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A SURVEY OF SOME RECENT INEQUALITIES FOR THE NORM AND NUMERICAL RADIUS OF OPERATORS IN HILBERT SPACES

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This paper is dedicated to Professor Themistocles. M. Rassias.

Submitted by F. Kittaneh

Abstract. Some recent inequalities for the norm and the numerical radius of linear operators in Hilbert spaces are surveyed.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [11, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}.$$

The following properties of $W(T)$ are immediate:

(i) $W(\alpha I+\beta T)=\alpha+\beta W(T)$ for $\alpha,\beta\in\mathbb{C}$;
(ii) $W(T^*)=\left\{\bar{\lambda},\lambda\in W(T)\right\}$, where T^* is the *adjoint operator* of T;
(iii) $W(U^*TU)=W(T)$ for any *unitary* operator U.

The following classical fact about the geometry of the numerical range [11, p. 4] may be stated:

Theorem 1.1 (Toeplitz-Hausdorff). *The numerical range of an operator is convex.*

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An important use of $W(T)$ is to bound the *spectrum* $\sigma(T)$ of the operator T [11, p. 6]:

Theorem 1.2 (Spectral inclusion)**.** *The spectrum of an operator is contained in the closure of its numerical range.*

The self-adjoint operators have their spectra bounded sharply by the numerical range [11, p. 7]:

Theorem 1.3. *The following statements hold true:*

(i) *T is self-adjoint iff $W(T)$ is real;*
(ii) *If T is self-adjoint and $W(T) = [m, M]$ (the closed interval of real numbers m, M), then $\|T\| = \max\{|m|, |M|\}$.*
(iii) *If $W(T) = [m, M]$, then $m, M \in \sigma(T)$.*

The *numerical radius* $w(T)$ of an operator T on H is given by [11, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $w(T) \geq 0$ for any $T \in B(H)$ and $w(T) = 0$ if and only if $T = 0$;
(ii) $w(\lambda T) = |\lambda| w(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $w(T + V) \leq w(T) + w(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds [11, p. 9]:

Theorem 1.4 (Equivalent norm)**.** *For any $T \in B(H)$ one has*

$$w(T) \leq \|T\| \leq 2w(T). \tag{1.2}$$

Let us now look at two extreme cases of the inequality (1.2). In the following $r(t) := \sup\{|\lambda|, \lambda \in \sigma(T)\}$ will denote the *spectral radius* of T and $\sigma_p(T) = \{\lambda \in \sigma(T),\ Tf = \lambda f \text{ for some } f \in H\}$ the *point spectrum* of T.

The following results hold [11, p. 10]:

Theorem 1.5. *We have*

(i) *If $w(T) = \|T\|$, then $r(T) = \|T\|$.*
(ii) *If $\lambda \in W(T)$ and $|\lambda| = \|T\|$, then $\lambda \in \sigma_p(T)$.*

To address the other extreme case $w(T) = \frac{1}{2}\|T\|$, we can state the following sufficient condition in terms of (see [11, p. 11])

$$R(T) := \{Tf,\ f \in H\} \quad \text{and} \quad R(T^*) := \{T^*f,\ f \in H\}.$$

Theorem 1.6. *If $R(T) \perp R(T^*)$, then $w(T) = \frac{1}{2}\|T\|$.*

It is well-known that the two-dimensional shift

$$S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

has the property that $w(T) = \frac{1}{2}\|T\|$.

The following theorem shows that some operators T with $w(T) = \frac{1}{2}\|T\|$ have S_2 as a component [11, p. 11]:

Theorem 1.7. *If $w(T) = \frac{1}{2}\|T\|$ and T attains its norm, then T has a two-dimensional reducing subspace on which it is the shift S_2.*

For other results on numerical radius, see [12], Chapter 11.

We recall some classical results involving the numerical radius of two linear operators A, B.

The following general result for the product of two operators holds [11, p. 37]:

Theorem 1.8. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then*

$$w(AB) \leq 4w(A)w(B).$$

In the case that $AB = BA$, then

$$w(AB) \leq 2w(A)w(B).$$

The following results are also well known [11, p. 38].

Theorem 1.9. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \leq w(B). \tag{1.3}$$

If A is an isometry and $AB = BA$, then (1.3) *also holds true.*

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$.

The following result holds [11, p. 38].

Theorem 1.10 (Double commute)**.** *If the operators A and B double commute, then*

$$w(AB) \leq w(B)\|A\|.$$

As a consequence of the above, we have [11, p. 39]:

Corollary 1.11. *Let A be a normal operator commuting with B. Then*

$$w(AB) \leq w(A)w(B).$$

For other results and historical comments on the above see [11, p. 39–41]. For more results on the numerical radius, see [12].

The main aim of this paper is to survey some inequalities for the norm and the numerical radius of bounded linear operators in complex Hilbert spaces obtained by the author in a sequence of recent works. For the sake of completeness and since not all involved results have yet been published, detailed proofs are given as well.

2. Reverse inequalities for one operator

The following results may be stated [6]:

Theorem 2.1. *Let $T : H \to H$ be a bounded linear operator on the complex Hilbert space H. If $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$ are such that*

$$\|T - \lambda I\| \leq r, \tag{2.1}$$

where $I : H \to H$ is the identity operator on H, then

$$(0 \leq) \|T\| - w(T) \leq \frac{1}{2} \cdot \frac{r^2}{|\lambda|}. \tag{2.2}$$

Proof. For $x \in H$ with $\|x\| = 1$, we have from (2.1) that

$$\|Tx - \lambda x\| \leq \|T - \lambda I\| \leq r,$$

giving

$$\|Tx\|^2 + |\lambda|^2 \leq 2\mathrm{Re}\left[\overline{\lambda} \langle Tx, x\rangle\right] + r^2 \leq 2|\lambda| |\langle Tx, x\rangle| + r^2. \tag{2.3}$$

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.3) we get the following inequality that is of interest in itself:

$$\|T\|^2 + |\lambda|^2 \leq 2w(T)|\lambda| + r^2. \tag{2.4}$$

Since, obviously,

$$\|T\|^2 + |\lambda|^2 \geq 2\|T\| |\lambda|, \tag{2.5}$$

hence by (2.4) and (2.5) we deduce the desired inequality (2.2). □

Remark 2.2. If the operator $T : H \to H$ is such that $R(T) \perp R(T^*)$, $\|T\| = 1$ and $\|T - I\| \leq 1$, then the equality case holds in (2.2). Indeed, by Theorem 1.6, we have in this case $w(T) = \frac{1}{2}\|T\| = \frac{1}{2}$ and since we can choose in Theorem 2.1, $\lambda = 1$, $r = 1$, then we get in both sides of (2.2) the same quantity $\frac{1}{2}$.

Problem 2.3. Find the bounded linear operators $T : H \to H$ with $\|T\| = 1$, $R(T) \perp R(T^*)$ and $\|T - \lambda I\| \leq |\lambda|^{\frac{1}{2}}$.

