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ON STEINER LOOPS AND POWER ASSOCIATIVITY

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by K. Ciesielski

Abstract. In this paper we investiagte Steiner loops introduced by N.S. Mendelsohn [Aeq. Math. 6 (1991), 228–230] and provide six (seven) equivalent identities to characterize it. We also prove the power associativity of Bol loops by using closure (Hexagonal) conditions.

1. Steiner loops

In [9] Mendelsohn has defined the concept of a generalized triple system as follows. Let S be a set of ν elements. Let T be a collection of b subsets of S, each of which contains three elements arranged cyclically, and such that any ordered pair of elements of S appears in exactly a cyclic triplet (note the cyclic triplet $\{a, b, c\}$ contains the ordered pairs ab, bc, ca but not ba, cb, ac). When such a configuration exists we will refer to it as a *generalized triple system*. If we ignore the cyclic order of the triples, the generalized triple system is a B.I.B.D.

There is one to one correspondence between generalized triple systems of order ν and quasigroups of order ν satisfying the identities $x^2 = e \cdot (xy)x = x(yx) = y$. The term *generalized Steiner quasigroup* means a quasigroup which satisfies the above identities.

Let G be a generalized Steiner Quasigroup of order ν. From G a loop G^* with operator $*$ is constructed as follows. The elements of G^* are the same as those of G together with an extra element e. Multiplication in G^* is defined as follows: $a * e = e * a = a$; $a * a = e$ and for $a, b \in G$, with $a \neq b$ define $a * b = a \cdot b$. It

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follows easily that G^* is a loop satisfying the identities $x * e = e * x = x$, $x * x = e$, $x * (y * x) = (x * y) * x = y$ for $x, y \in G$. Also, the correspondence between generalized Steiner quasigroups and generalized Steiner loops is a bijection.

A loop which satisfies the identities

$$xx = e, \quad xe = x = ex, \quad x \cdot yx = y = xy \cdot x \qquad \text{for } x, y \in G, \tag{1.1}$$

is called a *generalized Steiner loop* (g.s.l.). In [9] the identity (2) characterizing g.s.l. is given. Five (six) equivalent identities were found immediately afterwards in 1970 to characterize g.s.l. Now we present them in the following theorem:

Theorem 1.1. *A groupoid $G(\cdot)$ is a generalized Steiner loop if an only if G satisfies any one of the following identities:*

$$a \cdot [((bb) \cdot c) \cdot a] = c, \tag{1.2}$$

$$[a \cdot c(bb)] \cdot a = c, \tag{2a}$$

$$a \cdot (ca \cdot bb) = c, \tag{2b}$$

$$(a \cdot ca) \cdot bb = c, \tag{2c}$$

$$bb \cdot (a \cdot ca) = c, \tag{2d}$$

$$(bb \cdot a) \cdot (ca \cdot dd) = c, \tag{2e}$$

for $a, b, c, d \in G$.

Proof. First we consider (2) investigated in [9], here we present a different simpler proof to show that $G(\cdot)$ satisfying (2) is a g.s.l.

In (2) replace c by $(dd \cdot k) \cdot bb$ and use (2) to get

$$a \cdot ka = (dd \cdot k) \cdot bb \tag{1.3}$$

$$\text{and} \qquad bb \cdot (a \cdot ka) = k, \qquad \text{for } a, b, k, \in G. \tag{3a}$$

Suppose $\nu a = ua$. Then (3a) shows that $\nu = u$, that is, $(\cdot)$ is right cancellative (r.c.). Apply r.c. in (3a) to obtain $bb =$ constant $= e$ (say). Then (2) becomes

$$a \cdot (ec \cdot a) = c.$$

Put $c = e$ to obtain $a \cdot ea = e = ea \cdot ea$ implying $ea = a$. So $a \cdot ca = c$.

First $a = e$ in (2) yields $ce = c$ showing thereby that e is an identity and then replacing a by ac gives $ac \cdot (c \cdot ac) = c$, that is $ac \cdot a = c$. This proves g.s.l.

Second we prove the implication of the identities in the order written above; that is, we prove that

$$(2) \Rightarrow (2\text{a}) \Rightarrow (2\text{b}) \Rightarrow (2\text{c})\,((2\text{c}')) \Rightarrow (2\text{d}) \Rightarrow (2\text{e}) \text{ finally } \Rightarrow (2)$$

to complete the cycle.

To prove (2) ⇒ (2a)

Suppose (2) $a \cdot [(bb \cdot c) \cdot a] = c$ holds.

From the above prove we see that

$$bb = e, \qquad ae = a, \qquad ac \cdot a = e.$$

Now $[a \cdot c(bb)] \cdot a = (a \cdot ce) \cdot a = ac \cdot a = c$ which is (2a).

