

(UN)STABILITY AND BORDISM GROUPS IN PDE'S

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ABSTRACT. In this paper, by using the theory of integral bordism groups in PDE's, previously introduced by Prástaro, we give a new interpretation of the concept of (un)stability in the framework of the geometric theory of PDE's. A geometric criterium to identify stable PDE's and stable solutions of PDE's is given.

1. INTRODUCTION AND PRELIMINARIES

Problems of stability of dynamic systems are usually studied in the framework of the functional analysis and referring to functional equations. By the way, recently some geometric studies of such problems have been made by Prástaro in the framework of the geometric theory of PDE's, and related to integral bordism groups. (See [19, 20, 23].) There a k -order PDE is considered as a subset $E_k \subset JD^k(W)$ of the k -jet-derivative space $JD^k(W)$, built on some fiber bundle $\pi : W \rightarrow M$, in the category of smooth manifolds. Then, to investigate the stability of a regular solution $D^k s(M) \equiv V \subset E_k$, of $E_k \subset JD^k(W)$, one considers the linearization of E_k at the solution V . The solutions of the linearized equation

$$(D^k s)^* vTE_k \equiv E_k[s] \subset JD^k(s^*vTW) \equiv JD^k(E[s]),$$

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represent the admissible perturbations of the original solution. Then, if to a “small” initial Cauchy data for such linearized equation there correspond solutions (perturbations) that oscillate around the zero solution (of the linearized equation), or remain limited around such zero solution, then the solution s is said to be stable, otherwise s is called unstable. Taking into account that the linearized equation $E_k[s]$ belongs to a vector neighbourhood of E_k , at the solution V , and that solutions of $E_k[s]$ are infinitesimal vertical symmetries of E_k , it follows that such perturbations deform the original solution $V \subset E_k$ into solutions $\tilde{V} \subset E_k$ such that, if V is stable, remain into suitable neighbourhoods of the same V . When, instead the perturbations blow-up, then V is unstable. The blowing-up of the perturbation corresponds to the fact that such a solution of the linearized equation $E_k[s]$ is not regular in all of its points, but there are present singular points. Then in the cases where V is unstable, between the solutions of above type \tilde{V} , there are ones that are also singular and this fact just characterizes unstable solutions of E_k .

This approach to the stability is in some sense related to the Ljapunov concept of stability in functional analysis [9], and it is founded on the assumption that the possible perturbations can influence only the given solution, say $V \subset E_k$, but do not have any influence on the same equation E_k .

On the other hand, we can more generally assume that perturbations can change the same original equation. In such a case we can ask whether a given solution of the original equation can change for “little” perturbations of the same equation. Then we talk about (un)stable equations. This last approach is, instead, related to the concept of Ulam (un)stability for functional equations [34].

In this paper we aim to prove that all above point of views for stability in PDE’s can be unified in the geometric theory of PDE’s on the ground of integral bordism groups, as has been formulated by Prástaro [13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. (See also [1, 2] for related subjects.)¹ The main result of this paper is Theorem 3.5. This gives a criterium to identify functionally stable PDE’s, and a criterium to identify stable solutions of PDE’s. Both are founded on purely geometric arguments. Another result is Theorem 3.7 that by means of a compactification approach allows to apply the criterium for stability of solutions, given in the quoted Theorem 3.5, to recognize asymptotic stability of solutions for evolutionary PDE’s. Applications to some important PDE’s are given also.

2. STABLE AND SUPERSTABLE FUNCTIONAL EQUATIONS

In this section we shall recall some definition of (un)stability for functional equations and some our previous applications of such a concept of stability to PDE’s.

Definition 2.1. Let G_1 be a group and let G_2 be a metric group with a metric $d(.,.)$. We say that the functional equation $h(xy) = h(x)h(y)$ is *Ulam-stable* if given $\epsilon > 0$, there exists a $\delta > 0$ such that if a function $f : G_1 \rightarrow G_2$ satisfies the

¹For basic informations on the geometry of PDE’s see also the following [3, 5, 6, 15]. For basic informations on some subjects of differential topology and algebraic topology, related to this paper, see also [4, 7, 10, 11, 12, 25, 27, 28, 29, 30, 31, 32, 33, 35, 36].

inequality $d(f(xy), f(x)f(y)) < \delta$, for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with $d(f(x), H(x)) < \epsilon$, for all $x \in G_1$.

