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HARDY INEQUALITY OF FRACTIONAL ORDER

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This paper is dedicated to Professor Josip E. Pečarić

Submitted by M. Abel

ABSTRACT. We prove optimality of power-type weights in the Hardy inequality of fractional order.

1. Introduction and the main result

In [3] the following theorem was proved.

Theorem 1.1. *Let* $1 \le p < \infty$, $\delta \in (0,1) \cup (1,p)$ *and* u *be a locally integrable function on* $[0,\infty)$. *Let*

(i) *either* $0 < \delta < 1$ *and* $\lim_{t\to\infty} \frac{1}{t}\int_0^t u = 0$,

(ii) *or* $1 < \delta < p$ *and* $\lim_{t\to 0_+} \frac{1}{t}\int_0^t u = 0$.

Then

$$\int_0^\infty |u(x)|^p\, x^{-\delta}\,\mathrm{d}x \le C \int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{\delta+1}}\,\mathrm{d}x\,\mathrm{d}y, \tag{1.1}$$

where $C = (1 + p/|\delta - 1|)^p/2$.

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It is known that the restriction $\delta \in (0,1) \cup (1,p)$ is essential. Indeed, if either $\delta \le 0$ or $\delta \ge p$, then the integral on the right-hand side of (1.1) diverges for each nonzero function $u \in C_0^\infty(0,\infty)$. If $p > 1$ and $\delta = 1$, then there is no finite constant C such that inequality (1.1) holds for all functions in question. Indeed, inserting the functions

$$u_\varepsilon(t) = \frac{t-\varepsilon}{\varepsilon}\chi_{[\varepsilon,2\varepsilon)}(t) + \chi_{[2\varepsilon,1/2)}(t) + 2(1-t)\chi_{(1/2,1)}(t)$$

into (1.1) and letting $\varepsilon \to 0_+$, we obtain that the constant $C \to \infty$. (See [3, Remark 6].) Here the symbol χ_I stands for the characteristic function of an interval $I \subset \mathbb{R}$.

The aim of this paper is to show that power-type weights in inequality (1.1) are optimally chosen. This follows from the next result.

Theorem 1.2. *Let $1 \le p < \infty$. Suppose that $\delta \in (0,1) \cup (1,p)$, $\eta \in (0,p)$ and there is a positive constant C such that the inequality*

$$\int_0^\infty |u(x)|^p\, x^{-\delta}\, \mathrm{d}x \le C \int_0^\infty \int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{\eta+1}}\, \mathrm{d}x\, \mathrm{d}y \tag{1.2}$$

holds for all locally integrable functions u satisfying one of conditions (i), (ii) *of* Theorem 1.1. *Then $\eta = \delta$.*

The proof of Theorem 1.2 is based on some ideas developed in [1] and [2].

2. Proof of Theorem 1.2

To prove Theorem 1.2 we need several lemmas.

Lemma 2.1. *Let $0 < p < \infty$ and w be a measurable nonnegative even function. Then*

$$\int_0^\infty \int_0^\infty |g(x)-g(y)|^p\, w(x-y)\, \mathrm{d}x\, \mathrm{d}y = 2\int_0^\infty \Big(\int_0^\infty |g(y+h)-g(y)|^p\, \mathrm{d}y\Big) w(h)\, \mathrm{d}h, \tag{2.1}$$

provided that the left-hand side of the equality makes sense.

Proof. Using the change of variables $x = y+h$ in the inner integral and applying the Fubini theorem, we obtain

$$\begin{aligned}\int_0^\infty \int_0^\infty |g(x)-g(y)|^p\, w(x-y)\, \mathrm{d}x\, \mathrm{d}y &= \int_0^\infty \Big(\int_{-y}^\infty |g(y+h)-g(y)|^p\, w(h)\, \mathrm{d}h\Big)\, \mathrm{d}y \\ &= \int_0^\infty \Big(\int_0^\infty |g(y+h)-g(y)|^p\, \mathrm{d}y\Big) w(h)\, \mathrm{d}h \\ &\quad + \int_{-\infty}^0 \Big(\int_{-h}^\infty |g(y+h)-g(y)|^p\, \mathrm{d}y\Big) w(h)\, \mathrm{d}h. \end{aligned} \tag{2.2}$$

In the second term we replace h by k and y by z, then we make two changes of variables $h = -k$ and $z - h = y$ and use the fact that $w(-h) = w(h)$, to arrive at

$$\int\limits_{-\infty}^{0}\Big(\int\limits_{-h}^{\infty}|g(y+h)-g(y)|^p\,\mathrm{d}y\Big)w(h)\,\mathrm{d}h = \int\limits_{0}^{\infty}\Big(\int\limits_{0}^{\infty}|g(y+h)-g(y)|^p\,\mathrm{d}y\Big)w(h)\,\mathrm{d}h.$$

Together with (2.2), it gives (2.1). □

In what follows we write $A \lesssim B$ (or $A \gtrsim B$) if $A \le cB$ (or $cA \ge B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B. For $p \in [1,\infty]$, the conjugate number p' is defined by $1/p + 1/p' = 1$ with the convention that $1/\infty = 0$.

