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## THE HYPERBOLIC SQUARE AND MÖBIUS TRANSFORMATIONS

ABRAHAM A. UNGAR<sup>1</sup>

*This paper is dedicated to Professor Themistocles M. Rassias.*

Submitted by J. M. Isidro Gómez

ABSTRACT. Professor Themistocles M. Rassias' special predilection and contribution to the study of Möbius transformations is well known. Möbius transformations of the open unit disc of the complex plane and, more generally, of the open unit ball of any real inner product space, give rise to Möbius addition in the ball. The latter, in turn, gives rise to Möbius gyrovector spaces that enable the Poincaré ball model of hyperbolic geometry to be approached by gyrovector spaces, in full analogy with the common vector space approach to the standard model of Euclidean geometry. The purpose of this paper, dedicated to Professor Themistocles M. Rassias, is to employ the Möbius gyrovector spaces for the introduction of the hyperbolic square in the Poincaré ball model of hyperbolic geometry. We will find that the hyperbolic square is richer in structure than its Euclidean counterpart.

### 1. INTRODUCTION

Professor Themistocles M. Rassias' work in the areas of Möbius transformations appear in several papers, including [10, 11, 12, 13] and [25, 28, 27], along with essential mathematical developments found, for instance, in [3, 4, 5, 17, 18, 19, 20, 21, 22], which contain essential research Mathematics on geometric

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transformations including Möbius transformations. In addition, the author’s unified way of solving the quadratic, cubic and quartic equations was presented by Professor Themistocles M. Rassias and coauthors in [15].

This work demonstrates Professor Rassias’ special predilection and contribution to the study of Möbius transformations. The purpose of this article, dedicated to him, is to employ Möbius transformations in the presentation and the study of the hyperbolic square in the Poincaré ball model of hyperbolic geometry. In order to keep the exposition reasonably self-contained, and to set the stage for our study of the hyperbolic square, we present the modern use of Möbius transformations of the open disc of the complex plane, and of the open ball of any real inner product space, in the study of the Poincaré ball model of hyperbolic geometry. Following this presentation, in Sections 2–6, we will be in the position to present and study the hyperbolic square in the Poincaré ball model of hyperbolic geometry in Section 7. Thus, a large part of this article provides a research account of recent results that set the stage for the introduction and the study of the hyperbolic square, shown in Fig. 3. Basically, as in Euclidean geometry, the hyperbolic square is a hyperbolic quadrilateral with all four sides of equal hyperbolic length and all four hyperbolic angles of equal measure.

In Section 2 we show how the well known *polar decomposition* of Möbius transformation of the complex open unit disc leads to *Möbius addition* in the disc. Seemingly structureless, Möbius addition in the disc is neither commutative nor associative. However, we discover an unexpected grouplike structure that underlies Möbius addition, according to which Möbius addition is both *gyrocommutative* and *gyroassociative*, giving rise to the *gyrogroup* structure. These “gyro” variations create an elaborate “gyrolanguage” in which terms familiar from the Euclidean setting get their gyro-counterpart.

In Section 3 we extend Möbius addition and its gyrocommutative gyrogroup structure from a binary operation in the complex open unit disc into a binary operation in the open ball of any real inner product space, resulting in the Möbius ball gyrogroup. In Section 4 we extend the Möbius ball gyrogroup into a Möbius gyrovector space. In Section 5 we show that Möbius gyrovector spaces form the setting for the Poincaré ball model of hyperbolic geometry just as vector spaces form the setting for the standard model of Euclidean geometry. In Section 6 we present the hyperbolic triangle in the Poincaré ball model of hyperbolic geometry, which we naturally call a *gyrotriangle* in gyrolanguage, along with its standard notation and basic results. Only now, following Sections 1–6 we are in the position to introduce and study the *gyrosquare*, that is, the hyperbolic square. Guided by Euclidean geometry, a gyrosquare is a gyroquadrilateral with all four sides of equal gyrolength and all four gyroangles of equal measure. Gyrolanguage, thus, turns out to be the language we need to articulate analogies that the classical and the modern in this paper share.

## 2. MÖBIUS TRANSFORMATIONS OF THE DISC

Möbius transformations of the open unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  of the complex plane  $\mathbb{C}$  are well known. Ahlfors’ book [1], *Conformal Invariants*:

*Topics in Geometric Function Theory*, begins with a presentation of the Möbius self-transformation of the complex open unit disc  $\mathbb{D}$ ,

$$z \mapsto e^{i\theta} \frac{a+z}{1+\bar{a}z} = e^{i\theta}(a\oplus z) \quad (2.1)$$

where  $\bar{a}$  is the complex conjugate of  $a$ , and where  $a, z \in \mathbb{D}$  and  $\theta \in \mathbb{R}$  [6, p. 211] [9, p. 185] [16, pp. 177–178]. We present the Möbius transformation *polar decomposition* (2.1) in a form that suggests the *Möbius addition*,  $\oplus$ , defined by the equation

$$a\oplus z = \frac{a+z}{1+\bar{a}z} \quad (2.2)$$

Naturally, Möbius subtraction,  $\ominus$ , is given by  $a\ominus z = a\oplus(-z)$ , so that  $z\ominus z = 0$  and  $0z = 0\ominus z = 0\oplus(-z) = -z$ . Remarkably, Möbius addition possesses the *automorphic inverse property*

$$\ominus(a\oplus b) = \ominus a\ominus b \quad (2.3)$$

and the *left cancellation law*

$$\ominus a\oplus(a\oplus z) = z \quad (2.4)$$

for all  $a, b, z \in \mathbb{D}$ , [29, 30].

