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STRONG CONVERGENCE OF MONOTONE CQ ALGORITHM FOR RELATIVELY NONEXPANSIVE MAPPINGS

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ABSTRACT. X. Qin and Y. Su proved a strong convergence theorems of modified Ishikawa iteration by CQ method for relatively nonexpansive mappings in a Banach space [Xiaolong Qin, Yongfu Su, Nonlinear Anal. 67 (2007), no. 6, 1958–1965]. The result of this paper extends and improves the result of X. Qin and Y. Su in the two respects: (1). By using the monotone CQ method to modify the CQ method, so that the new method of proof is used. (2). Relax the restriction on T from uniformly continuous to continuous. The result of this paper also extends and improves the recent ones announced by Nakajo, Takahashi, Kim, Martinez-Yanes, Xu and some others.

1. INTRODUCTION

In an infinite-dimensional Hilbert space, Mann's iterative algorithm has only weak covergence, in general, even for nonexpansive mappings. Hence in order to have strong convergence, in recent years, the CQ iteration methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors [1, 2, 3, 4].

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In 2003, Nakajo and Takahashi [1] proposed the following modification of Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.1}$$

where C is a closed convex subset of H, P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.1) converges strongly to $P_{F(T)}(x_0)$. Where $F(T)$ denote the fixed points set of T.

In 2006, T.H. Kim and H.K. Xu [2] proposed the following modification of the Mann iteration method for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.2}$$

where C is bounded closed convex subset and

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(diam C)^2 \to 0 \quad as\ n \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$.

They also proposed the following modification of the Mann iteration method for asymptotically nonexpansive semigroup $\Im$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \overline{\theta}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where C is bounded closed convex subset and

$$\overline{\theta}_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1](diam C)^2 \to 0 \quad as\ n \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(\Im)}(x_0)$. Where $F(\Im)$ denote the common fixed points set of $\Im$.

In 2006, C. Martinez-Yanes and H.-K. Xu [3] proposed the following modification of the Ishikawa iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad +(1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where C is a closed convex subset of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and $\beta_n \to 0$, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}(x_0)$.

C. Martinez-Yanes and H.-K. Xu [3] proposed also the following modification of the Halpern iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad +\alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.5}$$

where C is a closed convex subset of H. They proved that if the sequence $\alpha_n \to 0$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}(x_0)$.

In 2005, S.-Y. Matsushita and W. Takahashi [4] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\} \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.6}$$

They proved the following convergence theorem .

Theorem 1.1 (MT). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.6), where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

Recent, X. Qin and Y. Su [13] proved a strong convergence theorems of modified Ishikawa iteration by CQ method for relatively nonexpansive mappings in a Banach space.

Theorem 1.2 (QS). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JTx_n) \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.7}$$

where j is the single-valued duality mapping on E. If T is uniformly continuous, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.

The result of this paper extends and improves the result of X. Qin and Y. Su in the two respects: (1). By using the monotone CQ method to modify the CQ method, so that new method of proof is used. (2). Relax the restriction on T from uniformly continuous to continuous. The result of this paper also extends and improves the recent ones announced by Nakajo, Takahashi, Kim, Martinez-Yanes, Xu and some others.

By using the monotone CQ method we can easy show the iteration sequence $\{x_n\}$ is Cauchy sequence, so that without use of the Kadec-Klee property, demiclosedness principle and Opial's condition or other about weak topological technologies.

2. Preliminaries

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [5] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as [5, 6] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \qquad (x, y \in E). \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x,y) = \|x-y\|^2, \ \ x,y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x),$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, for example, [5, 6, 7]). In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2 \quad (x, y \in E). \tag{2.2}$$

Remark 2.1. If E is a reflexive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (2.2), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definitions of j, we have $Jx = Jy$. That is, $x = y$; see [8, 9] for more details.

Let C be a closed convex subset of E, and Let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point of p in C is said to be an asymptotic fixed point of T [10] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [10, 11, 12] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

We need the following Lemmas for the proof of our main results.

Lemma 2.2. *[Kamimura and Takahashi* [7]*] Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.3. *[Alber* [5]*] Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \ge 0 \quad \textit{for } y \in C.$$

Lemma 2.4. *[Alber* [5]*] Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_c x) + \phi(\Pi_c x, x) \le \phi(y, x) \quad \textit{for all } y \in C.$$

Lemma 2.5. *(Matsushita and Takahashi* [4]*). Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

3. Main results

Now, we prove a strong convergence theorem for relatively nonexpansive mapping in a Banach space by using the monotone CQ method. This result extends and improves the result of Qin and Su [13] in several respects:

(1). By using the monotone CQ method to modify the CQ method, so that new method of proof is used.