The following corollary may be stated [6]:

Corollary 2.4. *Let $A : H \to H$ be a bounded linear operator and $\varphi, \phi \in \mathbb{C}$ with $\phi \neq -\varphi, \varphi$. If*

$$Re\langle \phi x - Ax, Ax - \varphi x\rangle \geq 0 \quad \text{for any} \quad x \in H, \ \|x\| = 1 \tag{2.6}$$

then

$$(0 \leq) \|A\| - w(A) \leq \frac{1}{4} \cdot \frac{|\phi - \varphi|^2}{|\phi + \varphi|}. \tag{2.7}$$

Proof. Utilising the fact that in any Hilbert space the following two statements are equivalent:

(i) $\mathrm{Re}\langle Z - x, x - z\rangle \geq 0$, $x, z, Z \in H$;
(ii) $\left\|x - \frac{z+Z}{2}\right\| \leq \frac{1}{2}\|Z - z\|$,

we deduce that (2.6) is equivalent to

$$\left\| Ax - \frac{\phi + \varphi}{2} \cdot Ix \right\| \leq \frac{1}{2} |\phi - \varphi|$$

for any $x \in H$, $\|x\| = 1$, which in its turn is equivalent with the operator norm inequality:

$$\left\| A - \frac{\phi + \varphi}{2} \cdot I \right\| \leq \frac{1}{2} |\phi - \varphi| .$$

Now, applying Theorem 2.1 for $T = A$, $\lambda = \frac{\varphi+\phi}{2}$ and $r = \frac{1}{2} |\Gamma - \gamma|$, we deduce the desired result (2.7). □

Remark 2.5. Following [11, p. 25], we say that an operator $B : H \to H$ is accretive, if $\operatorname{Re} \langle Bx, x \rangle \geq 0$ for any $x \in H$. One may observe that the assumption (2.6) above is then equivalent with the fact that the operator $(A^* - \bar{\varphi} I)(\phi I - A)$ is accretive.

Perhaps a more convenient sufficient condition in terms of positive operators is the following one:

Corollary 2.6. *Let $\varphi, \phi \in \mathbb{C}$ with $\phi \neq -\varphi, \varphi$ and $A : H \to H$ a bounded linear operator in H. If $(A^* - \bar{\varphi} I)(\phi I - A)$ is self-adjoint and*

$$(A^* - \bar{\varphi} I)(\phi I - A) \geq 0 \tag{2.8}$$

in the operator order, then

$$(0 \leq) \|A\| - w(A) \leq \frac{1}{4} \cdot \frac{|\phi - \varphi|^2}{|\phi + \varphi|}.$$

The following result may be stated as well:

Corollary 2.7. *Assume that T, λ, r are as in Theorem* 2.1*. If, in addition, there exists $\rho \geq 0$ such that*

$$||\lambda| - w(T)| \geq \rho, \tag{2.9}$$

then

$$(0 \leq) \|T\|^2 - w^2(T) \leq r^2 - \rho^2. \tag{2.10}$$

Proof. From (2.4) of Theorem 2.1, we have

$$\begin{aligned} \|T\|^2 - w^2(T) &\leq r^2 - w^2(T) + 2w(T)|\lambda| - |\lambda|^2 \\ &= r^2 - (|\lambda| - w(T))^2 . \end{aligned}$$

On utilising (2.4) and (2.9) we deduce the desired inequality (2.10). □

Remark 2.8. In particular, if $\|T - \lambda I\| \leq r$ and $|\lambda| = w(T)$, $\lambda \in \mathbb{C}$, then

$$(0 \leq) \|T\|^2 - w^2(T) \leq r^2.$$

The following result may be stated as well.

Theorem 2.9. *Let $T : H \to H$ be a nonzero bounded linear operator on H and $\lambda \in \mathbb{C} \setminus \{0\}$, $r > 0$ with $|\lambda| > r$. If*

$$\|T - \lambda I\| \leq r,$$

then

$$\sqrt{1 - \frac{r^2}{|\lambda|^2}} \leq \frac{w(T)}{\|T\|} \quad (\leq 1). \tag{2.11}$$

Proof. From (2.4) of Theorem 2.1, we have

$$\|T\|^2 + |\lambda|^2 - r^2 \leq 2|\lambda| w(T),$$

which implies, on dividing with $\sqrt{|\lambda|^2 - r^2} > 0$ that

$$\frac{\|T\|^2}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \leq \frac{2|\lambda| w(T)}{\sqrt{|\lambda|^2 - r^2}}. \tag{2.12}$$

By the elementary inequality

$$2\|T\| \leq \frac{\|T\|^2}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2}$$

and by (2.12) we deduce

$$\|T\| \leq \frac{w(T)|\lambda|}{\sqrt{|\lambda|^2 - r^2}},$$

which is equivalent to (2.11). □

Remark 2.10. Squaring (2.11), we get the inequality

$$(0 \leq) \|T\|^2 - w^2(T) \leq \frac{r^2}{|\lambda|^2}\|T\|^2.$$

Remark 2.11. Since for any bounded linear operator $T : H \to H$ we have that $w(T) \geq \frac{1}{2}\|T\|$, hence (2.11) would produce a refinement of this classic fact only in the case when

$$\frac{1}{2} \leq \left(1 - \frac{r^2}{|\lambda|^2}\right)^{\frac{1}{2}},$$

which is equivalent to $r/|\lambda| \leq \sqrt{3}/2$.

The following corollary holds [6].

Corollary 2.12. *Let $\varphi, \phi \in \mathbb{C}$ with $Re(\phi\bar{\varphi}) > 0$. If $T : H \to H$ is a bounded linear operator such that either (2.6) or (2.8) holds true, then:*

$$\frac{2\sqrt{Re(\phi\bar{\varphi})}}{|\phi + \varphi|} \leq \frac{w(T)}{\|T\|} \, (\leq 1) \tag{2.13}$$

and

$$(0 \leq) \|T\|^2 - w^2(T) \leq \left|\frac{\phi - \varphi}{\phi + \varphi}\right|^2 \|T\|^2.$$

Proof. If we consider $\lambda = \frac{\phi+\varphi}{2}$ and $r = \frac{1}{2}|\phi-\varphi|$, then $|\lambda|^2 - r^2 = \left|\frac{\phi+\varphi}{2}\right|^2 - \left|\frac{\phi-\varphi}{2}\right|^2 = \operatorname{Re}(\phi\bar{\varphi}) > 0$. Now, on applying Theorem 2.9, we deduce the desired result. □

Remark 2.13. If $|\phi-\varphi| \leq \frac{\sqrt{3}}{2}|\phi+\varphi|$, $\operatorname{Re}(\phi\bar{\varphi}) > 0$, then (2.13) is a refinement of the inequality $w(T) \geq \frac{1}{2}\|T\|$.

The following result may be of interest as well [6].

Theorem 2.14. *Let $T : H \to H$ be a nonzero bounded linear operator on H and $\lambda \in \mathbb{C}\setminus\{0\}$, $r > 0$ with $|\lambda| > r$. If*

$$\|T - \lambda I\| \leq r,$$

then

$$(0 \leq)\, \|T\|^2 - w^2(T) \leq \frac{2r^2}{|\lambda| + \sqrt{|\lambda|^2 - r^2}} w(T). \tag{2.14}$$

Proof. From the proof of Theorem 2.1, we have

$$\|Tx\|^2 + |\lambda|^2 \leq 2\operatorname{Re}\left[\bar{\lambda}\langle Tx, x\rangle\right] + r^2 \tag{2.15}$$

for any $x \in H$, $\|x\| = 1$.