Lemma 2.2. *Let w be a measurable nonnegative function, let $p \in [1,\infty)$, $\alpha \in (1,\infty)$ and $\alpha' := \alpha/(\alpha-1)$. Then*

$$\begin{aligned}\int_0^\infty\Big(\int_0^\infty|g(y+h)-g(y)|^p\,\mathrm{d}y\Big)w(h)\,\mathrm{d}h\\ \lesssim \int_0^\infty\Big(\int_0^{2h}|g(y)|^p\,\mathrm{d}y\Big)w(h)\,\mathrm{d}h + \int_0^\infty\Big(\int_h^\infty|g'(y)|^p\,\mathrm{d}y\Big)h^pw(h)\,\mathrm{d}h \quad (2.3)\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty\Big(\int_0^\infty|g(y+h)-g(y)|^p\,\mathrm{d}y\Big)w(h)\,\mathrm{d}h\\ \lesssim \int_0^\infty\Big(\int_0^h\Big(\int_y^\infty|g'(\tau)|^\alpha\,\mathrm{d}\tau\Big)^{p/\alpha}\mathrm{d}y\Big)h^{p/\alpha'}w(h)\,\mathrm{d}h\\ + \int_0^\infty\Big(\int_h^\infty|g'(y)|^p\,\mathrm{d}y\Big)h^pw(h)\,\mathrm{d}h \quad (2.4)\end{aligned}$$

for all locally absolutely continuous functions g on $[0,\infty)$.

Proof. Let $h > 0$. Then

$$\begin{aligned}\int_0^\infty|g(y+h)-g(y)|^p\,\mathrm{d}y\\ = \int_0^h|g(y+h)-g(y)|^p\,\mathrm{d}y + \int_h^\infty|g(y+h)-g(y)|^p\,\mathrm{d}y\\ =: N_1(h) + N_2(h). \quad (2.5)\end{aligned}$$

First, we estimate N_1:

$$\begin{aligned}N_1(h) = \int_0^h|g(y+h)-g(y)|^p\,\mathrm{d}y\\ \lesssim \int_0^h|g(y+h)|^p\,\mathrm{d}y + \int_0^h|g(y)|^p\,\mathrm{d}y = \int_0^{2h}|g(y)|^p\,\mathrm{d}y. \quad (2.6)\end{aligned}$$

For the alternative estimate, we use the Hölder inequality with the exponents α and α' to get, for all $y > 0$,

$$|g(y+h) - g(y)| = \left| \int_y^{y+h} g'(\tau)\,\mathrm{d}\tau \right| \\ \le h^{1/\alpha'} \Big(\int_y^{y+h} |g'(\tau)|^\alpha \,\mathrm{d}\tau \Big)^{1/\alpha} \le h^{1/\alpha'} \Big(\int_y^{\infty} |g'(\tau)|^\alpha \,\mathrm{d}\tau \Big)^{1/\alpha}.$$

Consequently,

$$N_1(h) \le h^{p/\alpha'} \int_0^h \Big(\int_y^\infty |g'(\tau)|^\alpha \,\mathrm{d}\tau \Big)^{p/\alpha} \,\mathrm{d}y. \tag{2.7}$$

Now, we estimate the second term N_2. We use the estimate $|g(y+h) - g(y)| \le h \int_0^1 |g'(y+\tau h)|\,\mathrm{d}\tau$, then the Hölder inequality, the Fubini theorem and the change of variables $y + \tau h = z$ to obtain

$$N_2(h) = \int_h^\infty |g(y+h) - g(y)|^p \,\mathrm{d}y \le \int_h^\infty h^p \Big(\int_0^1 |g'(y+\tau h)|\,\mathrm{d}\tau \Big)^p \,\mathrm{d}y \\ \le h^p \int_h^\infty \Big(\int_0^1 |g'(y+\tau h)|^p \,\mathrm{d}\tau \Big) \,\mathrm{d}y = h^p \int_0^1 \Big(\int_{h(1+\tau)}^\infty |g'(z)|^p \,\mathrm{d}z \Big) \,\mathrm{d}\tau \\ \le h^p \int_0^1 \Big(\int_h^\infty |g'(z)|^p \,\mathrm{d}z \Big) \,\mathrm{d}\tau = h^p \int_h^\infty |g'(y)|^p \,\mathrm{d}y. \tag{2.8}$$