Möbius addition gives rise to the Möbius disc groupoid  $(\mathbb{D}, \oplus)$ , recalling that a groupoid  $(G, \oplus)$  is a nonempty set,  $G$ , with a binary operation,  $\oplus$ , and that an automorphism of a groupoid  $(G, \oplus)$  is a bijective self map  $f$  of  $G$  that respects its binary operation  $\oplus$ , that is,  $f(a\oplus b) = f(a)\oplus f(b)$ . The set of all automorphisms of a groupoid  $(G, \oplus)$  forms a group, denoted  $Aut(G, \oplus)$ .

Möbius addition  $\oplus$  in the disc is neither commutative nor associative. To measure the extent to which Möbius addition  $\oplus$  in the disc  $\mathbb{D}$  deviates from associativity we define the *gyrator*

$$gyr : \mathbb{D} \times \mathbb{D} \rightarrow Aut(\mathbb{D}, \oplus) \quad (2.5)$$

by the equation

$$gyr[a, b]z = \ominus(a\oplus b)\oplus\{a\oplus(b\oplus z)\} \quad (2.6)$$

for all  $a, b, z \in \mathbb{D}$ .

The automorphisms

$$gyr[a, b] \in Aut(\mathbb{D}, \oplus) \quad (2.7)$$

$a, b \in \mathbb{D}$ , are called gyrations [34]. In order to emphasize that gyrations of  $\mathbb{D}$  are also automorphisms of  $(\mathbb{D}, \oplus)$ , as we will see below, they are also called *gyroautomorphisms*.

Clearly, in the special case when the binary operation  $\oplus$  in (2.6) is associative,  $gyr[a, b]$  reduces to the trivial automorphism,  $gyr[a, b]z = z$  for all  $a, b, z \in \mathbb{D}$ , so that, indeed, the self map  $gyr[a, b]$  of the disc  $\mathbb{D}$  measures the extent to which Möbius addition  $\oplus$  in the disc  $\mathbb{D}$  deviates from associativity.

One can rewrite (2.6) in terms of (2.2), obtaining

$$gyr[a, b]z = \frac{1+a\bar{b}}{1+\bar{a}b}z \quad (2.8)$$

so that the gyrations

$$\text{gyr}[a, b] = \frac{1 + a\bar{b}}{1 + \bar{a}b} = \frac{a \oplus b}{b \oplus a} \quad (2.9)$$

$a, b \in \mathbb{D}$ , are unimodular complex numbers. As such, gyrations represent rotations of the disc  $\mathbb{D}$  about its center, as shown in (2.8).

Gyrations are invertible. The inverse  $\text{gyr}^{-1}[a, b]$  of a gyration  $\text{gyr}[a, b]$  is the gyration  $\text{gyr}[b, a]$ ,

$$\text{gyr}^{-1}[a, b] = \text{gyr}[b, a] \quad (2.10)$$

Moreover, gyrations respect Möbius addition in the disc,

$$\text{gyr}[a, b](c \oplus d) = \text{gyr}[a, b]c \oplus \text{gyr}[a, b]d \quad (2.11)$$

for all  $a, b, c, d \in \mathbb{D}$ , so that gyrations of the disc are, in fact, automorphisms of the disc, as anticipated in (2.7).

Identity (2.9) can be written as

$$a \oplus b = \text{gyr}[a, b](b \oplus a) \quad (2.12)$$

thus giving rise to the *gyrocommutative law* of Möbius addition. Furthermore, Identity (2.6) can be manipulated, by mean of the left cancellation law (2.4), into the identity

$$a \oplus (b \oplus z) = (a \oplus b) \oplus \text{gyr}[a, b]z \quad (2.13)$$

thus giving rise to the *left gyroassociative law* of Möbius addition.

The gyrocommutative law, (2.12), and the left gyroassociative law, (2.13), of Möbius addition in the disc reveal the grouplike structure of Möbius groupoid  $(\mathbb{D}, \oplus)$ , that we naturally call a *gyrocommutative gyrogroup*. Taking the key features of Möbius groupoid  $(\mathbb{D}, \oplus)$  as axioms, and guided by analogies with group theory, we obtain the following definitions of gyrogroups and gyrocommutative gyrogroups, where we attach the prefix “gyro” to a classical term in the algebra of Euclidean geometry to mean the analogous term in the algebra of hyperbolic geometry.

**Definition 2.1. (Gyrogroups).** *A groupoid  $(G, \oplus)$  is a gyrogroup if its binary operation satisfies the following axioms. In  $G$  there is at least one element,  $0$ , called a left identity, satisfying*

$$(G1) \quad 0 \oplus a = a$$

*for all  $a \in G$ . There is an element  $0 \in G$  satisfying axiom (G1) such that for each  $a \in G$  there is an element  $\ominus a \in G$ , called a left inverse of  $a$ , satisfying*

$$(G2) \quad \ominus a \oplus a = 0.$$

*Moreover, for any  $a, b, c \in G$  there exists a unique element  $\text{gyr}[a, b]c \in G$  such that the binary operation obeys the left gyroassociative law*

$$(G3) \quad a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c.$$

*The map  $\text{gyr}[a, b] : G \rightarrow G$  given by  $c \mapsto \text{gyr}[a, b]c$  is an automorphism of the groupoid  $(G, \oplus)$ , that is,*

$$(G4) \quad \text{gyr}[a, b] \in \text{Aut}(G, \oplus),$$

*and the automorphism  $\text{gyr}[a, b]$  of  $G$  is called the gyroautomorphism, or the gyration, of  $G$  generated by  $a, b \in G$ . The operator  $\text{gyr} : G \times G \rightarrow \text{Aut}(G, \oplus)$  is*

called the gyrotator of  $G$ . Finally, the gyroautomorphism  $\text{gyr}[a, b]$  generated by any  $a, b \in G$  possesses the left loop property

$$(G5) \quad \text{gyr}[a, b] = \text{gyr}[a \oplus b, b].$$

The gyrogroup axioms (G1)–(G5) in Definition 2.1 are classified into three classes:

- (1) The first pair of axioms, (G1) and (G2), is reminiscent of the group axioms.
- (2) The last pair of axioms, (G4) and (G5), presents the gyrotator axioms.
- (3) The middle axiom, (G3), is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation  $a \ominus b = a \oplus (\ominus b)$  in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

**Definition 2.2. (Gyrocommutative Gyrogroups).** *A gyrogroup  $(G, \oplus)$  is gyrocommutative if its binary operation obeys the gyrocommutative law*

$$(G6) \quad a \oplus b = \text{gyr}[a, b](b \oplus a)$$

for all  $a, b \in G$ .

Gyrogroup theorems, some of which are analogous to group theorems, are presented in [29, 30]. Thus, without losing the flavor of the group structure we have generalized it into the gyrogroup structure to suit the needs of Möbius addition in the disc and, more generally, in the ball of any real inner product space, [32, 33]. Gyrogroups abound in group theory, as shown in [7] and [8], where finite and infinite gyrogroups, both gyrocommutative and non-gyrocommutative, are studied. The generalization of groups into gyrogroups that Möbius addition suggests bears an intriguing resemblance to the generalization of the rational numbers into the real ones. The beginner is initially surprised to discover an irrational number, like  $\sqrt{2}$ , but soon later he is likely to realize that there are more irrational numbers than rational ones. Similarly, the gyrogroup structure of Möbius addition initially comes as a surprise. But, interested explorers may soon realize that there are more non-group gyrogroups than groups.

### 3. MÖBIUS ADDITION IN THE BALL

If we identify complex numbers of the complex plane  $\mathbb{C}$  with vectors of the Euclidean plane  $\mathbb{R}^2$  in the usual way,

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2 \tag{3.1}$$

then the inner product and the norm in  $\mathbb{R}^2$  are given by the equations

$$\begin{aligned} \bar{u}v + u\bar{v} &= 2\mathbf{u} \cdot \mathbf{v} \\ |u| &= \|\mathbf{u}\| \end{aligned} \tag{3.2}$$

These, in turn, enable us to translate Möbius addition from the open complex unit disc  $\mathbb{D}$  into the open unit disc  $\mathbb{R}_{s=1}^2 = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < s = 1\}$  of  $\mathbb{R}^2$  as

follows:

$$\begin{aligned}
\mathbb{D} \ni u \oplus v &= \frac{u + v}{1 + \bar{u}v} \\
&= \frac{(1 + u\bar{v})(u + v)}{(1 + \bar{u}v)(1 + u\bar{v})} \\
&= \frac{(1 + \bar{u}v + u\bar{v} + |v|^2)u + (1 - |u|^2)v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2} \\
&= \frac{(1 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \mathbf{u} \oplus \mathbf{v} \in \mathbb{R}_{s=1}^2
\end{aligned} \tag{3.3}$$

for all  $u, v \in \mathbb{D}$  and all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}_{s=1}^2$ . The last equation in (3.3) is a vector equation, so that its restriction to the ball of the Euclidean two-dimensional space is a mere artifact. As such, it survives unimpaired in higher dimensions, suggesting the following definition of Möbius addition in the ball of any real inner product space.

**Definition 3.1. (Möbius Addition in the Ball).** *Let  $\mathbb{V}$  be a real inner product space [14], and let  $\mathbb{V}_s$  be the  $s$ -ball of  $\mathbb{V}$ ,*

$$\mathbb{V}_s = \{\mathbb{V}_s \in \mathbb{V} : \|\mathbf{v}\| < s\} \tag{3.4}$$

for any fixed  $s > 0$ . Möbius addition  $\oplus$  in the ball is a binary operation in  $\mathbb{V}_s$  given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{(1 + \frac{2}{s^2}\mathbf{u} \cdot \mathbf{v} + \frac{1}{s^2}\|\mathbf{v}\|^2)\mathbf{u} + (1 - \frac{1}{s^2}\|\mathbf{u}\|^2)\mathbf{v}}{1 + \frac{2}{s^2}\mathbf{u} \cdot \mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{3.5}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$ , where  $\cdot$  and  $\|\cdot\|$  are the inner product and norm that the ball  $\mathbb{V}_s$  inherits from its space  $\mathbb{V}$ .

Without loss of generality, one may select  $s = 1$  in Definition 3.1. We, however, prefer to keep  $s$  as a free positive parameter in order to exhibit the result that in the limit as  $s \rightarrow \infty$ , the ball  $\mathbb{V}_s$  expands to the whole of its real inner product space  $\mathbb{V}$ , and Möbius addition  $\oplus$  reduces to vector addition in  $\mathbb{V}$ .

Möbius addition in the ball  $\mathbb{V}_s$  is known in the literature as a *hyperbolic translation* [2, 23]. Following the discovery of the gyrocommutative gyrogroup structure in 1988 [24], Möbius hyperbolic translation in the ball  $\mathbb{V}_s$  now deserves the title ‘‘Möbius addition’’ in the ball  $\mathbb{V}_s$ , in full analogy with the standard vector addition in the space  $\mathbb{V}$  that contains the ball.