If we divide (2.15) by $|\lambda|\,|\langle Tx, x\rangle|$, (which, by (2.15), is positive) then we obtain

$$\frac{\|Tx\|^2}{|\lambda|\,|\langle Tx,x\rangle|} \leq \frac{2\operatorname{Re}\left[\bar{\lambda}\langle Tx,x\rangle\right]}{|\lambda|\,|\langle Tx,x\rangle|} + \frac{r^2}{|\lambda|\,|\langle Tx,x\rangle|} - \frac{|\lambda|}{|\langle Tx,x\rangle|} \tag{2.16}$$

for any $x \in H$, $\|x\| = 1$.

If we subtract in (2.16) the same quantity $\frac{|\langle Tx,x\rangle|}{|\lambda|}$ from both sides, then we get

$$\begin{aligned}
&\frac{\|Tx\|^2}{|\lambda|\,|\langle Tx,x\rangle|} - \frac{|\langle Tx,x\rangle|}{|\lambda|} \\
&\leq \frac{2\operatorname{Re}\left[\bar{\lambda}\langle Tx,x\rangle\right]}{|\lambda|\,|\langle Tx,x\rangle|} + \frac{r^2}{|\lambda|\,|\langle Tx,x\rangle|} - \frac{|\langle Tx,x\rangle|}{|\lambda|} - \frac{|\lambda|}{|\langle Tx,x\rangle|} \\
&= \frac{2\operatorname{Re}\left[\bar{\lambda}\langle Tx,x\rangle\right]}{|\lambda|\,|\langle Tx,x\rangle|} - \frac{|\lambda|^2 - r^2}{|\lambda|\,|\langle Tx,x\rangle|} - \frac{|\langle Tx,x\rangle|}{|\lambda|} \\
&= \frac{2\operatorname{Re}\left[\bar{\lambda}\langle Tx,x\rangle\right]}{|\lambda|\,|\langle Tx,x\rangle|} - \left(\frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{|\lambda|\,|\langle Tx,x\rangle|}} - \frac{\sqrt{|\langle Tx,x\rangle|}}{\sqrt{|\lambda|}}\right)^2 \frac{|\langle Tx,x\rangle|}{|\lambda|} \\
&\quad - 2\frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|}.
\end{aligned} \tag{2.17}$$

Since

$$\operatorname{Re}\left[\bar{\lambda}\langle Tx,x\rangle\right] \leq |\lambda|\,|\langle Tx,x\rangle|$$

and

$$\left(\frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{|\lambda| |\langle Tx, x\rangle|}} - \frac{\sqrt{|\langle Tx, x\rangle|}}{\sqrt{|\lambda|}}\right)^2 \geq 0$$

hence by (2.17) we get

$$\frac{\|Tx\|^2}{|\lambda| |\langle Tx, x\rangle|} - \frac{|\langle Tx, x\rangle|}{|\lambda|} \leq \frac{2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)}{|\lambda|}$$

which gives the inequality

$$\|Tx\|^2 \leq |\langle Tx, x\rangle|^2 + 2|\langle Tx, x\rangle| \left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get

$$\begin{aligned}\|T\|^2 &\leq \sup\left\{|\langle Tx, x\rangle|^2 + 2|\langle Tx, x\rangle| \left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)\right\} \\ &\leq \sup\left\{|\langle Tx, x\rangle|^2\right\} + 2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right) \sup\left\{|\langle Tx, x\rangle|\right\} \\ &= w^2(T) + 2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right) w(T),\end{aligned}$$

which is clearly equivalent to (2.14). □

Corollary 2.15. *Let $\varphi, \phi \in \mathbb{C}$ with $Re(\phi\bar{\varphi}) > 0$. If $A : H \to H$ is a bounded linear operator such that either* (2.6) *or* (2.8) *hold true, then:*

$$(0 \leq) \|A\|^2 - w^2(A) \leq \left[|\phi + \varphi| - 2\sqrt{Re(\phi\bar{\varphi})}\right] w(A).$$

Remark 2.16. If $M \geq m > 0$ are such that either $(A^* - mI)(MI - A)$ is accretive, or, sufficiently, $(A^* - mI)(MI - A)$ is self-adjoint and

$$(A^* - mI)(MI - A) \geq 0 \quad \text{in the operator order,}$$

then, by (2.13) we have:

$$(1 \leq) \frac{\|A\|}{w(A)} \leq \frac{M + m}{2\sqrt{mM}},$$

which is equivalent to

$$(0 \leq) \|A\| - w(A) \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} w(A),$$

while from (2.14) we have

$$(0 \leq) \|A\|^2 - w^2(A) \leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(A).$$

Also, the inequality (2.7) becomes

$$(0 \leq) \|A\| - w(A) \leq \frac{1}{4} \cdot \frac{(M-m)^2}{M+m}.$$

3. Other inequalities for one operator

The following result may be stated as well [7]:

Theorem 3.1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \leq \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{3.1}$$

The constant $\frac{1}{2}$ is best possible in (3.1).

Proof. We need the following refinement of Schwarz's inequality obtained by the author in 1985 [2, Theorem 2] (see also [9] and [5]):

$$\|a\| \|b\| \geq |\langle a,b\rangle - \langle a,e\rangle \langle e,b\rangle| + |\langle a,e\rangle \langle e,b\rangle| \geq |\langle a,b\rangle|, \tag{3.2}$$

provided a, b, e are vectors in H and $\|e\| = 1$.

Observing that

$$|\langle a,b\rangle - \langle a,e\rangle \langle e,b\rangle| \geq |\langle a,e\rangle \langle e,b\rangle| - |\langle a,b\rangle|,$$

hence by the first inequality in (3.2) we deduce

$$\frac{1}{2}\left(\|a\| \|b\| + |\langle a,b\rangle|\right) \geq |\langle a,e\rangle \langle e,b\rangle|. \tag{3.3}$$

This inequality was obtained in a different way earlier by M.L. Buzano in [1].

Now, choose in (3.3), $e = x$, $\|x\| = 1$, $a = Tx$ and $b = T^*x$ to get

$$\frac{1}{2}\left(\|Tx\| \|T^*x\| + \left|\left\langle T^2x, x\right\rangle\right|\right) \geq |\langle Tx, x\rangle|^2 \tag{3.4}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (3.4) over $x \in H$, $\|x\| = 1$, we deduce the desired inequality (2.1).

Now, if we assume that (3.1) holds with a constant $C > 0$, *i.e.*,

$$w^2(T) \leq C\left[w\left(T^2\right) + \|T\|^2\right] \tag{3.5}$$

for any $T \in B(H)$, then if we choose T a normal operator and use the fact that for normal operators we have $w(T) = \|T\|$ and $w(T^2) = \|T^2\| = \|T\|^2$, then by (3.5) we deduce that $2C \geq 1$ which proves the sharpness of the constant. □

Remark 3.2. From the above result (3.1) we obviously have

$$\begin{aligned} w(T) &\leq \left\{\frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]\right\}^{1/2} \\ &\leq \left\{\frac{1}{2}\left(\left\|T^2\right\| + \|T\|^2\right)\right\}^{1/2} \leq \|T\| \end{aligned}$$

and

$$\begin{aligned} w(T) &\leq \left\{ \frac{1}{2} \left[w\left(T^2\right) + \|T\|^2 \right] \right\}^{1/2} \\ &\leq \left\{ \frac{1}{2} \left(w^2(T) + \|T\|^2 \right) \right\}^{1/2} \leq \|T\|, \end{aligned}$$

that provide refinements for the first inequality in (1.2).