(2). Relax the restriction on T from uniformly continuous to continuous.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) JTx_n) \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTz_n), \\ C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \phi(z, y_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n)\}, \\ C_0 = \{z \in C : \phi(z, y_0) \leq \alpha_0 \phi(z, x_0) + (1-\alpha_0)\phi(z, z_0)\}, \\ Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0) \end{cases}$$

where J is the duality mapping on E. If T continuous, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \geq 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \geq 0$. We show that C_n is convex for any $n \geq 0$. For $v_1, v_2 \in C_n$ and $t \in (0,1)$, put $v = tv_1 + (1-t)v_2$. It is sufficient to show that $v \in C_n$. First, we show that

$$\phi(z, y_n) \leq \alpha_n \phi(z, x_n) + (1-\alpha_n)\phi(z, z_n) \tag{3.1}$$

is equivalent to

$$\begin{aligned} &2\alpha_n \langle z, Jx_n\rangle + 2(1-\alpha_n)\langle z, Jz_n\rangle - 2\langle z, Jy_n\rangle \\ &\qquad \leq \alpha_n \|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2. \end{aligned} \tag{3.2}$$

Indeed, from (2.1) we have

$$\phi(z, y_n) = \|z\|^2 - 2\langle z, Jy_n\rangle + \|y_n\|^2,$$

$$\phi(z, x_n) = \|z\|^2 - 2\langle z, Jx_n\rangle + \|x_n\|^2,$$

$$\phi(z, z_n) = \|z\|^2 - 2\langle z, Jz_n\rangle + \|z_n\|^2$$

which combined with (3.1) yield that (3.1) is equivalent to (3.2). Therefore, we have

$$\begin{aligned}
&2\alpha_n\langle z, Jx_n\rangle + 2(1-\alpha_n)\langle z, Jz_n\rangle - 2\langle z, Jy_n\rangle \\
&= 2\alpha_n\langle tv_1 + (1-t)v_2, Jx_n\rangle + 2(1-\alpha_n)\langle tv_1 + (1-t)v_2, Jz_n\rangle \\
&\quad - 2\langle tv_1 + (1-t)v_2, Jy_n\rangle \\
&= 2t\alpha_n\langle v_1, Jx_n\rangle + 2(1-t)\alpha_n\langle v_2, Jx_n\rangle + 2(1-\alpha_n)t\langle v_1, Jz_n\rangle \\
&\quad + 2(1-\alpha_n)(1-t)\langle v_2, Jz_n\rangle - 2t\langle v_1, Jy_n\rangle - 2(1-t)\langle v_2, Jy_n\rangle \\
&\le \alpha_n\|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2.
\end{aligned}$$

This implies $v \in C_n$, hence C_n is convex. Next, we show that $F(T) \subset C_n$ for all $n \ge 0$. Indeed, we have, for all $p \in F(T)$ that

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JTz_n\rangle \\
&\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tz_n\|^2 \\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, Tz_n) \\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n).
\end{aligned}$$

That is, $p \in C_n$ for all $n \ge 0$.

Next, we show that $F(T) \subset Q_n$ for all $n \ge 0$, we prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.3 we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \ge 0, \quad \forall\, z \in C_n \cap Q_n.$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumptions, the last inequality holds, in particular, for all $z \in F(T)$. this together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$.

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $C_n \bigcap Q_n \subset C_{n-1} \bigcap Q_{n-1}$ for all $n \ge 1$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0) \tag{3.3}$$

for all $n \ge 0$. Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. In addition, it follows from definition of Q_n and Lemma 2.3 that $x_n = \Pi_{Q_n} x_0$. Therefore by Lemma 2.4 we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0),$$

for each $p \in F(T) \subset Q_n$ for all $n \ge 0$. Therefore, $\phi(x_n, x_0)$ is bounded, this together with (3.3) implies that the limit of $\{\phi(x_n, x_0)\}$ exists. Put

$$\lim_{n\to\infty} \phi(x_n, x_0) = d. \tag{3.4}$$

From Lemma 2.4, we have, for any positive integer m, that

$$\begin{aligned}
\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \le \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+m}, x_0) - \phi(x_n, x_0), \qquad (3.5)
\end{aligned}$$

for all $n \ge 0$. Therefore

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0. \tag{3.6}$$

We claim that $\{x_n\}$ is a Cauchy sequence, if not, there exists a positive real number $\varepsilon_0 > 0$ and the subsequence $\{n_k\}, \{m_k\} \subset \{n\}$ such that

$$\|x_{n_k+m_k} - x_{n_k}\| \geq \varepsilon_0,$$

for all $k \geq 1$.