Möbius addition in the ball  $\mathbb{V}_s$  satisfies the *gamma identity*

$$\gamma_{\mathbf{u} \oplus \mathbf{v}} = \gamma_{\mathbf{u}} \gamma_{\mathbf{v}} \sqrt{1 + \frac{2}{s^2}\mathbf{u} \cdot \mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{3.6}$$

for all  $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$ , where  $\gamma_{\mathbf{u}}$  is the gamma factor

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}} \tag{3.7}$$

in the  $s$ -ball  $\mathbb{V}_s$ . Let  $\mathbf{v} \in \mathbb{V}$ . Then,  $\mathbf{v} \in \mathbb{V}_s$  if and only if  $\gamma_{\mathbf{v}}$  is real. Hence, it follows from the gamma identity (3.6) that  $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s \implies \mathbf{u} \oplus \mathbf{v} \in \mathbb{V}_s$ , so that Möbius addition is, indeed, a binary operation in the ball  $\mathbb{V}_s$ .

#### 4. MÖBIUS SCALAR MULTIPLICATION IN THE BALL

A Möbius gyrogroup  $(\mathbb{V}_s, \oplus)$  admits scalar multiplication,  $\otimes$ , between a real number  $r \in \mathbb{R}$  and a vector  $\mathbf{v} \in \mathbb{V}_s$ , turning the gyrogroup into a Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$ . The Möbius scalar multiplication definition follows.

**Definition 4.1. (Möbius Scalar Multiplication).** *Let  $(\mathbb{V}_s, \oplus)$  be a Möbius gyrogroup. Then its corresponding Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$  involves the Möbius scalar multiplication  $r \otimes \mathbf{v} = \mathbf{v} \otimes r$  in  $\mathbb{V}_s$ , given by the equation*

$$\begin{aligned} r \otimes \mathbf{v} &= s \frac{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r - \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^r + \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s \tanh\left(r \tanh^{-1} \frac{\|\mathbf{v}\|}{s}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \quad (4.1)$$

where  $r \in \mathbb{R}$ ,  $\mathbf{v} \in \mathbb{V}_s$ ,  $\mathbf{v} \neq \mathbf{0}$ ; and  $r \otimes \mathbf{0} = \mathbf{0}$ .

As examples, for any natural number  $n$ ,  $n \otimes \mathbf{v} = \mathbf{v} \oplus \dots \oplus \mathbf{v}$  ( $n$  terms), and the ‘‘Möbius half’’ is obtained from (4.1) with  $r = 1/2$ , resulting in the identity

$$\frac{1}{2} \otimes \mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} \mathbf{v} \quad (4.2)$$

Indeed,  $(1/2) \otimes \mathbf{v} \oplus (1/2) \otimes \mathbf{v} = \mathbf{v}$ , as one can check.

Principles analogous to those of vector space approach to Euclidean geometry are at play in gyrovector space approach to hyperbolic geometry, as shown in [30, Chap. 6]. The first two basic examples are the gyroline and the gyroangle, presented in Sec. 5.

#### 5. MÖBIUS GYROLINE AND GYROANGLE

In full analogy with straight lines in the standard vector space approach to Euclidean geometry, let us consider the *gyroline* equation in the ball  $\mathbb{V}_s$ ,

$$L_{AB} = A \oplus (\ominus A \oplus B) \otimes t \quad (5.1)$$

$A, B \in \mathbb{V}_s$ ,  $t \in \mathbb{R}$ . The point  $A$  of the gyroline  $L_{AB}$  corresponds to  $t = 0$  and, owing to the left cancellation law (2.4), the point  $B$  of the gyroline  $L_{AB}$  corresponds to  $t = 1$ . The gyrosegment  $AB$  that links the points  $A$  and  $B$  is obtained from (5.1) with  $0 \leq t \leq 1$ , as shown in Fig. 1 (left).

The *gyromidpoint*  $M_{AB}$  of the points  $A$  and  $B$  in a Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$  corresponds to the parameter  $t = 1/2$  of the gyroline  $L_{AB}$  [26],

$$M_{AB} = A \oplus (\ominus A \oplus B) \otimes \frac{1}{2} \quad (5.2)$$

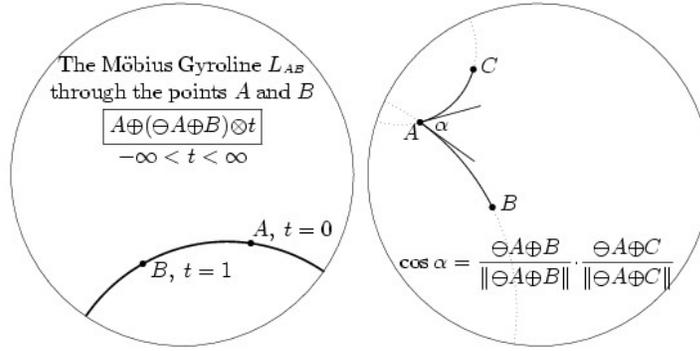


FIGURE 1. Möbius gyroline and gyroangle in a Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$ . The equations of the gyroline  $L_{AB}$  and the gyroangle measure  $\alpha$  in the gyrovector space approach to the Poincaré ball model of hyperbolic geometry are fully analogous to their classical counterparts in the common vector space approach to the standard model of Euclidean geometry. Interestingly, the measure of the gyroangle between two intersecting gyrolines is equal to the measure of the Euclidean angle between corresponding intersecting tangent lines, as shown in the right figure.