The following result may be stated [7].

Theorem 3.3. *Let $T : H \to H$ be a bounded linear operator on the Hilbert space H and $\lambda \in \mathbb{C}\setminus\{0\}$. If $\|T\| \leq |\lambda|$, then*

$$\|T\|^{2r} + |\lambda|^{2r} \leq 2\|T\|^{r-1} |\lambda|^r w(T) + r^2 |\lambda|^{2r-2} \|T - \lambda I\|^2, \tag{3.6}$$

where $r \geq 1$.

Proof. We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [10] (see also [4]):

$$\|a\|^{2r} + \|b\|^{2r} - 2\|a\|^r \|b\|^r \frac{\operatorname{Re}\langle a, b\rangle}{\|a\|\,\|b\|} \leq \begin{cases} r^2 \|a\|^{2r-2} \|a - b\|^2 & \text{if } r \geq 1, \\ \|b\|^{2r-2} \|a - b\|^2 & \text{if } r < 1, \end{cases} \tag{3.7}$$

provided $r \in \mathbb{R}$ and $a, b \in H$ with $\|a\| \geq \|b\|$.

Now, let $x \in H$ with $\|x\| = 1$. From the hypothesis of the theorem, we have that $\|Tx\| \leq |\lambda|\,\|x\|$ and applying (3.7) for the choices $a = \lambda x$, $\|x\| = 1$, $b = Tx$, we get

$$\|Tx\|^{2r} + |\lambda|^{2r} - 2\|Tx\|^{r-1} |\lambda|^r |\langle Tx, x\rangle| \leq r^2 |\lambda|^{2r-2} \|Tx - \lambda x\|^2 \tag{3.8}$$

for any $x \in H$, $\|x\| = 1$ and $r \geq 1$.

Taking the supremum in (3.8) over $x \in H$, $\|x\| = 1$, we deduce the desired inequality (3.6). □

The following result may be stated as well [7]:

Theorem 3.4. *Let $T : H \to H$ be a bounded linear operator on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then for any $\alpha \in [0, 1]$ and $t \in \mathbb{R}$ one has the inequality:*

$$\|T\|^2 \leq \left[(1-\alpha)^2 + \alpha^2\right] w^2(T) + \alpha \|T - tI\|^2 + (1-\alpha)\|T - itI\|^2.$$

Proof. We use the following inequality obtained by the author in [5]:

$$\begin{aligned} &\left[\alpha \|tb - a\|^2 + (1-\alpha)\|itb - a\|^2\right] \|b\|^2 \\ &\qquad \geq \|a\|^2 \|b\|^2 - \left[(1-\alpha)\operatorname{Im}\langle a, b\rangle + \alpha \operatorname{Re}\langle a, b\rangle\right]^2 (\geq 0) \end{aligned}$$

to get:

$$\|a\|^2\|b\|^2 \le [(1-\alpha)\operatorname{Im}\langle a,b\rangle + \alpha\operatorname{Re}\langle a,b\rangle]^2 + \left[\alpha\|tb-a\|^2 + (1-\alpha)\|itb-a\|^2\right]\|b\|^2 \le \left[(1-\alpha)^2+\alpha^2\right]|\langle a,b\rangle|^2 + \left[\alpha\|tb-a\|^2 + (1-\alpha)\|itb-a\|^2\right]\|b\|^2 \tag{3.9}$$

for any $a,b \in H$, $\alpha \in [0,1]$ and $t \in \mathbb{R}$.

Choosing in (3.9) $a = Tx$, $b = x$, $x \in H$, $\|x\| = 1$, we get

$$\|Tx\|^2 \le \left[(1-\alpha)^2+\alpha^2\right]|\langle Tx,x\rangle|^2 + \alpha\|tx-Tx\|^2 + (1-\alpha)\|itx-Tx\|^2. \tag{3.10}$$

Finally, taking the supremum over $x \in H$, $\|x\| = 1$ in (3.10), we deduce the desired result. □

The following particular cases may be of interest [7].

Corollary 3.5. *For any T a bounded linear operator on H, one has:*

$$(0 \le)\ \|T\|^2 - w^2(T) \le \begin{cases} \inf_{t\in\mathbb{R}} \|T-tI\|^2 \\ \\ \inf_{t\in\mathbb{R}} \|T-itI\|^2 \end{cases} \tag{3.11}$$

and

$$\|T\|^2 \le \frac{1}{2}w^2(T) + \frac{1}{2}\inf_{t\in\mathbb{R}}\left[\|T-tI\|^2 + \|T-itI\|^2\right].$$

Remark 3.6. The inequality (3.11) can in fact be improved taking into account that for any $a,b \in H$, $b \neq 0$, (see for instance [3]) the bound

$$\inf_{\lambda\in\mathbb{C}}\|a-\lambda b\|^2 = \frac{\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2}{\|b\|^2}$$

actually implies that

$$\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2 \le \|b\|^2\|a-\lambda b\|^2 \tag{3.12}$$

for any $a,b \in H$ and $\lambda \in \mathbb{C}$.

Now if in (3.12) we choose $a = Tx$, $b = x$, $x \in H$, $\|x\| = 1$, then we obtain

$$\|Tx\|^2 - |\langle Tx,x\rangle|^2 \le \|Tx-\lambda x\|^2$$

for any $\lambda \in \mathbb{C}$, which, by taking the supremum over $x \in H$, $\|x\| = 1$, implies that

$$(0 \le)\ \|T\|^2 - w^2(T) \le \inf_{\lambda\in\mathbb{C}}\|T-\lambda I\|^2.$$

Remark 3.7. If we take $a = x$, $b = Tx$ in (3.12), then we obtain

$$\|Tx\|^2 \le |\langle Tx,x\rangle|^2 + \|Tx\|^2\|x-\mu Tx\|^2 \tag{3.13}$$

for any $x \in H$, $\|x\| = 1$ and $\mu \in \mathbb{C}$. Now, if we take the supremum over $x \in H$, $\|x\| = 1$ in (3.13), then we get

$$(0 \leq) \|T\|^2 - w^2(T) \leq \|T\|^2 \inf_{\mu \in \mathbb{C}} \|I - \mu T\|^2 .$$

Finally and from a different view point we may state [7]:

Theorem 3.8. *Let $T : H \to H$ be a bounded linear operator on H. If $p \geq 2$, then:*

$$\|T\|^p + |\lambda|^p \leq \frac{1}{2} (\|T + \lambda I\|^p + \|T - \lambda I\|^p), \tag{3.14}$$

for any $\lambda \in \mathbb{C}$.

Proof. We use the following inequality obtained by Dragomir and Sándor in [9]:

$$\|a + b\|^p + \|a - b\|^p \geq 2 (\|a\|^p + \|b\|^p)$$

for any $a, b \in H$ and $p \geq 2$.