To prove (2a) ⇒ (2b)

Now $[a \cdot c(bb)] \cdot a = c$ holds.

Replace c by $bb \cdot (c \cdot dd)$ in (2b) and use (2b) to get

$$[a \cdot \underbrace{\{(bb \cdot (c \cdot dd)) \cdot bb\}}] \cdot a = bb \cdot (c \cdot dd)$$

that is

$$ac \cdot a = bb \cdot (c \cdot dd) \tag{1.4}$$

$$(ac \cdot a) \cdot bb = [bb \cdot (c \cdot dd)] \cdot bb = c \qquad \text{by (2a).} \tag{4a}$$

Let $ac = au$. (4a) yields $c = u$ implying l.c. (left cancellative). (4) is $ac \cdot a = bb \cdot (c \cdot dd) = bb \cdot (c \cdot ww)$ giving $dd = \text{constant} = e$ (say) by l.c. (4) becomes

$$ac \cdot a = e \cdot (ce). \tag{4b}$$

Let $ac = uc$. Then (4b) shows that $u = a$, that is , $(\cdot)$ is r.c. $c = e$ in (4b) gives $ae \cdot a = e = a \cdot a$, that is, $ae = a$ (using r.c.). Then (4a) gives $ac \cdot a = c$. Now $a \cdot (ca \cdot bb) = a \cdot (ca \cdot e) = a \cdot ca = c$ which is (2b). This proves that (2a) $\Rightarrow$ (2b).

Next we prove (2b) $\Rightarrow$ (2c)

(2b) $a \cdot (ca \cdot bb) = c$ holds.

Let $ca = da$. This in (2b) shows $c = d$, that is r.c. Let $ca = cd$. Then $a \cdot (ca \cdot bb) = d \cdot (cd \cdot bb)$ implying $a = d$, (use r.c.), that is, l.c. l.c. in (2b) gives $bb = e$ (say). Thus, $a \cdot (ca \cdot e) = c$.

$$c = a \quad \text{gives } ae = a \quad \text{and } a \cdot ca = c. \tag{1.5}$$

Now from (5) results $(a \cdot ca) \cdot bb = (a \cdot ca) \cdot e = a \cdot ca = e$ which is (2c).

Remark 1.2. Instead of (2c) we consider

$$(ac \cdot a) \cdot bb = c. \tag{2c'}$$

From (5) $a \cdot ca = c$, replacing c by ac we get $a \cdot (ac \cdot a) = ac$, that is $ac \cdot a = c$ (use l.c.. Then

$$(ac \cdot a) \cdot bb = (ac \cdot a) \cdot e = ac \cdot a = c$$

which is (2c'). That is (2b) $\Rightarrow$ (2c').

Next we take (2c) $\Rightarrow$ (2d)

(2c) $(a \cdot ca) \cdot bb = c$ holds.

Set $ca = da$ in (2c) to obtain $c = d$, that is, r.c. Let $ca = cd$. Then (2c) and r.c. yields l.c. l.c. in (2c) gives $bb = \text{constant} = e$ (say). Hence (2c) becomes

$$(a \cdot ca) \cdot e = c \tag{1.6}$$

$$c = e \quad \text{in (6) gives} \quad (a \cdot ea) \cdot e = e \Rightarrow a \cdot ea = e \Rightarrow ea = a. \tag{6a}$$

With $a = e$, (6) shows that $ce \cdot e = c$ and

$$a \cdot ca = ce.$$

Now c replaced by ac gives

$$a \cdot (ac \cdot a) = ac \cdot e,$$

$$\text{that is,} \qquad a \cdot ce = ac \cdot e.$$

Then $c = e$ gives $ae = ae \cdot e = a$ by (6).

Now, $bb \cdot (a \cdot ca) = e \cdot (a \cdot ca) = a \cdot ca = ce = e$ which is (2d).

Remark 1.3. Suppose (2c′) holds.

With $ac = au$, (2c′) gives l.c. Then changing b in (2c′) yields bb = constant $= e$.

$c = e$ in (2c′) gives $ae \cdot a = e$, that is, $ae = a$. With $a = e$ (2c′) shows $ec = c$, $ac \cdot a = c$ and $a \cdot ca = c$ (replace c by ca). Now $bb \cdot (a \cdot ca) = a \cdot ca = c$ which is (2d). Thus (2c′) ⇒ (2d).

Next we tackle (2d) ⇒ (2e)

(2d) $bb \cdot (a \cdot ca) = c$ holds.

$ca = da$ in (2d) gives $c = d$, that is, r.c. Then (2d) yields bb = constant $= e$ (using r.c.).

(2d) is $e \cdot (a \cdot ca) = c$. $c = e$ gives $a \cdot ea = e$ or $ea = a$ and $a \cdot ca = c$. Then $a = e$ results to $ce = c$.