Estimate (2.3) follows from (2.5), (2.6) and (2.8), estimate (2.4) is a consequence of (2.5), (2.7) and (2.8). □

Take $R \in (0, \infty)$ and put

$$u_R(x) := \varphi_R(x) \int_0^x \chi_{(R,2R)}(t)\, t^{-2} \,\mathrm{d}t, \quad x \in (0,\infty), \tag{2.9}$$

where $\varphi_R \in C^\infty[0,\infty)$ is a cut-off function such that

$$\operatorname{supp} \varphi_R \subset [0, 4R], \quad 0 \le \varphi_R \le 1, \\ \varphi_R(x) = 1 \text{ for } x \in [0, 3R], \quad \varphi_R(x) = 0 \text{ for } x \in [4R, \infty], \\ |\varphi_R'| \lesssim R^{-1} \chi_{[3R,4R]}.$$

Obviously,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t u_R(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \lim_{t\to 0_+} \frac{1}{t} \int_0^t u_R(x)\,\mathrm{d}x = 0, \tag{2.10}$$

$$|u_R'(x)| \lesssim R^{-2} \chi_{[3R,4R]}(x) + \chi_{(R,2R)}(x)\, x^{-2} \quad \text{for all } x \in (0,\infty). \tag{2.11}$$

Lemma 2.3. *Let $1 \le p < \infty$ and $\delta \in (0,1) \cup (1,p)$. Assume that u_R is given by* (2.9). *Then*

$$\int_0^\infty |u_R(x)|^p\, x^{-\delta} \,\mathrm{d}x \gtrsim R^{1-p-\delta} \quad \textit{for all } R \in (0,\infty). \tag{2.12}$$

Proof. Since

$$u_R(x) = \begin{cases} 0 & \text{if} \quad x \in [0, R], \\ 1/R - 1/x & \text{if} \quad x \in (R, 2R], \\ 1/(2R) & \text{if} \quad x \in (2R, 3R), \end{cases} \tag{2.13}$$

we obtain

$$\int_0^\infty |u_R(x)|^p \, x^{-\delta} \, \mathrm{d}x \geq \int_{2R}^{3R} \big(1/(2R)\big)^p \, x^{-\delta} \, \mathrm{d}x \approx R^{1-p-\delta}$$

and (2.12) is verified. □

Lemma 2.4. *Suppose that* $1 \leq p < \infty$ *and* $\eta \in (0, p)$*. Let* u_R *be given by* (2.9)*. Then*

$$\int_0^\infty \int_0^\infty \frac{|u_R(x) - u_R(y)|^p}{|x-y|^{\eta+1}} \, \mathrm{d}x \, \mathrm{d}y \lesssim R^{1-p-\eta} \quad \textit{for all } R \in (0, \infty). \tag{2.14}$$

Proof. We start with some auxiliary estimates. If $\beta \in [1, \infty)$, then, by (2.11),

$$\int_h^\infty |u_R'(t)|^\beta \, \mathrm{d}t \lesssim \begin{cases} R^{1-2\beta} & \text{if} \quad h \in [0, 4R], \\ 0 & \text{if} \quad h \in (4R, \infty). \end{cases} \tag{2.15}$$

Using this estimate with $\beta = p$, the facts that $p \in [1, \infty)$ and $\eta \in (0, p)$, we obtain

$$\begin{aligned} \int_0^\infty \Big(\int_h^\infty |u_R'(t)|^p \, \mathrm{d}t \Big) h^{p-\eta-1} \, \mathrm{d}h &\lesssim \int_0^\infty R^{1-2p} \, \chi_{(0,4R]}(h) \, h^{p-\eta-1} \, \mathrm{d}h \\ &= R^{1-2p} \int_0^{4R} h^{p-\eta-1} \, \mathrm{d}h \approx R^{1-p-\eta} \quad \text{for all } R \in (0, \infty). \end{aligned} \tag{2.16}$$