Remark 2.2. The meaning of the Ulam stability for groups is essentially that homomorphisms can be “approximated” by functions, and this property depends on the metric structure fixed on the group where these mappings are evaluated. Generalizations of Definition 2.1 have been given by D. H. Hyers [8] and Th.M. Rassias [26]. Furthermore, in [24], in order to apply the concept of Ulam stability to study stability in the Navier-Stokes equation, the following more general definition has been given.

Definition 2.3. Let F be a functional space, i.e., a space of suitable applications $f : X \rightarrow Y$ between finite dimensional Riemannian manifolds X and Y . Let E be a Banach space and S a subset of X^n . Let us consider a functional equation:

$$G(f, q^1, \dots, q^n) = 0, \quad \forall (q^1, \dots, q^n) \in S \subset X^n \quad (2.1)$$

defined by means of a mapping

$$G : F \times X^n \rightarrow E, \quad (f, (q^1, \dots, q^n)) \mapsto G(f, (q^1, \dots, q^n)).$$

We say that such a functional equation is *Ulam-extended stable* if for any function $\tilde{f} \in F$, satisfying the inequality

$$\|G(\tilde{f}, (q^1, \dots, q^n))\| \leq \varphi(q^1, \dots, q^n), \quad \forall (q^1, \dots, q^n) \in X^n \quad (2.2)$$

with $\varphi : X^n \rightarrow [0, \infty)$ fixed, there exists a solution f of 2.1 such that

$$d_Y(\tilde{f}(q), f(q)) \leq \Phi(q), \quad (q \in X),$$

for suitable $\Phi : X \rightarrow [0, \infty)$. Here $d_Y(., .)$ is the metric induced by the Riemannian structure of Y .² If each solution $\tilde{f} \in F$ of the inequality 2.2 is either a solution of the functional equation 2.1 or satisfies some stronger conditions, then we say that equation 2.1 is *Ulam-extended-superstable*.

Remark 2.4. A geometric way to determine the (un)stability of solutions of a PDE is to vertically linearize this equation at such solutions and to study the so-obtained linear equation, where the dependent variables represent the components of the perturbations. (For example for the Navier-Stokes equation, it is well known that this analysis gets to the conclusion that the unstability is related to critical values of some adimensional numbers. (For a modern geometric analysis see [19, 23].)) From the point of view of the characteristic flow, this corresponds to the lost of the diffeomorphism correspondence between space-like regions related

²More precisely, $d_Y(a, b) = \inf \int_{[0,1]} \sqrt{g_Y(\dot{\gamma}(t), \dot{\gamma}(t))} dt$, $\forall a, b \in Y$, where $\gamma \in C^1([0, 1], Y)$, with $\gamma(0) = a$, $\gamma(1) = b$ and g_Y the Riemannian metric on Y . One has the following important propositions: (i) If $\gamma \in C^1([0, 1], Y)$, such that $\gamma(0) = a$, $\gamma(1) = b$, and $\int_{[0,1]} \sqrt{g_Y(\dot{\gamma}(t), \dot{\gamma}(t))} dt = d_Y(a, b)$, then $\gamma([0, 1]) \subset Y$ is a geodesic. If in addition, γ has constant speed, then γ is a C^∞ geodesic. (ii)(de Rham) When Y is geodesically complete, then given $a, b \in Y$, there exists at least one geodesic γ in Y connecting a to b with $\int_{[0,1]} \sqrt{g_Y(\dot{\gamma}(t), \dot{\gamma}(t))} dt = d_Y(a, b)$. (iii)(Hopf, Rinov) The geodesic completeness of Y is equivalent to the completeness of Y as a metric space, which is equivalent to the statement that a subset of Y is compact iff it is closed and bounded. (iv) Y is complete whenever it is compact.

by such flows, assumed that at the beginning they were laminar. More precisely we get tunnel effects, as described in [15], where we talk of contractions of cells. Furthermore, in [24] Prástaro related such a behaviour to the concept of stability of functional equations, introducing a new functional equation (*functional Navier-Stokes equation*), that is Ulam-extended stable, and superstable. In fact, the stability of the functional Navier-Stokes equation is just in correspondence with the fact that there are flows with singularities in the neighbourhood of laminar flows. In the following section we want give a more geometric interpretation of the concept of Ulam-stability, that generalizes previous one, and can be applied to any PDE.