As one would expect from a midpoint, it is symmetric,  $M_{AB} = M_{BA}$ , satisfying the gyrodistance equality

$$d(A, M_{AB}) = d(B, M_{AB}) \quad (5.3)$$

where  $d(A, B)$  is the gyrodistance from  $A$  to  $B$  in the Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$ , given by the equation

$$d(A, B) = \|B \ominus A\| \quad (5.4)$$

The gyrodistance of a point  $A$  in the ball from the origin,  $O = \mathbf{0}$ , of the ball coincides with its Euclidean counterpart,

$$d(O, A) = \|A \ominus O\| = \|A\| \quad (5.5)$$

In the special case when  $\mathbb{V}_s = \mathbb{R}_s^2$ , the gyroline  $L_{AB}$ , shown in Fig. 1 (left), turns out to be a circular arc that intersects the boundary of the  $s$ -disc  $\mathbb{R}_s^2$  orthogonally. This gyroline is known in hyperbolic geometry as the unique geodesic that passes through the points  $A$  and  $B$  in the Poincaré disc model of hyperbolic geometry. This and other relationships between gyrovector spaces and various models of hyperbolic geometry are studied in [31] and [30, Ch. 7].

The gyroangle (that is, the hyperbolic angle) included by the gyrosegments  $AB$  and  $AC$  that emanate from the point  $A$ , denoted  $\angle BAC$ , has the measure  $\alpha$  given by the equation [30, 29]

$$\cos \alpha = \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|} \quad (5.6)$$

$A, B, C \in \mathbb{V}_s$ . The point  $A$  is the vertex of the gyroangle  $\angle BAC$ . A gyroangle with vertex at the origin,  $O = \mathbf{0}$ , of the ball coincides with its Euclidean

counterpart,

$$\cos \alpha = \frac{\ominus O \oplus B}{\|\ominus O \oplus B\|} \cdot \frac{\ominus O \oplus C}{\|\ominus O \oplus C\|} = \frac{B}{\|B\|} \cdot \frac{C}{\|C\|} \quad (5.7)$$

The measure of a gyroangle is invariant under the motions of hyperbolic geometry, which are left gyrotranslations and rotations. Interestingly, the measure of the gyroangle between two intersecting gyrolines is equal to the measure of the Euclidean angle between corresponding intersecting tangent lines, shown in Fig. 1 (right).

## 6. MÖBIUS GYROTRIANGLE

A Möbius gyrotriangle along with its standard notation and some basic identities is presented in Fig. 2.

Let  $ABC$  be a gyrotriangle in a Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$  with vertices  $A, B, C \in \mathbb{V}_s$ , sides  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_s$  and side gyrolengths  $a, b, c \in (-s, s)$ ,

$$\begin{aligned} \mathbf{a} &= \ominus B \oplus C, & a &= \|\mathbf{a}\|, & a_s &= \frac{a}{s} \\ \mathbf{b} &= \ominus C \oplus A, & b &= \|\mathbf{b}\|, & b_s &= \frac{b}{s} \\ \mathbf{c} &= \ominus A \oplus B, & c &= \|\mathbf{c}\|, & c_s &= \frac{c}{s} \end{aligned} \quad (6.1)$$

The gyroangle measures  $\alpha, \beta$  and  $\gamma$  of the gyroangles at the vertices  $A, B$  and  $C$  are given by the equations

$$\begin{aligned} \cos \alpha &= \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|} \\ \cos \beta &= \frac{\ominus B \oplus A}{\|\ominus B \oplus A\|} \cdot \frac{\ominus B \oplus C}{\|\ominus B \oplus C\|} \\ \cos \gamma &= \frac{\ominus C \oplus A}{\|\ominus C \oplus A\|} \cdot \frac{\ominus C \oplus B}{\|\ominus C \oplus B\|} \end{aligned} \quad (6.2)$$

In Euclidean geometry the triangle angles do not determine its side lengths. In contrast, in hyperbolic geometry the gyrotriangle gyroangles determine uniquely its side gyrolengths according to the following theorem [30, Theorem 8.48]:

**Theorem 6.1.** *Let  $ABC$  be a gyrotriangle in a Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$  with vertices  $A, B, C$ , corresponding gyroangles  $\alpha, \beta, \gamma$ ,  $0 < \alpha + \beta + \gamma < \pi$ , and side gyrolengths  $a, b, c$ , as shown in Fig. 2. The side gyrolengths of the gyrotriangle  $ABC$  are determined by its gyroangles according to the AAA to SSS conversion equations*

$$\begin{aligned} a_s^2 &= \frac{\cos \alpha + \cos(\beta + \gamma)}{\cos \alpha + \cos(\beta - \gamma)} \\ b_s^2 &= \frac{\cos \beta + \cos(\alpha + \gamma)}{\cos \beta + \cos(\alpha - \gamma)} \\ c_s^2 &= \frac{\cos \gamma + \cos(\alpha + \beta)}{\cos \gamma + \cos(\alpha - \beta)} \end{aligned} \quad (6.3)$$

