It is not necessary to have complementary formulas (in the centered case $0 < \delta \le 1$, in uncentered case $\delta > 1/2$) since it is easy to see that the following identities hold

$$C^{unc}(n,p;\delta) = C^{unc}(n,p;1-\delta),\ \delta\le 1/2, \tag{4.1}$$

$$C(n,p;\delta) = \delta^{-\frac{n}{p}}C\left(n,p;\frac{1}{\delta}\right), \tag{4.2}$$

$$C^{unc}(n,p;\delta) = \delta^{-\frac{n}{p}}C\left(n,p;\frac{1}{\delta}-1\right),\ 0<\delta<1. \tag{4.3}$$

In some cases we can explicitly calculate the above constants as functions of δ. The easiest case is $p = \frac{n}{n-2}$. This is the case when the function $\mathbf{x}\mapsto|\mathbf{x}|^{-\frac{n}{p}}$ is a harmonic function. We get $C(n,p=\frac{n}{n-2};\delta)=1,\ 0<\delta\le 1$ and $C(n,p=\frac{n}{n-2};\delta)=\delta^{2-n},\ \delta\ge 1$. Also, $C^{unc}(n,p=\frac{n}{n-2};\delta)=\delta^{2-n},\ \delta\ge 1/2$ and $C^{unc}(n,p=\frac{n}{n-2};\delta)=(1-\delta)^{2-n},\ \delta\le 1/2$. Note that $\sup_{\delta>0}C(n,p=\frac{n}{n-2};\delta)=1$ and $\sup_{\delta}C^{unc}(n,p=\frac{n}{n-2};\delta)=2^{n-2}$. It is easy to see using (4.2) that in harmonic case $p=n/(n-2)$ and in super-harmonic case $p>n/(n-2)$, we get $\sup_{\delta>0}C(n,p;\delta)=1$. Only in sub-harmonic cases $n/(n-1)<p<n/(n-2)$ we obtain non-trivial lower bounds. For example, $C(p=2,n=3;\delta)=\frac{\sqrt{1+\delta}-\sqrt{|1-\delta|}}{\delta}$, so $\sup_{\delta>0}C(p=2,n=3;\delta)=\sqrt{2}$.

The identities (4.1), (4.2), (4.3) and previous examples suggest to consider $C(n,p;1)$ and $C^{unc}(n,p;1/2)$ in order to obtain the best possible lower bounds for operator norms for appropriate maximal functions. We easily get

$$C(n,p;1) = 2^{\frac{n}{p'}-2}\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}B\left(\frac{n}{2p'}-\frac{1}{2},\frac{n-1}{2}\right),\ p>n'=\frac{n}{n-1},$$

and

$$C^{unc}(n,p;\frac{1}{2}) = 2^{n-2}\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}B\left(\frac{n}{2p'}-\frac{1}{2},\frac{n-1}{2}\right),\ p>n'=\frac{n}{n-1}.$$

Also,

$$C_2(n;1) = 2^{-n}\exp\left[\frac{n}{2}\left(H(n-2)-H\left(\frac{n-3}{2}\right)\right)\right],$$

and

$$C_2^{unc}(n;1) = \exp\left[\frac{n}{2}\left(H(n-2) - H\left(\frac{n-3}{2}\right)\right)\right],$$

where $H = H(s),\ s > -1$, are harmonic numbers.

Finally, using Stirling asymptotic formula $\Gamma(x) \sim e^{-x}x^{x-\frac{1}{2}}\sqrt{2\pi}$, we can give asymptotic behavior of the above constants for fixed $p > 1$ and large n. For the similar discussion in the case of balls see [8]. Straightforward calculation gives that $C(n,p;\delta)$ asymptotically behaves as

$$\left(\frac{4^{\frac{1}{p'}}\left(\frac{1}{p'}-\frac{1}{n}\right)^{\frac{1}{p'}}}{\left(\frac{1}{p'}+1-\frac{2}{n}\right)^{1+\frac{1}{p'}}}\right)^{\frac{n}{2}},$$

which shows that $C(n,p;\delta)$ has exponential decay, since by Bernoulli inequality $4/p' < (1+1/p')^{1+p'}$. Analogous arguing gives that $C^{unc}(n,p;\delta)$ asymptotically behaves as

$$\left(\frac{4\left(\frac{1}{p'}-\frac{1}{n}\right)^{\frac{1}{p'}}}{\left(\frac{1}{p'}+1-\frac{2}{n}\right)^{1+\frac{1}{p'}}}\right)^{\frac{n}{2}},$$

which shows that $C^{unc}(n,p;\delta)$ has exponential growth, since $4^{p'}/p' > (1+1/p')^{1+p'}$. Using $\lim_{n\to\infty}(H(2k) - H(k)) = \log 2$, we get that $C_2(n;1)$ asymptotically behaves as $2^{-n/2}$ and $C_2^{unc}(n;1/2)$ behaves as $2^{n/2}$.

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