Banach J. Math. Anal. 1 (2007), no. 2, 212–215

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

THE SCHUR-CONVEXITY OF STOLARSKY AND GINI MEANS

JÓZSEF SÁNDOR[1]

This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by S. S. Dragomir

ABSTRACT. We study in a unitary way the Schur-convexity or concavity of the Stolarsky and Gini means $D_{a,b}(x,y)$ and $S_{a,b}(x,y)$, for fixed $x, y > 0$, $x \neq y$.

1. INTRODUCTION

Let $x, y > 0$, $x \neq y$. The Stolarsky means $D_{a,b}(x,y)$, introduced in [15, 16], are defined for $a, b \in \mathbb{R}$ and $x > 0,\ y > 0$ by

$$D_{a,b}(x,y) = \begin{cases} \left[\dfrac{b(x^a - y^a)}{a(x^b - y^b)}\right]^{1/(a-b)}, & ab(a-b) \neq 0 \\ \exp\left(-\dfrac{1}{a} + \dfrac{x^a \ln x - y^a \ln y}{x^a - y^a}\right), & a = b \neq 0 \\ \left[\dfrac{x^a - y^a}{a(\ln x - \ln y)}\right]^{1/a}, & a \neq 0,\ b = 0 \\ \sqrt{xy}, & a = b = 0. \end{cases} \tag{1.1}$$

Means (1.1) are sometimes called the "difference means", or "extended means" (see, e.g. [3, 6, 7]).

Date: Received: 28 February 2007; Revised: 8 March 2007; Accepted: 29 October 2007.
2000 *Mathematics Subject Classification.* 26D15, 26D99, 26B25.
Key words and phrases. Stolarsky and Gini means, convexity, Schur-convexity, inequalities.

The identric, logarithmic, and power means of order a ($a \neq 0$) will be denoted by I_a, L_a and A_a, respectively. They are all contained in the above family of means. We have $I_a = D_{a,a}$; $L_a = D_{a,0}$, and $A_a = D_{2a,a}$. When $a = 1$, we write I, L, and A instead of I_1, L_1 and A_1, obtaining the identric, logarithmic, and arithmetic means (see e.g. [10, 13]). There is a simple relationship between means of order a ($a \neq 1$) and those of order one. Namely, we have

$$I_a(x,y) = (I(x^a, y^a))^{1/a} \tag{1.2}$$

with similar formulas for the remaining means mentioned above. Note that for the geometric mean of x and y, $\sqrt{xy} = G(x,y)$ we have $G(x,y) = D_{0,0}(x,y)$.

The second family of bivariate means studied here was introduced by C. Gini [2]. They are defined as follows:

$$S_{a,b}(x,y) = \begin{cases} \left(\dfrac{x^a + y^a}{x^b + y^b}\right)^{1/(a-b)}, & a \neq b \\ \exp\left(\dfrac{x^a \ln x + y^a \ln y}{x^a + y^a}\right), & a = b \neq 0 \\ \sqrt{xy}, & a = b = 0 \end{cases} \tag{1.3}$$

Gini means are also called the "sum means". It follows from (1.3) that $S_{0,-1} = H$ - the harmonic mean, $S_{0,0} = G$, and $S_{1,0} = A$. The mean $S_{1,1}$ denoted by $S_{1,1} = J$ will play an important role in what follows. Put

$$J_a(x,y) = (J(x^a, y^a))^{1/a} \tag{1.4}$$

The basic properties of these means, as well as their comparison theorems, and inequalities are studied in papers [2, 3, 5, 6, 15]. See also the survey monograph on inequalities [17].

The following integral representations will be important in what follows:

Lemma 1.1. *If $a \neq b$, then*

$$\ln D_{a,b} = \frac{1}{b-a} \int_a^b \ln I_t dt, \tag{1.5}$$

and

$$\ln S_{a,b} = \frac{1}{b-a} \int_a^b \ln J_t dt. \tag{1.6}$$

Formula (1.5) is derived in [15], while the proof of (1.6) is an elementary exercise in calculus. See also [5].

Recall now the definition of Schur-convex functions. Let I be an interval with nonempty interior, and let $f : I^n \to \mathbb{R}$. Then f is called Schur-convex on I^n ($n \geq 2$) if $f(x) \leq f(y)$ for each two n-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of I^n, such that $x \prec y$ holds. The relationship of majorization $x \prec y$ means that

$$\sum_{i=1}^{n} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $1 \leq k \leq n-1$, and $x_{[i]}$ denotes the ith largest component of x.