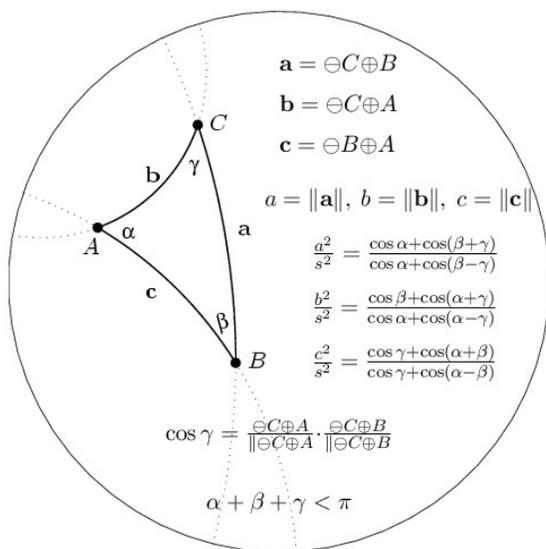


FIGURE 2. Möbius gyrotriangle and its standard notation and basic identities in a Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$ . Along with obvious analogies with Euclidean geometry there is a remarkable disanalogy. The gyrotriangle side gyrolengths  $a, b, c$  are uniquely determined by its gyroangles  $\alpha, \beta, \gamma$ .

Interestingly, in the limit as  $s \rightarrow \infty$  (i) each of the identities in (6.3) reduces to an identity equivalent to the Euclidean identity  $\alpha + \beta + \gamma = \pi$  for the Euclidean triangle angle sum, and (ii) each of the identities in (6.2) reduces to its Euclidean counterpart.

## 7. MÖBIUS GYROSQUARE

The special case when  $\beta = \gamma = \alpha/2$  in (6.3) and in Fig. 2 is of particular interest in our study of the gyrosquare. In this special case, the system of equations (6.3) reduces to the system

$$\begin{aligned} a_s^2 &= \frac{2 \cos \alpha}{1 + \cos \alpha} \\ b_s^2 = c_s^2 &= \frac{\cos \frac{\alpha}{2} + \cos \frac{3\alpha}{2}}{2 \cos \frac{\alpha}{2}} = \cos \alpha \end{aligned} \tag{7.1}$$

expressed in the notation of Fig. 2.

Interestingly, in the limit as  $s \rightarrow \infty$  each of the identities in (7.1) reduces to  $\cos \alpha = 0$ , implying  $\alpha = \pi/2$ , as expected in Euclidean geometry, where  $\alpha + \beta + \gamma = 2\alpha = \pi$ .

The gyrosquare  $ABCD$  in the Möbius gyrovector space  $(\mathbb{V}_s, \oplus, \otimes)$  is a gyroquadrilateral with all four sides of equal gyrolength,  $\|A \ominus B\| = \|B \ominus C\| = \|C \ominus D\| = \|D \ominus A\| =: a$ , and all four gyroangles of equal measure,  $\angle DAB = \angle ABC = \angle BCD = \angle CDA =: \theta > 0$ . The gyrosquare defect is  $\delta := 2\pi - 4\theta$ .

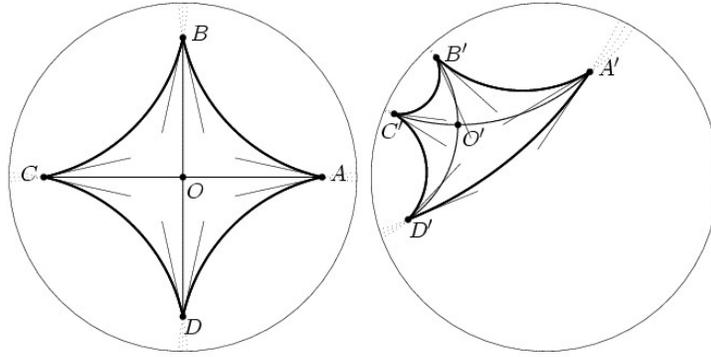


FIGURE 3. The Möbius gyrosquare in a Möbius gyrovector plane  $(\mathbb{R}_s^2, \oplus, \otimes)$ . The Möbius gyrosquare  $ABCD$  in the left figure is situated in a special position in the disc, so that the equality of its side gyrolengths and its gyroangles is visible to the readers, since they are inhabitants of Euclidean geometry. The Möbius gyrosquare  $A'B'C'D'$  in the right figure is obtained from the one in the left by a left gyrotranslation. Hence, for inhabitants of the Poincaré ball model of hyperbolic geometry the two Möbius gyrosquares in the left and right figures are indistinguishable.

The gyrosquare gyrocenter is the point where its gyrodiagonals intersect. A gyrosquare is in a *special position* in the ball if its gyrocenter coincides with the origin,  $O = \mathbf{0}$ , of the ball.

A gyrosquare  $ABCD$  in a special position in the Möbius disc  $(\mathbb{R}_s^2, \oplus, \otimes)$  is shown in Fig. 3 (left). This gyrosquare has been moved into the gyrosquare  $A'B'C'D'$  by a left gyrotranslation by some vector  $\mathbf{v} \in \mathbb{R}_s^2$ , shown in Fig. 3 (right), so that

$$\begin{aligned} A' &= \mathbf{v} \oplus A \\ B' &= \mathbf{v} \oplus B \\ C' &= \mathbf{v} \oplus C \\ D' &= \mathbf{v} \oplus D \end{aligned} \tag{7.2}$$

Since the measure of a Möbius gyroangle between two intersecting gyrolines equals the measure of the Euclidean angle between corresponding intersecting tangent line, it is easy to see graphically in Fig. 3 that the gyroangles of a gyrosquare are invariant under left gyrotranslations of the gyrosquare, as expected from the study of the gyroangle in [30, Chap. 8]. In the language of differential geometry [31] we say that the Möbius gyroangle is *conformal* to the Euclidean angle.