3. INTEGRAL AND QUANTUM BORDISM GROUPS VS. STABILITY

In this section we prove the main result of this paper (Theorem 3.5), i.e., a geometric criterium that allows us to recognize PDE's stable in functional sense, and also to relate stability of solutions to the concept of functional stability. In particular we give also a geometric criterium to recognize asymptotic stable solutions of PDE's, (Theorem 3.7).

Theorem 3.1. *Let $E_k \subset JD^k(W)$ be a k -order PDE on the fiber bundle $\pi : W \rightarrow M$ in the category of smooth manifolds, $\dim W = m + n$, $\dim M = n$. Let $s : M \rightarrow W$ be a section, solution of E_k , and let $\nu : M \rightarrow s^*vTW \equiv E[s]$ be a solution of the linearized equation $E_k[s] \subset JD^k(E[s])$. Then to ν is associated a flow $\{\phi_\lambda\}_{\lambda \in J}$, where $J \subset \mathbb{R}$ is a neighbourhood of $0 \in \mathbb{R}$, that transforms V into a new solution $\tilde{V} \subset E_k$.*

Proof. Let (x^α, y^j) be fibered coordinates on W . Let $\nu = \nu^j \partial y_j : M \rightarrow s^*vTW$ a vertical vector field on W along the section $s : M \rightarrow W$. Then ν is a solution of $E_k[s]$ iff the following diagram is commutative:

$$\begin{array}{ccccc}
 E_k[s] & \xlongequal{\sim} & (D^k s)^* vTE_k & \longrightarrow & vTE_k & (3.1) \\
 \downarrow & & \downarrow & & \downarrow & \\
 JD^k(s^*vTW) & \xlongequal{\sim} & (D^k s)^* vTJD^k(W) & \longrightarrow & vTJD^k(W) & \\
 \downarrow & & \downarrow & \swarrow & \downarrow \pi'_* & \\
 s^*vTW & \longrightarrow & vTW & & JD^k(W) & \\
 \uparrow \nu & & \downarrow \pi' & \swarrow & & \\
 M & \xrightarrow{s} & W & & & \\
 & & & \searrow & & \\
 & & & & & JD^k(W)
 \end{array}$$

$D^k \nu$ (curved arrow from M to $E_k[s]$)
 $D^k s$ (curved arrow from M to $JD^k(W)$)

Then $D^k \nu(p)$ identifies, for any $p \in M$, a vertical vector on E_k in the point $q = D^k s(p) \in V = D^k s(M) \subset E_k$. On the other hand infinitesimal vertical

symmetries on E_k are locally written in the form

$$\begin{cases} \zeta = \sum_{0 \leq |\alpha| \leq k} Y_\alpha^j \partial y_j^\alpha, & 0 = \zeta.F^I = 0 \\ Y_\alpha^j = Z_\alpha^{(0)} Y^j, & Z_\alpha^{(0)} = \partial_\alpha + y_\alpha^s \partial_s \\ Y_{\alpha_1 \dots \alpha_r}^j = Z_i^{(r)} Y_{\alpha_1 \dots \alpha_r}^j, & Z_i^{(r)} = Z_i^{(r-1)} + y_{\gamma_1 \dots \gamma_r}^s \partial_s^{\gamma_1 \dots \gamma_r} \end{cases} \quad (3.2)$$