On the other hand, from (3.4) and (3.5) we have

$$\begin{aligned}\phi(x_{n_k+m_k}, x_{n_k}) &\leq \phi(x_{n_k+m_k}, x_0) - \phi(x_{n_k}, x_0)\\ &\leq |\phi(x_{n_k+m_k}, x_0) - d| + |d - \phi(x_{n_k}, x_0)| \to 0,\ k \to \infty.\end{aligned}$$

Because from (3.4) we known the $\phi(x_n, x_0)$ is bounded implies the $\{x_n\}$ is also bounded, by using Lemma 2.2, we obtain

$$\lim_{k\to\infty} \|x_{n_k+m_k} - x_{n_k}\| = 0.$$

This is a contradiction, so that $\{x_n\}$ is a Cauchy sequence, therefore there exists a point $p \in C$ such that $\{x_n\}$ converges strongly to p.

Since $x_{n+2} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n, we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_n) + (1 - \alpha_n)\phi(x_{n+1}, z_n), \tag{3.7}$$

and

$$\begin{aligned}\phi(x_{n+1}, z_n) &= \phi(x_{n+1}, J^{-1}(\beta_n Jx_n + (1 - \beta_n)JTx_n))\\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n Jx_n + (1 - \beta_n)JTx_n\rangle\\ &\quad + \|\beta_n Jx_n + (1 - \beta_n)JTx_n\|^2\\ &\leq \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, Jx_n\rangle - 2(1 - \beta_n)\langle x_{n+1}, JTx_n\rangle\\ &\quad + \beta_n\|x_n\|^2 + (1 - \beta_n)\|Tx_n\|^2\\ &= \beta_n\phi(x_{n+1}, x_n) + (1 - \beta_n)\phi(x_{n+1}, Tx_n).\end{aligned}$$

Since $\lim_{n\to\infty} \beta_n = 1$ and $\{x_n\}$ is bounded, hence

$$\phi(x_{n+1}, z_n) \to 0. \tag{3.8}$$

It follows from (3.6), (3.7) and (3.8) that

$$\phi(x_{n+1}, y_n) \to 0.$$

By using Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.9}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, then we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.10}$$

On the other hand, it follows from

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|$$

that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Noticing that

$$\begin{aligned}\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JTz_n)\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - JTz_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JTz_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JTz_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|,\end{aligned}$$

which implies that

$$\|Jx_{n+1} - JTz_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|).$$

This together with (3.10)) and $\limsup_{n\to\infty} \alpha_n < 1$ implies that $\lim_{n\to\infty} \|Jx_{n+1} - JTz_n\| = 0$.

Since J^{-1} is also uniformly norm-to-norm continuous on any bounded sets, then we have

$$\lim_{n\to\infty} \|x_{n+1} - Tz_n\| = 0. \tag{3.11}$$

Observe that

$$\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tz_n\| + \|Tz_n - Tx_n\|.$$

It follows from (3.9), (3.11), $x_n \to p,\ z_n \to p$ and T is continuous, that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This together with the definition of relatively nonexpansive mapping implies that, p is a fixed point of T.

Finally, we prove that $p = \Pi_{F(T)}x_0$, from Lemma 2.4 and Lemma 2.5, we have

$$\phi(p, \Pi_{F(T)}x_0) + \phi(\Pi_{F(T)}x_0, x_0) \leq \phi(p, x_0). \tag{3.12}$$

On the other hand, since $x_{n+1} = \Pi_{C_n \bigcap Q_n}$ and $C_n \bigcap Q_n \supset F(T)$, for all n. Also from Lemma 2.4, we have

$$\phi(\Pi_{F(T)}x_0, x_{n+1}) + \phi(x_{n+1}, x_0) \leq \phi(\Pi_{F(T)}x_0, x_0). \tag{3.13}$$

By the definition of $\phi(x, y)$, we know that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_0) = \phi(p, x_0). \tag{3.14}$$

Combining (3.12), (3.13) and (3.14), we know that

$$\phi(p, x_0) = \phi(\Pi_{F(T)}x_0, x_0).$$

Therefore, it follows from the uniqueness of $\Pi_{F(T)}x_0$ that $p = \Pi_{F(T)}x_0$. This completes the proof. □

Remark 3.2. In recent years, the CQ iteration methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors. In fact that, all CQ iteration methods can be replaced (or modified) by monotone CQ iteration methods respectively. On the other hand, by using the monotone CQ method we can easy show the iteration sequence $\{x_n\}$ is Cauchy sequence, so that without use of the Kadec-Klee property, demiclosedness principle and Opial's condition or other about weak topological technologies.

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