Let the gyrolength of each side of the Möbius gyrosquare in Fig. 3 be  $a$ , and let the measure of each of its gyroangles be  $\theta$ . Then, the Möbius gyrosquare  $ABCD$  contains each of the four gyrotriangles  $ABC$ ,  $BCD$ ,  $CDA$  and  $DAB$ . Each of these gyrotriangles is isosceles, having the gyroangles  $\theta/2$ ,  $\theta/2$  and  $\theta$ . Hence, by applying (7.1) and using the notation of Fig. 3, the side gyrolength  $a$  and the

gyroangle  $\theta$  of the Möbius gyrosquare  $ABCD$  in Fig. 3 are related by the equation

$$a_s = \sqrt{\cos \theta} \quad (7.3)$$

Here  $a_s = a/s$ , and  $a = \|A\ominus B\| = \|B\ominus C\| = \|C\ominus D\| = \|D\ominus A\|$ , and  $\theta = \angle DAB = \angle ABC = \angle BCD = \angle CDA$ , shown in Fig. 3 (left). Similarly,  $a = \|A'\ominus B'\| = \|B'\ominus C'\| = \|C'\ominus D'\| = \|D'\ominus A'\|$ , and  $\theta = \angle D'A'B' = \angle A'B'C' = \angle B'C'D' = \angle C'D'A'$ , shown in Fig. 3 (right).

The gyrolength  $d$  of each of the two gyrodiagonals  $AC$  and  $BD$  of the gyrosquare  $ABCD$  in Fig. 3 is

$$d_s = \sqrt{\frac{2 \cos \theta}{1 + \cos \theta}} \quad (7.4)$$

where  $d_s = d/s$  and  $d = \|A\ominus C\| = \|B\ominus D\|$ , shown in Fig. 3 (left). Similarly,  $d = \|A'\ominus C'\| = \|B'\ominus D'\|$ , shown in Fig. 3 (right).

The gyrodiagonal “half gyrolength”  $(1/2)\otimes d$  of a gyrosquare  $ABCD$  is important since it gives the gyrodistance of each vertex  $A, B, C, D$  from the gyrosquare gyrocenter. Following “Möbius half” (4.2), with  $\gamma_d = (1 - d_s^2)^{-1/2}$ , we thus have from (7.4),

$$\frac{1}{2}\otimes d = \frac{\gamma_d}{1 + \gamma_d} d = s \frac{\sqrt{2 \cos \theta}}{\sqrt{1 + \cos \theta} + \sqrt{1 - \cos \theta}} \quad (7.5)$$

To insure simultaneously the reality of the gyrolengths  $a$  in (7.3) and  $d$  in (7.4), the non-negative gyrosquare gyroangle  $\theta$  must obey the condition  $\cos \theta \geq 0$ . For nonnegative  $\theta < \pi$  this condition is equivalent to the condition

$$0 \leq \theta \leq \frac{\pi}{2} \quad (7.6)$$

The two extreme values of  $\theta$  in (7.6) are excluded. The lower extreme value  $\theta = 0$  is excluded since it corresponds to the limiting case when the gyrosquare vertices  $A, B, C, D$  lie on the boundary of the disc, and the upper extreme value  $\theta = \pi/2$  is excluded since it corresponds to the limiting case when the gyrosquare side gyrolength vanishes.

We thus see that the gyrosquare gyroangle  $\theta$  in a Möbius grovector space  $(\mathbb{V}_s, \oplus, \otimes)$  can have any value in the range

$$0 < \theta < \frac{\pi}{2} \quad (7.7)$$

and its side gyrolength  $a$  is uniquely determined by  $\theta$  according to (7.3),

$$a = s\sqrt{\cos \theta} \quad (7.8)$$

In contrast, the square angle  $\theta$  in Euclidean geometry has the unique value  $\theta = \pi/2$ , but the square side length  $a$  can assume any positive value. Like the square diagonals, the gyrosquare gyrodiagonals intersect each other orthogonally at their gyromidpoints.

Finally, extending the gyrosquare definition to the three-dimensional gyrocube is straightforward. The study of internal gyrolengths and gyroangles of the gyrocube is more complicated than that of the gyrosquare, but can be done in a similar way.

Instructively, we present two illustrative examples of the search for gyrosquares with a given gyroangle.

**Example 1:** Let us calculate the vertices  $A, B, C, D$  of a gyrosquare  $ABCD$  with a gyroangle

$$\theta = \frac{\pi}{4} \quad (7.9)$$

situated in a special position of the open unit disc.

It follows from (7.9) and (7.3)–(7.4) with  $s = 1$  that the side gyrolength  $a$  and the gyrodiagonal gyrolength  $d$  of the gyrosquare  $ABCD$  are given by the equations

$$\begin{aligned} a &= \sqrt{\cos \frac{\pi}{4}} = \sqrt{\frac{1}{\sqrt{2}}} \\ d &= \sqrt{\frac{2}{1 + \sqrt{2}}} \end{aligned} \quad (7.10)$$

so that

$$\gamma_d = 1 + \sqrt{2} \quad (7.11)$$

where  $\gamma_d = (1 - d^2)^{-1/2}$ . Hence, by “Möbius half” in (4.2),

$$\frac{1}{2} \otimes d = \frac{\gamma_d}{1 + \gamma_d} d = \frac{1}{\sqrt{1 + \sqrt{2}}} \quad (7.12)$$

where  $(1/2) \otimes d$  gives the gyrodistance of each vertex of the gyrosquare  $ABCD$  from its gyrocenter.