Note that such vector fields cannot be characteristic vector fields, since these last are of the type $\xi = X^\alpha(\partial x_\alpha + \sum_{0 \leq |\beta| \leq k} Y_{\alpha\beta}^j \partial y_j^\beta)$. Therefore, the flows of above vertical vector fields, transform regular solutions V of E_k into new solutions of E_k . Solutions of the linearized equation $E_k[s]$ give initial conditions for the determination of such vertical flows. \square

Definition 3.2. Let $E_k \subset J_n^k(W)$, where $\pi : W \rightarrow M$ is a fiber bundle, in the category of smooth manifolds. We say that E_k is *functionally stable* if for any compact regular solution $V \subset E_k$, such that $\partial V = N_0 \cup P \cup N_1$ one has quantum solutions $\tilde{V} \subset J_n^{k+s}(W)$, $s \geq 0$, such that $\pi_{k+s,0}(\tilde{N}_0 \cup \tilde{N}_1) = \pi_{k,0}(N_0 \cup N_1) \equiv X \subset W$, where $\partial \tilde{V} = \tilde{N}_0 \cup \tilde{P} \cup \tilde{N}_1$.³

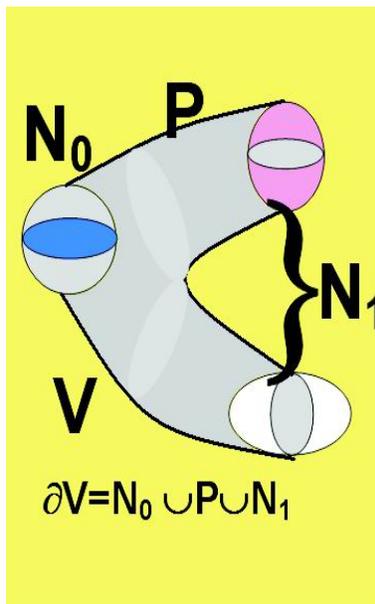


Fig.1 Bordism V between compact manifolds N_0 and N_1 .

³Here and in the following we shall denote the boundary ∂V of a compact n -dimensional manifold V , split in the form $\partial V = N_0 \cup P \cup N_1$, where N_0 and N_1 are two disjoint $(n - 1)$ -dimensional submanifolds of V , that are not necessarily closed, and P is another $(n - 1)$ -dimensional submanifold of V . For example, if $V = S \times I$, where $I \equiv [0, 1] \subset \mathbb{R}$, one has $N_0 = S \times \{0\}$, $N_1 = S \times \{1\}$, $P = \partial S \times I$. In the particular case that $\partial S = \emptyset$, one has also $P = \emptyset$. (See Fig.1.) Let us also recall that with the term *quantum solutions* we mean integral bordisms relating Cauchy hypersurfaces of E_{k+s} , contained in $J_n^{k+s}(W)$, but not necessarily contained into E_{k+s} . (For details see [13, 15, 16, 17, 18, 19].)

We call the set $\Omega[V]$ of such solutions \tilde{V} the *full quantum situs* of V . We call also each element $\tilde{V} \in \Omega[V]$ a *quantum fluctuation* of V .⁴

Definition 3.3. We call *infinitesimal bordism* of a regular solution $V \subset E_k \subset JD^k(W)$ an element $\tilde{V} \in \Omega[V]$, defined in the proof of Theorem 3.1. We denote by $\Omega_0[V] \subset \Omega[V]$ the set of infinitesimal bordisms of V . We call $\Omega_0[V]$ the *infinitesimal situs* of V .

Definition 3.4. Let $E_k \subset J_n^k(W)$, where $\pi : W \rightarrow M$ is a fiber bundle, in the category of smooth manifolds. We say that a regular solution $V \subset E_k$, $\partial V = N_0 \cup P \cup N_1$, is *functionally stable* if the infinitesimal situs $\Omega_0[V] \subset \Omega[V]$ of V does not contain singular infinitesimal bordisms.

Theorem 3.5. Let $E_k \subset J_n^k(W)$, where $\pi : W \rightarrow M$ is a fiber bundle, in the category of smooth manifolds. If E_k is formally integrable and completely integrable, then it is functionally stable as well as Ulam-extended superstable.

A regular solution $V \subset E_k$ is stable iff it is functionally stable.