A function f is called Schur-concave if $-f$ is Schur-convex. The following two characterizations are often used in the theory of Schur-convex functions.

Lemma 1.2. *Let I be an open interval. Then a continuously differentiable function $f : I^2 \to \mathbb{R}$ is Schur-convex iff it is symmetric and satisfies the relation*

$$\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right)(y - x) > 0 \text{ for all } x, y \in I,\ x \neq y.$$

See e.g. [4, 9] for more general results, with applications.

The next result appears in [1]:

Lemma 1.3. *Let f be a continuous function on I. Then $F : I^2 \to \mathbb{R}$, defined by*

$$F(a,b) = \begin{cases} \dfrac{1}{b-a}\displaystyle\int_a^b f(t)dt, & a \neq b \\ f(a), & a = b \end{cases} \tag{1.7}$$

is Schur-convex on I^2 iff f is convex on I.

2. Main results

In a recent paper, F. Qi [7] has proved the following result:

Theorem 2.1. *For fixed x, y with $x, y > 0$, $x \neq y$, the mean values $D_{a,b}(x,y)$ are Schur-concave on $\mathbb{R}_+^2 = [0,+\infty) \times [0,+\infty)$, and Schur-convex on $\mathbb{R}_-^2 = (-\infty, 0] \times (-\infty, 0]$, with respect to (a,b).*

Our aim in what follows is to offer a new proof of a more complete result:

Theorem 2.2. *For fixed x, y with $x, y > 0$, $x \neq y$, the mean values $D_{a,b}(x,y)$ and $S_{a,b}(x,y)$ are Schur-concave on $\mathbb{R}_+^2$, and Schur-convex on $\mathbb{R}_-^2$, with respect to (a,b).*

Proof. In paper [12] it is proved (by using certain inequalities established in [10]) that the function $t \to I_t$ of (1.2) is log-concave for $t > 0$ and log-convex for $t < 0$. The similar property of the function $t \to J_t$ of (1.4) has been proved in paper [5]. Now, Lemma 1, combined with Lemma 3 and the above results, imply that $\ln D_{a,b}$ and $\ln S_{a,b}$ are Schur-concave for $a, b > 0$, and Schur-convex for $a, b < 0$ (for fixed $x, y > 0$, $x \neq y$). This in turn implies Theorem 2, as $\ln D(a,b)$ is Schur-convex (concave) iff $D(a,b)$ is Schur-convex (concave), etc. □

Remark 2.3. (1) The Schur-convexity problem of $D_{a,b}(x,y)$ for fixed a, b with respect to $x, y > 0$ is considered in [8, 14]. In this case the results are not so nice as in Theorem 1, 2. The similar problems for $S_{a,b}(x,y)$ are still open.

(2) As a corollary of Theorem 1, in [7] the following inequality is stated: For $x, y > 0$, $x \neq y$ one has when $r > 0$:

$$\left(\frac{1}{2r} \cdot \frac{y^{2r} - x^{2r}}{\ln y - \ln x}\right)^{1/2r} \leq \frac{1}{e^{1/r}}(x^{x^r}/y^{y^r})^{1/(x^r - y^r)} \tag{2.1}$$

For $r < 0$, inequality (2.1) reverses. We wish to note here that these reduce in fact to known inequalities. Indeed, for $r > 0$, (2.1) becomes $L(x^{2r}, y^{2r}) \leq (I(x^r, y^r))^2$, or by letting $x^r = u$, $y^r = v$:

$$L(u^2, v^2) \leq (I(u, v))^2 \tag{2.2}$$

It is easy to see that, by homogeneity considerations, for $r < 0$, (2.1) reduces again to (2.2).

Since $L(u^2, v^2) = L(u, v)A(u, v)$ (see e.g. [11] for such identities), inequality (2.2) reduces to

$$\sqrt{L \cdot A} \leq I \tag{2.3}$$

This is a consequence of relation (1.7) of [10], namely: $\sqrt{L \cdot A} \leq A_{2/3} \leq I$. For other refinements of (2.3) (involving e.g. the arithmetic-geometric mean of Gauss), see [13, 15].

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[1] Babeş-Bolyai University of Cluj, Romania.
E-mail address: jjsandor@hotmail.com