If $\eta \in (1, p)$, we use (2.13) to get

$$\begin{aligned} \int_0^\infty \Big(\int_0^{2h} |u_R(t)|^p \, \mathrm{d}t \Big) h^{-\eta-1} \, \mathrm{d}h &\lesssim \int_{R/2}^\infty \Big(\int_0^{2h} R^{-p} \, \mathrm{d}t \Big) h^{-\eta-1} \, \mathrm{d}h \\ &\approx R^{-p} \int_{R/2}^\infty h^{-\eta} \, \mathrm{d}h \approx R^{1-p-\eta} \quad \text{for all } R \in (0, \infty). \end{aligned} \tag{2.17}$$

Now, assume that $\eta \in (0, 1]$ and $\alpha \in (1, \infty)$ is such that $\alpha' > p/\eta$. Then

$$0 > p/\alpha' - \eta > -1. \tag{2.18}$$

Using (2.15) with $\beta = \alpha$, we get

$$\begin{aligned} \int_0^h \Big(\int_y^\infty |u_R'(t)|^\alpha \, \mathrm{d}t \Big)^{p/\alpha} \mathrm{d}y &\lesssim \int_0^h \big(R^{1-2\alpha} \, \chi_{(0,4R]}(y) \big)^{p/\alpha} \, \mathrm{d}y \\ &\leq R^{(1-2\alpha)p/\alpha} \, \min\{h, 4R\} \quad \text{for all } h, R \in (0, \infty). \end{aligned} \tag{2.19}$$

Thus, if $\eta \in (0,1]$, then (2.19) and (2.18) imply that

$$\int_0^\infty \left(\int_0^h \left(\int_y^\infty |u_R'(\tau)|^\alpha \, \mathrm{d}\tau\right)^{p/\alpha} \mathrm{d}y\right) h^{p/\alpha'-\eta-1} \, \mathrm{d}h$$
$$\lesssim R^{(1-2\alpha)p/\alpha} \int_0^{4R} h^{p/\alpha'-\eta} \, \mathrm{d}h + R^{(1-2\alpha)p/\alpha+1} \int_{4R}^\infty h^{p/\alpha'-\eta-1} \, \mathrm{d}h$$
$$\approx R^{1-p-\eta} \quad \text{for all } R \in (0,\infty). \quad (2.20)$$

Now, we are able to prove (2.14). To this end, we distinguish two cases.

(i) Let $\eta \in (1,p)$. Then, (2.1) with $w(h) := |h|^{-\eta-1}$, (2.3), (2.17) and (2.16) yield

$$\int_0^\infty \int_0^\infty \frac{|u_R(x) - u_R(y)|^p}{|x-y|^{\eta+1}} \, \mathrm{d}x \, \mathrm{d}y$$
$$\lesssim \int_0^\infty \left(\int_0^{2h} |u_R(y)|^p \, \mathrm{d}y\right) h^{-\eta-1} \, \mathrm{d}h + \int_0^\infty \left(\int_h^\infty |u_R'(y)|^p \, \mathrm{d}y\right) h^{p-\eta-1} \, \mathrm{d}h$$
$$\lesssim R^{1-p-\eta} \text{ for all } R \in (0,\infty).$$

(ii) Let $\eta \in (0,1]$. Choose $\alpha \in (1,\infty)$ such that $\alpha' > p/\eta$. Then, (2.1) with $w(h) := |h|^{-\eta-1}$, (2.4), (2.20) and (2.16) imply that

$$\int_0^\infty \int_0^\infty \frac{|u_R(x) - u_R(y)|^p}{|x-y|^{\eta+1}} \, \mathrm{d}x \, \mathrm{d}y$$
$$\lesssim \int_0^\infty \left(\int_0^h \left(\int_y^\infty |u_R'(\tau)|^\alpha \, \mathrm{d}\tau\right)^{p/\alpha} \mathrm{d}y\right) h^{p/\alpha'-1-\eta} \, \mathrm{d}h$$
$$+ \int_0^\infty \left(\int_h^\infty |u_R'(y)|^p \, \mathrm{d}y\right) h^{p-1-\eta} \, \mathrm{d}h \lesssim R^{1-p-\eta} \quad \text{for all } R \in (0,\infty).$$

□

Now, we can prove Theorem 1.2.

Proof of Theorem 1.2. By (2.10), the test function u_R satisfies both of conditions (i), (ii) of Theorem 1.1. We obtain from (1.2), (2.12) and (2.14) that

$$R^{1-p-\delta} \lesssim C\, R^{1-p-\eta} \quad \text{for all } R \in (0,\infty).$$

Since the constant C is independent of R, the last estimate implies that $\eta = \delta$. □

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