Proof. In fact, if E_k is formally integrable and completely integrable, we can consider, for any compact regular solution $V \subset E_k$, its s -th prolongation $V^{(s)} \subset (E_k)_{+s} \subset J_n^{k+s}(W)$. Since one has the following short exact sequence

$$\Omega_{n-1}^{(E_k)_{+s}} \longrightarrow \Omega_{n-1}((E_k)_{+s}) \longrightarrow 0 \quad (3.3)$$

between quantum bordism groups $\Omega_{n-1}((E_k)_{+s})$ ⁵ and integral bordism groups $\Omega_{n-1}^{(E_k)_{+s}}$, we get that there exists a solution $\tilde{V} \subset J_n^{k+s}(W)$ such that

$$\begin{cases} \partial \tilde{V} = \tilde{N}_0 \cup \tilde{P} \cup \tilde{N}_1 \\ \partial V^{(s)} = N_0^{(s)} \cup P^{(s)} \cup N_1^{(s)} \\ \tilde{N}_0 = N_0^{(s)} \\ \tilde{N}_1 = N_1^{(s)} \end{cases} \quad (3.4)$$

Then, as a by-product we get also: $\pi_{k+s,0}(\tilde{N}_0 \cup \tilde{N}_1) = \pi_{k,0}(N_0 \cup N_1) \subset W$. Therefore, E_k is functionally stable. Furthermore, E_k is also Ulam-extended superstable, since the integral bordism group $\Omega_{n-1}^{E_k}$ for smooth solutions and the integral bordism group $\Omega_{n-1,s}^{E_k}$ for singular solutions, are related by the following short exact sequence:

$$0 \longrightarrow K_{n-1,s}^{E_k} \longrightarrow \Omega_{n-1}^{E_k} \longrightarrow \Omega_{n-1,s}^{E_k} \longrightarrow 0 \quad (3.5)$$

This implies that in the neighbourhood of each smooth solution there are singular solutions.

Finally a regular solution $V \subset E_k$ is stable iff the set of solutions of the corresponding linearized equation $E_k[V]$ does not contains singular solutions. As a

⁴Let us emphasize that to $\Omega[V]$ belong also (non necessarily regular) solutions $V' \subset E_k$ such that $N'_0 \cup N'_1 = N_0 \cup N_1$, where $\partial V' = N'_0 \cup P' \cup N'_1$.

⁵Here the considered bordism groups are for admissible non-necessarily closed Cauchy hypersurfaces.

by-product we get then also $\Omega_0[V]$ does not contains singular solutions, therefore V is functionally stable. \square

Remark 3.6. Let us emphasize that the definition of functionally stable PDE interprets in pure geometric way the definition of Ulam superstable functional equation just adapted to PDE's.

Let us also remark that in evolutionary PDE's, i.e., PDE's built on a fiber bundle $\pi : W \rightarrow M$, over a "space-time" M , $\{x^\alpha, y^j\}_{0 \leq \alpha \leq n, 1 \leq j \leq m} \mapsto \{x^\alpha\}_{0 \leq \alpha \leq n}$, where $x^0 = t$ represents the time coordinate, one can consider "asymptotic stability", i.e., the behaviour of perturbations of global solutions for $t \rightarrow \infty$. In such cases we can recast our formulation on the corresponding compactified space-times.

Theorem 3.7. *Let $E_k \subset JD^k(W)$ be a k -order PDE, over a fiber bundle $\pi : W \rightarrow M$, $\dim W = m + n$, $\dim M = n$, where $M \simeq D^n$, is identified with the n -dimensional disk. We call the boundary ∂M the asymptotic situs of M . Set $\overset{\circ}{M} = M \setminus \partial M$ and $\overset{\circ}{W} \equiv W|_{\overset{\circ}{M}}$. One has the canonical fiber bundle embedding $(j, i) : (\overset{\circ}{W}, \overset{\circ}{M}) \hookrightarrow (W, M)$. This induces an embedding $j^{(k)} : JD^k(\overset{\circ}{W}) \rightarrow JD^k(W)$ over (j, i) . Set*

$$\overset{\circ}{E}_k \equiv E_k \cap JD^k(\overset{\circ}{W}). \quad (3.6)$$

Therefore, one has the canonical embedding $j^{(k)} : \overset{\circ}{E}_k \rightarrow E_k$. A global regular solution $V \subset W$, of E_k , is considered the asymptotic completion of $\overset{\circ}{V} = V \cap \overset{\circ}{E}_k$. $\overset{\circ}{V}$ is said asymptotically stable if V is stable. Then $\overset{\circ}{V}$ is asymptotically stable iff V is functionally stable.