Now, if we choose $a = Tx$, $b = \lambda x$, then we get

$$\|Tx + \lambda x\|^p + \|Tx - \lambda x\|^p \geq 2 (\|Tx\|^p + |\lambda|^p) \tag{3.15}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (3.15) over $x \in H$, $\|x\| = 1$, we get the desired result (3.14). □

Remark 3.9. For $p = 2$, we have the simpler result:

$$\|T\|^2 + |\lambda|^2 \leq \frac{1}{2} \left(\|T + \lambda I\|^2 + \|T - \lambda I\|^2\right)$$

for any $\lambda \in \mathbb{C}$. This can easily be obtained from the parallelogram identity as well.

4. Reverse inequalities for two operators

The following result may be stated [8]:

Theorem 4.1. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If $r > 0$ and*

$$\|A - B\| \leq r, \tag{4.1}$$

then

$$\left\| \frac{A^*A + B^*B}{2} \right\| \leq w(B^*A) + \frac{1}{2} r^2. \tag{4.2}$$

Proof. For any $x \in H$, $\|x\| = 1$, we have from (4.1) that

$$\|Ax\|^2 + \|Bx\|^2 \leq 2\mathrm{Re} \langle Ax, Bx \rangle + r^2. \tag{4.3}$$

However

$$\begin{aligned} \|Ax\|^2 + \|Bx\|^2 &= \langle (A^*A) x, x \rangle + \langle (B^*B) x, x \rangle \\ &= \langle (A^*A + B^*B) x, x \rangle \end{aligned}$$

and by (4.3) we obtain

$$\langle (A^*A + B^*B) x, x \rangle \leq 2 |\langle (B^*A) x, x \rangle| + r^2 \tag{4.4}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (4.4) we get

$$w(A^*A + B^*B) \leq 2w(B^*A) + r^2 \tag{4.5}$$

and since the operator $A^*A + B^*B$ is self-adjoint, hence

$$w(A^*A + B^*B) = \|A^*A + B^*B\|$$

and by (4.5) we deduce the desired inequality (4.2). □

Remark 4.2. We observe that, from the proof of the above theorem, we have the inequalities

$$0 \leq \left\| \frac{A^*A + B^*B}{2} \right\| - w(B^*A) \leq \frac{1}{2} \|A - B\|^2, \tag{4.6}$$

provided that A, B are bounded linear operators in H.

The second inequality in (4.6) is obvious while the first inequality follows by the fact that

$$\begin{aligned} \langle (A^*A + B^*B) x, x \rangle &= \|Ax\|^2 + \|Bx\|^2 \\ &\geq 2 \|Ax\| \|Bx\| \geq 2 |\langle (B^*A) x, x \rangle| \end{aligned}$$

for any $x \in H$.

The inequality (4.2) is obviously a reach source of particular inequalities of interest.

Indeed, if we assume, for $\lambda \in \mathbb{C}$ and a bounded linear operator T, that we have

$$\|T - \lambda T^*\| \leq r,$$

for a given positive number r, then by (4.6) we deduce the inequality

$$0 \leq \left\| \frac{T^*T + |\lambda|^2 TT^*}{2} \right\| - |\lambda| w(T^2) \leq \frac{1}{2} r^2.$$

Now, if we assume that for $\lambda \in \mathbb{C}$ and a bounded linear operator V we have that

$$\|V - \lambda I\| \leq r,$$

where I is the identity operator on H, then by (4.2) we deduce the inequality

$$0 \leq \left\| \frac{V^*V + |\lambda|^2 I}{2} \right\| - |\lambda| w(V) \leq \frac{1}{2} r^2.$$

As a dual approach, the following result may be noted as well [8]:

Theorem 4.3. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space H. Then*

$$\left\| \frac{A + B}{2} \right\|^2 \leq \frac{1}{2} \left[\left\| \frac{A^*A + B^*B}{2} \right\| + w(B^*A) \right]. \tag{4.7}$$

Proof. We obviously have

$$\begin{aligned}\|Ax+Bx\|^2 &= \|Ax\|^2 + 2\mathrm{Re}\langle Ax, Bx\rangle + \|Bx\|^2\\ &\le \langle (A^*A+B^*B)x,x\rangle + 2\left|\langle (B^*A)x,x\rangle\right|\end{aligned}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get

$$\begin{aligned}\|A+B\|^2 &\le w(A^*A+B^*B) + 2w(B^*A)\\ &= \|A^*A+B^*B\| + 2w(B^*A),\end{aligned}$$

from where we get the desired inequality (4.7). □

Remark 4.4. The inequality (4.7) can generate some interesting particular results such as the following inequality

$$\left\|\frac{T+T^*}{2}\right\|^2 \le \frac{1}{2}\left[\left\|\frac{T^*T+TT^*}{2}\right\| + w\left(T^2\right)\right],$$

holding for each bounded linear operator $T : H \to H$.

The following result may be stated as well [8].

Theorem 4.5. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space H and $p \ge 2$. Then*

$$\left\|\frac{A^*A+B^*B}{2}\right\|^{\frac{p}{2}} \le \frac{1}{4}\left[\|A-B\|^p + \|A+B\|^p\right]. \tag{4.8}$$

Proof. We use the following inequality for vectors in inner product spaces obtained by Dragomir and Sándor in [9]:

$$2\left(\|a\|^p + \|b\|^p\right) \le \|a+b\|^p + \|a-b\|^p \tag{4.9}$$

for any $a, b \in H$ and $p \ge 2$.

Utilising (4.9) we may write

$$2\left(\|Ax\|^p + \|Bx\|^p\right) \le \|Ax+Bx\|^p + \|Ax-Bx\|^p \tag{4.10}$$

for any $x \in H$.

Now, observe that

$$\|Ax\|^p + \|Bx\|^p = \left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}}$$

and by the elementary inequality:

$$\frac{\alpha^q+\beta^q}{2} \ge \left(\frac{\alpha+\beta}{2}\right)^q, \quad \alpha,\beta \ge 0 \ \text{ and } \ q \ge 1$$

we have

$$\begin{aligned}\left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}} &\ge 2^{1-\frac{p}{2}}\left(\|Ax\|^2 + \|Bx\|^2\right)^{\frac{p}{2}}\\ &= 2^{1-\frac{p}{2}}\left[\langle (A^*A+B^*B)x,x\rangle\right]^{\frac{p}{2}}.\end{aligned} \tag{4.11}$$

Combining (4.10) with (4.11) we get

$$\frac{1}{4}\left[\|Ax-Bx\|^{p}+\|Ax+Bx\|^{p}\right] \geq \left|\left\langle\left(\frac{A^{*}A+B^{*}B}{2}\right)x,x\right\rangle\right|^{\frac{p}{2}}$$

for any $x\in H$, $\|x\|=1$. Taking the supremum over $x\in H$, $\|x\|=1$, and taking into account that

$$w\left(\frac{A^{*}A+B^{*}B}{2}\right)=\left\|\frac{A^{*}A+B^{*}B}{2}\right\|,$$

we deduce the desired result (4.8). □

Remark 4.6. If $p=2$, then we have the inequality:

$$\left\|\frac{A^{*}A+B^{*}B}{2}\right\| \leq \left\|\frac{A-B}{2}\right\|^{2}+\left\|\frac{A+B}{2}\right\|^{2},$$

for any A,B bounded linear operators. This result can also be obtained directly on utilising the parallelogram identity.