Now $(bb \cdot a) \cdot (ca \cdot dd) = a \cdot ca = c$ which is (2e).

Finally, to complete the cycle, we prove that (2e) ⇒ (2)

(2e) $(bb \cdot a) \cdot (ca \cdot dd) = c$ holds.

$ca = ua$ in (2e) gives r.c. and bb = constant $= e$.

(2e) is $ea \cdot (ca \cdot e) = c$.

First $c = e$ yields $ea = ea \cdot e$. Second $c = a$ gives $ea \cdot e = a = ea$ and then $ae = a$. Further $a \cdot ca = c$. Now $a \cdot [(bb \cdot c) \cdot a] = a \cdot ca = c$ which is (2).

This completes the proof of the theorem. □

2. Bol loop and power associativity

There are several closure conditions in Quasigroups and Loops theory [1, 2, 3, 4, 5, 6, 7, 8] of which R-condition (Reidemeister condition) connected to groups, T-condition (Thomsen condition) connected to Abelian groups, H-condition (Hexagonal condition) connected to power associativity are well known.

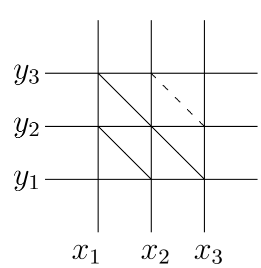
H-condition

For $x_1, x_2, x_3, y_1, y_2, y_3$ in G, a groupoid if

$$x_1y_2 = x_2y_1, \quad x_1y_3 = x_2y_2 = x_3y_1 \quad \text{implies} \quad x_2y_3 = x_3y_2, \tag{2.1}$$

then G is said to satisfy the *closure condition* known as *Hexagonal condition.*

Geometrically, H-condition means the following:



H-condition implies power associativity, that is,

$$x \cdot y^{n+m} = xy^n \cdot y^m$$

for all $x, y \in G$ and all $m, n \in Z$, integers.

(Left) Bol Loop

A loop $G(\cdot)$ is said to be a (left) Bol loop provided

$$x \cdot (y \cdot xz) = (x \cdot yx) \cdot z, \qquad \text{for } x, y, z \in G \tag{2.2}$$

holds.

It is well known that left (right) Bol loop is power associative. Here we prove it by using H-condition.

Theorem 2.1. *The left Bol loop is power associative (by using the hexagonal closure condition).*

Proof. Suppose (8) holds.

Set $y = x^{-1}$ (inverse of x) in (9) to obtain $x \cdot (x^{-1} \cdot xz) = xz \Rightarrow x^{-1} \cdot xz = z$, that is $G(\cdot)$ satisfies l.i.p. (left inverse property) or $G(\cdot)$ is a left inverse property loop. Suppose

$$x_1, y_2 = x_2 y_1, \quad x_1 y_3 = x_2 y_2 = x_3 y_1 \qquad \text{holds in } G. \tag{7'}$$

First use (7') to get

$$x_3^{-1} x_2 = y_1^{-1} y_2, \quad x_1 = x_2 y_1^{-1} y_2 = x_2 (x_3^{-1} x_2).$$

Now

$$x_1 y_3 = (x_2 (x_3^{-1} x_2)) \cdot y_3 \overset{\text{by (8)}}{=} x_2 (x_3^{-1} \cdot x_2 y_3) \overset{\text{also}}{=} x_2 y_2.$$

Thus

$$x_3^{-1} \cdot x_2 y_3 = y_2 \qquad \text{or} \qquad x_2 y_3 = x_3 y_2 \quad \text{(using l.i.p.)}.$$

Thus H-condition holds. Hence G is power associative. □

A loop in which $xy \cdot x = x \cdot yx$ holds is said to satisfy *elasticity law*. In passing, we mention that a loop satisfying elasticity law is power associative.

References

1. J. Aczel, *Quasigroups, nets and nomographsm*, Advances in Math. **1** (1965), 383–450.
2. V.D. Belousov, *Theory of quasigroups and loops* (Russian), 1967.
3. V.D. Belousov, *Algebraic nets and quasigroups* (Russian), 1971.
4. R.H. Bruck, *A survey of binary system*, Springer-Verlag, 1966.
5. R.H. Bruck, *What is a loop?*, Studies in Modern Algebra, Vol. 2, edited by A. Albert.
6. Pl. Kannappan, *Groupoids and groups*, Iber. Deutsch. Math. Verein **75** (1972), 95–100.
7. Pl. Kannappan, *On some identities*, Math. Student **40** (1972), 260–264.
8. Pl. Kannappan and M.A. Taylor, *On closure conditions*, Canad. Math. Bull. **19** (1976), 291–296.
9. N.S. Mendelsohn, *A single groupoid identity for Steiner loops*, Aeq. Math. **6** (1991), 228–230.

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