Proof. If V is a regular solution of $E_k \subset JD^k(W)$, we can identify a regular solution $\overset{\circ}{V}$ of $\overset{\circ}{E}_k$, such that the following diagram

$$\begin{array}{ccc}
 \overset{\circ}{E}_k & \xrightarrow{j^{(k)}} & E_k \\
 \downarrow & & \downarrow \\
 JD^k(\overset{\circ}{W}_k) & \xrightarrow{j^{(k)}} & JD^k(W) \\
 \downarrow & & \downarrow \\
 \overset{\circ}{W}_k & \xrightarrow{j} & W \\
 \downarrow & \curvearrowright \overset{\circ}{s}_k & \downarrow \\
 \overset{\circ}{M}_k & \xrightarrow{i} & M
 \end{array}
 \quad (3.7)$$

$\overset{\circ}{D}^k \overset{\circ}{s}_k$ (left side) $D^k s$ (right side)

is commutative. Therefore, to study the asymptotic behaviour of regular global solutions of $\overset{\circ}{E}_k$, we can consider the differential equation $E_k \subset JD^k(W)$, and directly study admissible global solutions of E_k . Then by considering the infinitesimal situs $\Omega_0[V]$ of $V = s(M)$, global regular solution of E_k , we can say that

$\overset{\circ}{V}$ is asymptotically stable iff V is stable. From Theorem 3.5 one has that V is stable iff it is functionally stable, i.e., $\Omega_0[V]$ does not contain singular solutions of E_k . So the proof is complete. \square

Example 3.8. (*d'Alembert equation*). Let $(d'A) \subset JD^2(\mathbb{R}^2, \mathbb{R})$ be the following PDE of the second order $uu_{xy} - u_xu_y = 0$, over the trivial fiber bundle $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $(x, y, u) \mapsto (x, y)$. In the subset $u \neq 0$, this equation is formally integrable and completely integrable. (See [18, 19].) Then, according to Theorem 3.5 we get that $(d'A) \setminus \{u = 0\}$ is functionally stable and Ulam-extended superstable too.

Asymptotic stability of solutions of the d'Alembert equation can be studied considering their functional stability for the corresponding regular solution of the d'Alembert equation written on the fiber bundle $\pi : W \equiv D^2 \times \mathbb{R} \rightarrow D^2$.

Example 3.9. (*Navier-Stokes equation*). The non-isothermal Navier-Stokes equation is a second order PDE (NS) on the affine fiber bundle $\pi : W \equiv M \times \mathbf{I} \times \mathbb{R}^2 \rightarrow M$, $(x^\alpha, \dot{x}^i, \lambda, \mu) \mapsto (x^\alpha)$, where M is the 4-dimensional space-time, \mathbf{I} is the affine space of the time-like velocities of the flows, and \mathbb{R}^2 is the space of the isobaric pressure and temperature respectively. One can prove that such an equation is not formally integrable, but we can associate to it another equation (\widehat{NS}) that is formally integrable and completely integrable. (See [19].) Then, from Theorem 3.5 we get that (\widehat{NS}) is functionally stable and Ulam-extended superstable too. Instead for (NS) we cannot state the same. This agrees with the same result given in [24], but, now, our conclusions are founded only on geometric arguments.

Asymptotic stability of solutions of the Navier-Stokes equation can be studied considering their functional stability for the corresponding regular solution of the Navier-Stokes equation written on the fiber bundle $\pi : W \equiv D^4 \times \mathbf{I} \times \mathbb{R}^2 \rightarrow D^4$.

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