We also should observe that for $A=T$ and $B=T^{*}$, T a normal operator, the inequality (4.8) becomes

$$\|T\|^{p} \leq \frac{1}{4}\left[\|T-T^{*}\|^{p}+\|T+T^{*}\|^{p}\right],$$

where $p\geq 2$.

The following result may be stated as well [8].

Theorem 4.7. *Let $A,B:H\to H$ be two bounded linear operators on the Hilbert space H and $r\geq 1$. If $A^{*}A\geq B^{*}B$ in the operator order or, equivalently, $\|Ax\|\geq \|Bx\|$ for any $x\in H$, then:*

$$\left\|\frac{A^{*}A+B^{*}B}{2}\right\|^{r} \leq \|A\|^{r-1}\|B\|^{r-1}w\left(B^{*}A\right)+\frac{1}{2}r^{2}\|A\|^{2r-2}\|A-B\|^{2}. \quad (4.12)$$

Proof. We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [10]:

$$\|a\|^{2r}+\|b\|^{2r} \leq 2\|a\|^{r-1}\|b\|^{r-1}\operatorname{Re}\langle a,b\rangle+r^{2}\|a\|^{2r-2}\|a-b\|^{2}, \quad (4.13)$$

where $r\geq 1$, $a,b\in H$ and $\|a\|\geq\|b\|$.

Utilising (4.13) we can state that:

$$\|Ax\|^{2r}+\|Bx\|^{2r} \leq 2\|Ax\|^{r-1}\|Bx\|^{r-1}\left|\langle Ax,Bx\rangle\right|+r^{2}\|Ax\|^{2r-2}\|Ax-Bx\|^{2}, \quad (4.14)$$

for any $x\in H$.

As in the proof of Theorem 4.5, we also have

$$2^{1-r}\left[\langle\left(A^{*}A+B^{*}B\right)x,x\rangle\right]^{r} \leq \|Ax\|^{2r}+\|Bx\|^{2r}, \quad (4.15)$$

for any $x\in H$.

Therefore, by (4.14) and (4.15) we deduce

$$\left[\left\langle\left(\frac{A^*A+B^*B}{2}\right)x,x\right\rangle\right]^r \le \|Ax\|^{r-1}\|Bx\|^{r-1}|\langle Ax,Bx\rangle|+\frac{1}{2}r^2\|A\|^{2r-2}\|Ax-Bx\|^2 \quad (4.16)$$

for any $x \in H$.

Taking the supremum in (4.16) we obtain the desired result (4.12). □

Remark 4.8. Following [11, p. 156], we recall that the bounded linear operator V is hyponormal, if

$$\|V^*x\| \le \|Vx\| \text{ for all } x \in H.$$

Now, if we choose in (4.12) $A = V$ and $B = V^*$, then, on taking into account that for hyponormal operators $w(V^2) = \|V\|^2$, we get the inequality

$$\left\|\frac{V^*V+VV^*}{2}\right\|^r \le \|V\|^{2r-2}\left[\|V\|^2+\frac{1}{2}r^2\|V-V^*\|^2\right],$$

holding for any hyponormal operator V and any $r \ge 1$.

5. Further inequalities for an invertible operator

In this section we assume that $B : H \to H$ is an invertible bounded linear operator and let $B^{-1} : H \to H$ be its inverse. Then, obviously,

$$\|Bx\| \ge \frac{1}{\|B^{-1}\|}\|x\| \quad \text{for any} \quad x \in H, \tag{5.1}$$

where $\|B^{-1}\|$ denotes the norm of the inverse B^{-1}.

The following result holds true [8]:

Theorem 5.1. *Let $A, B : H \to H$ be two bounded linear operators on H and B is invertible such that, for a given $r > 0$,*

$$\|A-B\| \le r. \tag{5.2}$$

Then:

$$\|A\| \le \left\|B^{-1}\right\|\left[w(B^*A)+\frac{1}{2}r^2\right]. \tag{5.3}$$

Proof. The condition (5.2) is obviously equivalent to:

$$\|Ax\|^2+\|Bx\|^2 \le 2\mathrm{Re}\langle(B^*A)x,x\rangle+r^2 \tag{5.4}$$

for any $x \in H$, $\|x\| = 1$.

Since, by (5.1),

$$\|Bx\|^2 \ge \frac{1}{\|B^{-1}\|^2}\|x\|^2, \quad x \in H$$

and $\mathrm{Re}\langle(B^*A)x,x\rangle \le |\langle(B^*A)x,x\rangle|$, hence by (5.4) we get

$$\|Ax\|^2+\frac{\|x\|^2}{\|B^{-1}\|^2} \le 2|\langle(B^*A)x,x\rangle|+r^2 \tag{5.5}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (5.5), we have

$$\|A\|^2 + \frac{1}{\|B^{-1}\|^2} \le 2w(B^*A) + r^2. \tag{5.6}$$

By the elementary inequality

$$\frac{2\|A\|}{\|B^{-1}\|} \le \|A\|^2 + \frac{1}{\|B^{-1}\|^2} \tag{5.7}$$

and by (5.6) we then deduce the desired result (5.3). □

Remark 5.2. If we choose above $B = \lambda I$, $\lambda \neq 0$, then we get the inequality

$$(0 \le) \|A\| - w(A) \le \frac{1}{2|\lambda|} r^2,$$

provided $\|A - \lambda I\| \le r$. This result has been obtained in [6].

Also, if we assume that $B = \lambda A^*$, A is invertible, then we obtain

$$\|A\| \le \left\|A^{-1}\right\| \left[w\left(A^2\right) + \frac{1}{2|\lambda|} r^2 \right],$$

provided $\|A - \lambda A^*\| \le r$, $\lambda \neq 0$.

The following result may be stated as well [8]:

Theorem 5.3. *Let $A, B : H \to H$ be two bounded linear operators on H. If B is invertible and for $r > 0$,*

$$\|A - B\| \le r, \tag{5.8}$$

then

$$(0 \le) \|A\| \|B\| - w(B^*A) \le \frac{1}{2} r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}. \tag{5.9}$$

Proof. The condition (5.8) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \le 2\mathrm{Re}\langle Ax, Bx\rangle + r^2$$

for any $x \in H$, which is clearly equivalent to

$$\|Ax\|^2 + \|B\|^2 \le 2\mathrm{Re}\langle B^*Ax, x\rangle + r^2 + \|B\|^2 - \|Bx\|^2. \tag{5.10}$$

Since

$$\mathrm{Re}\langle B^*Ax, x\rangle \le |\langle B^*Ax, x\rangle|, \quad \|Bx\|^2 \ge \frac{1}{\|B^{-1}\|^2} \|x\|^2$$

and

$$\|Ax\|^2 + \|B\|^2 \ge 2\|B\| \|Ax\|$$

for any $x \in H$, hence by (5.10) we get

$$2\|B\| \|Ax\| \le 2|\langle B^*Ax, x\rangle| + r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ we deduce the desired result (5.9). □

Remark 5.4. If we choose in Theorem 5.3, $B = \lambda A^*$, $\lambda \neq 0$, A is invertible, then we get the inequality:

$$(0 \leq) \|A\|^2 - w\left(A^2\right) \leq \frac{1}{2|\lambda|} r^2 + |\lambda| \cdot \frac{\|A\|^2 \|A^{-1}\|^2 - 1}{\|A^{-1}\|^2}$$

provided $\|A - \lambda A^*\| \leq r$.