Hence, the vertices of a gyrosquare  $ABCD$  situated in a special position in the disc, with a gyrosquare gyroangle  $\pi/4$ , are

$$\begin{aligned} A &= \left( \frac{1}{\sqrt{1 + \sqrt{2}}}, 0 \right) \\ B &= \left( 0, \frac{1}{\sqrt{1 + \sqrt{2}}} \right) \\ C &= \left( -\frac{1}{\sqrt{1 + \sqrt{2}}}, 0 \right) \\ D &= \left( 0, -\frac{1}{\sqrt{1 + \sqrt{2}}} \right) \end{aligned} \quad (7.13)$$

shown in Fig. 4 (left).

One can now check that each gyroangle,  $\theta$ , of the gyrosquare  $ABCD$  in Fig. 4 (left) is  $\pi/4$ . Indeed, for instance,

$$\cos \theta = \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus D}{\|\ominus A \oplus D\|} = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4} \quad (7.14)$$

for  $A, B, D$  of (7.13).

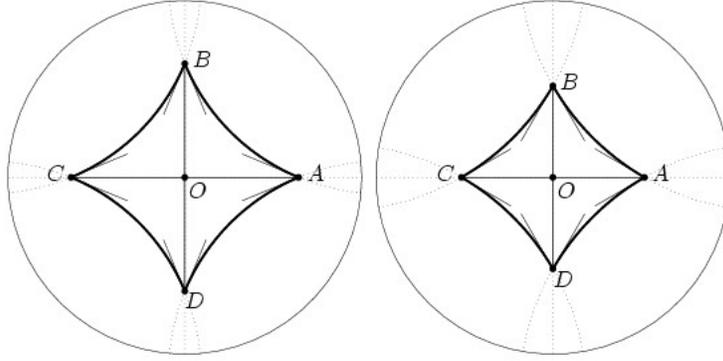


FIGURE 4. Illustrating Examples 1 and 2: The Möbius gyrovector plane  $(\mathbb{R}_s^2, \oplus, \otimes)$  (i) with a gyrosquare gyroangle  $\theta = \pi/4$  and defect  $\delta = \pi$  (left), and (ii) with a gyrosquare gyroangle  $\theta = \pi/3$  and defect  $\delta = 2\pi/3$  (right).

Similarly, one can check that each side gyrolength,  $a$ , of the gyrosquare  $ABCD$  in Fig. 4 (left) is  $a = \sqrt{\cos(\pi/4)} = \sqrt{1/\sqrt{2}}$ . Indeed, for instance,

$$a^2 = \|\ominus A \oplus B\|^2 = \frac{1}{\sqrt{2}} = \cos \frac{\pi}{4} \quad (7.15)$$

for  $A, B$  of (7.13).

**Example 2:** Let us calculate the vertices  $A, B, C, D$  of a gyrosquare  $ABCD$  with a gyroangle

$$\theta = \frac{\pi}{3} \quad (7.16)$$

situated in a special position of the open unit disc.

It follows from (7.16) and (7.3)–(7.4) with  $s = 1$  that the side gyrolength  $a$  and the gyrodiagonal gyrolength  $d$  of the gyrosquare  $ABCD$  are given by the equations

$$\begin{aligned} a &= \sqrt{\cos \frac{\pi}{3}} = \sqrt{\frac{1}{2}} \\ d &= \sqrt{\frac{2}{3}} \end{aligned} \quad (7.17)$$

so that

$$\gamma_d = \sqrt{3} \quad (7.18)$$

and hence, by “Möbius half” in (4.2),

$$\frac{1}{2} \otimes d = \frac{\gamma_d}{1 + \gamma_d} d = \frac{\sqrt{2}}{1 + \sqrt{3}} \quad (7.19)$$

where  $(1/2) \otimes d$  gives the gyrodistance of each vertex of the gyrosquare  $ABCD$  from its gyrocenter.

Hence, the vertices of a gyrosquare  $ABCD$  situated in a special position in the disc, with a gyrosquare gyroangle  $\pi/3$ , are

$$\begin{aligned} A &= \left( \frac{\sqrt{2}}{1 + \sqrt{3}}, 0 \right) \\ B &= \left( 0, \frac{\sqrt{2}}{1 + \sqrt{3}} \right) \\ C &= \left( -\frac{\sqrt{2}}{1 + \sqrt{3}}, 0 \right) \\ D &= \left( 0, -\frac{\sqrt{2}}{1 + \sqrt{3}} \right) \end{aligned} \tag{7.20}$$

shown in Fig. 4 (right).

One can now check that each gyroangle,  $\theta$ , of the gyrosquare  $ABCD$  in Fig. 4 (right) is  $\pi/3$ . Indeed, for instance,

$$\cos \theta = \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus D}{\|\ominus A \oplus D\|} = \frac{1}{2} = \cos \frac{\pi}{3} \tag{7.21}$$

for  $A, B, D$  of (7.20).

Similarly, one can check that each side gyrolength,  $a$ , of the gyrosquare  $ABCD$  in Fig. 4 (right) is  $a = \sqrt{\cos(\pi/3)} = \sqrt{1/2}$ . Indeed, for instance,

$$a^2 = \|\ominus A \oplus B\|^2 = \frac{1}{2} = \cos \frac{\pi}{3} \tag{7.22}$$

for  $A, B$  of (7.20).

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<sup>1</sup> DEPARTMENT OF MATHEMATICS, NORTH DAKOTA STATE UNIVERSITY, FARGO, ND 58105, USA.

*E-mail address:* [abraham.ungar@ndsu.edu](mailto:abraham.ungar@ndsu.edu)