The following result may be stated as well [8].

Theorem 5.5. *Let $A, B : H \to H$ be two bounded linear operators on H. If B is invertible and for $r > 0$ we have*

$$\|A - B\| \leq r < \|B\|, \tag{5.11}$$

then

$$\|A\| \leq \frac{1}{\sqrt{\|B\|^2 - r^2}} \left(w(B^* A) + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{2 \|B^{-1}\|^2} \right). \tag{5.12}$$

Proof. The first part of condition (5.11) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \leq 2\mathrm{Re}\langle Ax, Bx\rangle + r^2$$

for any $x \in H$, which is clearly equivalent to

$$\|Ax\|^2 + \|B\|^2 - r^2 \leq 2\mathrm{Re}\langle B^* Ax, x\rangle + \|B\|^2 - \|Bx\|^2. \tag{5.13}$$

Since

$$\mathrm{Re}\langle B^* Ax, x\rangle \leq |\langle B^* Ax, x\rangle|,$$

$$\|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2$$

and, by the second part of (5.11),

$$\|Ax\|^2 + \|B\|^2 - r^2 \geq 2\sqrt{\|B\|^2 - r^2} \|Ax\|,$$

for any $x \in H$, hence by (5.13) we get

$$2\|Ax\| \sqrt{\|B\|^2 - r^2} \leq 2|\langle B^* Ax, x\rangle| + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2} \tag{5.14}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (5.14), we deduce the desired inequality (5.12). □

Remark 5.6. The above Theorem 5.5 has some particular cases of interest. For instance, if we choose $B = \lambda I$, with $|\lambda| > r$, then (5.11) is obviously fulfilled and by (5.12) we get

$$\|A\| \leq \frac{w(A)}{\sqrt{1 - \left(\frac{r}{|\lambda|}\right)^2}},$$

provided $\|A - \lambda I\| \leq r$. This result has been obtained in [6].

On the other hand, if in the above we choose $B = \lambda A^*$ with $\|A\| \geq \frac{r}{|\lambda|}$ $(\lambda \neq 0)$, then by (5.12) we get

$$\|A\| \leq \frac{1}{\sqrt{\|A\|^2 - \left(\frac{r}{|\lambda|}\right)^2}} \left[w\left(A^2\right) + |\lambda| \cdot \frac{\|A\|^2 \|A^{-1}\|^2 - 1}{2\|A^{-1}\|^2}\right],$$

provided $\|A - \lambda A^*\| \leq r$.

The following result may be stated as well [8].

Theorem 5.7. *Let* A, B *and* r *be as in Theorem* 5.1. *Moreover, if*

$$\left\|B^{-1}\right\| < \frac{1}{r}, \tag{5.15}$$

then

$$\|A\| \leq \frac{\|B^{-1}\|}{\sqrt{1 - r^2\|B^{-1}\|^2}} w\left(B^* A\right). \tag{5.16}$$

Proof. Observe that, by (5.6) we have

$$\|A\|^2 + \frac{1 - r^2\|B^{-1}\|^2}{\|B^{-1}\|^2} \leq 2w\left(B^* A\right). \tag{5.17}$$

Utilising the elementary inequality

$$2\frac{\|A\|}{\|B^{-1}\|}\sqrt{1 - r^2\|B^{-1}\|^2} \leq \|A\|^2 + \frac{1 - r^2\|B^{-1}\|^2}{\|B^{-1}\|^2}, \tag{5.18}$$

which can be stated since (5.15) is assumed to be true, hence by (5.17) and (5.18) we deduce the desired result (5.16). □

Remark 5.8. If we assume that $B = \lambda A^*$ with $\lambda \neq 0$ and A an invertible operator, then, by applying Theorem 5.7, we get the inequality:

$$\|A\| \leq \frac{\|A^{-1}\| w\left(A^2\right)}{\sqrt{|\lambda|^2 - r^2\|A^{-1}\|^2}},$$

provided $\|A - \lambda A^*\| \leq r$ and $\|A^{-1}\| \leq \frac{|\lambda|}{r}$.

The following result may be stated as well.

Theorem 5.9. *Let* $A, B : H \to H$ *be two bounded linear operators. If* $r > 0$ *and* B *is invertible with the property that* $\|A - B\| \leq r$ *and*

$$\frac{1}{\sqrt{r^2 + 1}} \leq \left\|B^{-1}\right\| < \frac{1}{r}, \tag{5.19}$$

then

$$\|A\|^2 \leq w^2\left(B^* A\right) + 2w\left(B^* A\right) \cdot \frac{\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|}. \tag{5.20}$$

Proof. Let $x \in H$, $\|x\| = 1$. Then by (5.5) we have

$$\|Ax\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2\left|\langle B^*Ax, x\rangle\right| + r^2, \tag{5.21}$$

and since

$$\frac{1}{\|B^{-1}\|^2} - r^2 > 0,$$

we can conclude that $|\langle B^*Ax, x\rangle| > 0$ for any $x \in H$, $\|x\| = 1$.

Dividing in (5.21) with $|\langle B^*Ax, x\rangle| > 0$, we obtain

$$\frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} \leq 2 + \frac{r^2}{|\langle B^*Ax, x\rangle|} - \frac{1}{\|B^{-1}\|^2 |\langle B^*Ax, x\rangle|}. \tag{5.22}$$

Subtracting $|\langle B^*Ax, x\rangle|$ from both sides of (5.22), we get

$$\begin{aligned}
&\frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} - |\langle B^*Ax, x\rangle| \\
&\leq 2 - |\langle B^*Ax, x\rangle| - \frac{1 - r^2\|B^{-1}\|^2}{|\langle B^*Ax, x\rangle| \|B^{-1}\|^2} \\
&= 2 - \frac{2\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|} \\
&\quad - \left(\sqrt{|\langle B^*Ax, x\rangle|} - \frac{\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|\sqrt{|\langle B^*Ax, x\rangle|}}\right)^2 \\
&\leq 2\left(\frac{\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|}\right),
\end{aligned}$$

which gives:

$$\|Ax\|^2 \leq |\langle B^*Ax, x\rangle|^2 + 2|\langle B^*Ax, x\rangle| \frac{\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|}. \tag{5.23}$$

We also remark that, by (5.19) the quantity

$$\left\|B^{-1}\right\| - \sqrt{1 - r^2\|B^{-1}\|^2} \geq 0,$$

hence, on taking the supremum in (5.23) over $x \in H$, $\|x\| = 1$, we deduce the desired inequality. □

Remark 5.10. It is interesting to remark that if we assume $\lambda \in \mathbb{C}$ with $0 < r \leq |\lambda| \leq \sqrt{r^2 + 1}$ and $\|A - \lambda I\| \leq r$, then by (5.2) we can state the following inequality:

$$\|A\|^2 \leq |\lambda|^2 w\left(A^2\right) + 2|\lambda|\left(1 - \sqrt{|\lambda|^2 - r^2}\right) w(A).$$

Also, if $\|A - A^*\| \le r$, A is invertible and $\frac{1}{\sqrt{r^2+1}} \le \|A^{-1}\| \le \frac{1}{r}$, then, by (5.20) we also have

$$\|A\|^2 \le w^2\left(A^2\right) + 2w\left(A^2\right) \cdot \frac{\|A^{-1}\| - \sqrt{1 - r^2\|A^{-1}\|^2}}{\|A^{-1}\|}.$$

One can also prove the following result [8].

Theorem 5.11. *Let $A, B : H \to H$ be two bounded linear operators. If $r > 0$ and B is invertible with the property that $\|A - B\| \le r$ and $\|B^{-1}\| \le \frac{1}{r}$, then*

$$\begin{aligned}(0 \le)\ &\|A\|^2\|B\|^2 - w^2(B^*A) \\ &\le 2w(B^*A) \cdot \frac{\|B\|}{\|B^{-1}\|}\left(\|B\|\left\|B^{-1}\right\| - \sqrt{1 - r^2\|B^{-1}\|^2}\right).\end{aligned} \tag{5.24}$$

Proof. We subtract the quantity $\frac{|\langle B^*Ax, x\rangle|}{\|B\|^2}$ from both sides of (5.22) to obtain

$$\begin{aligned}0 &\le \frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} - \frac{|\langle B^*Ax, x\rangle|}{\|B\|^2} \\ &\le 2 - \frac{|\langle B^*Ax, x\rangle|}{\|B\|^2} - \frac{1 - r^2\|B^{-1}\|^2}{|\langle B^*Ax, x\rangle|\,\|B^{-1}\|^2} \\ &= 2 - 2 \cdot \frac{\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B\|\,\|B^{-1}\|} \\ &\qquad - \left(\frac{\sqrt{|\langle B^*Ax, x\rangle|}}{\|B\|} - \frac{\sqrt{1 - r^2\|B^{-1}\|^2}}{\sqrt{|\langle B^*Ax, x\rangle|}\,\|B^{-1}\|}\right)^2 \\ &\le 2 \cdot \frac{\left(\|B\|\,\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}\right)}{\|B\|\,\|B^{-1}\|},\end{aligned}$$

which is equivalent with

$$\begin{aligned}(0 \le)\ &\|Ax\|^2\|B\|^2 - |\langle B^*Ax, x\rangle|^2 \\ &\le 2\frac{\|B\|}{\|B^{-1}\|}|\langle B^*Ax, x\rangle|\left(\|B\|\left\|B^{-1}\right\| - \sqrt{1 - r^2\|B^{-1}\|^2}\right)\end{aligned} \tag{5.25}$$

for any $x \in H$, $\|x\| = 1$.

The inequality (5.25) also shows that $\|B\|\,\|B^{-1}\| \ge \sqrt{1 - r^2\|B^{-1}\|^2}$ and then, by (5.25), we get

$$\begin{aligned}\|Ax\|^2\|B\|^2 &\le |\langle B^*Ax, x\rangle|^2 \\ &\quad + 2\frac{\|B\|}{\|B^{-1}\|}|\langle B^*Ax, x\rangle|\left(\|B\|\left\|B^{-1}\right\| - \sqrt{1 - r^2\|B^{-1}\|^2}\right)\end{aligned} \tag{5.26}$$

for any $x \in X$, $\|x\| = 1$.

Taking the supremum in (5.26) we deduce the desired inequality (5.24). □

Remark 5.12. The above Theorem 5.11 has some particular instances of interest as follows. If, for instance, we choose $B = \lambda I$ with $|\lambda| \geq r > 0$ and $\|A - \lambda I\| \leq r$, then by (5.24) we obtain the inequality

$$(0 \leq) \|A\|^2 - w^2(A) \leq 2|\lambda| w(A) \left(1 - \sqrt{1 - \frac{r^2}{|\lambda|^2}}\right).$$

Also, if A is invertible, $\|A - \lambda A^*\| \leq r$ and $\|A^{-1}\| \leq \frac{|\lambda|}{r}$, then by (5.24) we can state:

$$\begin{aligned}&(0 \leq) \|A\|^4 - w^2(A^2) \\ &\leq 2|\lambda| w(A^2) \cdot \frac{\|A\|}{\|A^{-1}\|} \left(\|A\| \|A^{-1}\| - \sqrt{1 - \frac{r^2}{|\lambda|^2} \|A^{-1}\|^2}\right).\end{aligned}$$

References

1. M.L. Buzano, *Generalizzazione della disiguaglianza di Cauchy-Schwaz*(Italian), Rend. Sem. Mat. Univ. e Politech. Torino, **31** (1971/73), 405-409 (1974).
2. S.S. Dragomir, *Some refinements of Schwarz inequality*, Simposional de Matematică şi Aplicaţii, Polytechnical Institute Timişoara, Romania, 1-2 Nov., 1985, 13-16. ZBL 0594:46018.
3. S.S. Dragomir, *Some Grüss type inequalities in inner product spaces*, J. Ineq. Pure & Appl. Math., **4**(2) (2003), Art. 42. [ONLINE http://jipam.vu.edu.au/article.php?sid=280].
4. S.S. Dragomir, *A potpourri of Schwarz related inequalities in inner product spaces (I)*, J. Ineq. Pure & Appl. Math., **6**(3) (2005), Art. 59. [ONLINE http://jipam.vu.edu.au/article.php?sid=532].
5. S.S. Dragomir, *A potpourri of Schwarz related inequalities in inner product spaces (II)*, J. Ineq. Pure & Appl. Math.,**7** (1) (2006), Art. 14. [ONLINE http://jipam.vu.edu.au/article.php?sid=619].
6. S.S. Dragomir, *Reverse inequalities for the numerical radius of linear operators in Hilbert spaces*, Bull. Austral. Math. Soc., **73**(2006), 255-262. Preprint available on line at RGMIA Res. Rep. Coll. **8**(2005), Supplement, Article 9, [ONLINE http://rgmia.vu.edu.au/v8(E).html].
7. S.S. Dragomir, *Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces*, Demonstratio Mathematica (Poland), **XL**(2007), No. 2, 411-417. Preprint available on line at RGMIA Res. Rep. Coll.**8**(2005), Supplement, Article 10, [ONLINE http://rgmia.vu.edu.au/v8(E).html].
8. S.S. Dragomir, *Inequalities for the norm and numerical radius of composite operators in Hilbert spaces*, Preprint available on line at RGMIA Res. Rep. Coll. **8**(2005), Supplement, Article 11, [ONLINE http://rgmia.vu.edu.au/v8(E).html].
9. S.S. Dragomir and J. Sándor, *Some inequalities in prehilbertian spaces*, Studia Univ. "Babeş-Bolyai"- Mathematica, **32**(1) (1987), 71-78.
10. A. Goldstein, J.V. Ryff and L.E. Clarke, *Problem 5473*, Amer. Math. Monthly, **75**(3) (1968), 309.
11. K.E. Gustafson and D.K.M. Rao, *Numerical Range*, Springer-Verlag, New York, Inc., 1997.
12. P.R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Chelsea Pub. Comp, New York, N.Y., 1972.

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