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ON THE CAUCHY–SCHWARZ INEQUALITY AND ITS REVERSE IN SEMI-INNER PRODUCT C^*-MODULES

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This paper is dedicated to Professor Themistocles M. Rassias.

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ABSTRACT. There are many known Cauchy–Schwarz-type inequalities which are valid in different frameworks. In this paper we consider the A-valued Cauchy–Schwarz inequality and its reverse in a semi-inner product A-module over the C^*-algebra A. Some remarks on the A-valued Cauchy–Schwarz inequality in a semi-inner product A-module over the H^*-algebra A are also given.

1. INTRODUCTION AND PRELIMINARIES

The Cauchy–Schwarz inequality plays an important role in the theory of inner product spaces. It is one of the classical inequalities (see [5, 6]). It is well known that in a semi-inner product space $(\mathcal{H}, \langle\, .\, , .\, \rangle)$ the Cauchy–Schwarz inequality has the form

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \qquad (x, y \in \mathcal{H}).$$

Furthermore, the following additive reverse of the Cauchy–Schwarz inequality holds ([1, Theorem 1]).

Theorem 1.1. *Let $(\mathcal{H}, \langle\, .\, , .\, \rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and let $\|\, .\, \|$ be the norm in $\mathcal{H}$ induced by $\langle\, .\, , .\, \rangle$. If $\alpha, \beta \in \mathbb{K}$ and $x, y \in \mathcal{H}$ are such*

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that

$$Re\langle\, \alpha y - x\,, x - \beta y\,\rangle \geq 0,$$

or, equivalently,

$$\left\| x - \frac{\alpha+\beta}{2} y \right\| \leq \frac{1}{2}|\alpha - \beta|\|y\|$$

holds, then we have the inequality

$$0 \leq \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4}|\alpha - \beta|^2\|y\|^4.$$

The constant $\frac{1}{4}$ *is the best possible.*

A natural way to generalize these inequalities is to consider them in more abstract structures. In this paper, the framework for our considerations is a semi-inner product C^*-module.

In the rest of this section we define and give some properties of a semi-inner product C^*-module.

A C^*-algebra is a complex Banach $*$-algebra $(A, \|\,.\,\|)$ such that $\|a^*a\| = \|a\|^2$ for every $a \in A$. A C^*-algebra A can be embedded into a unital C^*-algebra $\tilde{A}$ containing A as an ideal. The elements of $\tilde{A}$ are pairs (a, λ) with $a \in A$ and $\lambda \in \mathbb{C}$. For every C^*-algebra $(A, \|\,.\,\|)$ there is a generalized sequence $\{e_\alpha\}$ in A such that

$$\lim_{\alpha} \|{e_\alpha}^* a e_\alpha - a\| = 0 \qquad (a \in A) \tag{1.1}$$

(see [2, Remark 2.4] and examples afterwards). The centre of A is denoted by $Z(A)$.

An element $a \in A$ is called positive (we write $a \geq 0$) if $a = b^*b$ for some $b \in A$. The relation '$\leq$' on A is given by $a \leq b$ if and only if $b - a \geq 0$. If $a \in A$ is positive, then there exists a unique positive $b \in A$ such that $a = b^2$; such an element b is called the positive square root of a. For every $a \in A$, $|a|$ denotes the positive square root of a^*a.

A (right) semi-inner product C^*-module (a semi-inner product A-module over the C^*-algebra A) is an algebraic right A-module H which is a complex linear space such that $(\lambda f)a = f(\lambda a) = \lambda(fa)$ for all $f \in H$, $a \in A$, $\lambda \in \mathbb{C}$, together with a generalized semi-inner product, that is with an A-valued mapping $[\,.\,,\,.\,]$ on $H \times H$, having the following properties:

(i) $[f, \lambda g + \mu h] = \lambda[f, g] + \mu[f, h]$,
(ii) $[f, ga] = [f, g]a$,
(iii) $[f, g]^* = [g, f]$,
(iv) $[f, f] \geq 0$

for all $f, g, h \in H$, $a \in A$, $\lambda, \mu \in \mathbb{C}$. Since $[f, f]$ is a positive element in A, there is the positive square root of $[f, f]$ and it is denoted by $|f|$.

Any semi-inner product A-module may be regarded as a semi-inner product $\tilde{A}$-module if we put $f(a, \lambda) = fa + \lambda f$ for all $f \in H$, $a \in A$, $\lambda \in \mathbb{C}$.

Every C^*-algebra A can be understood as a semi-inner product A-module via $[a, b] = a^*b$ for all $a, b \in A$.

If H is a semi-inner product A-module, then it is well-known (e.g. [3, Proposition 1.1]) that the following A-valued Cauchy–Schwarz inequality holds:

$$|[f,g]|^2 \leq \| \, |f| \, \|^2 \cdot |g|^2 \qquad (f,g \in H). \tag{1.2}$$

In Section 2 we sharpen this inequality in the case when $|f| \in Z(A)$, and in Section 3 we give an additive reverse of the obtained inequality. Some remarks on the Cauchy–Schwarz inequality in semi-inner product H^*-modules are given in Section 4.

2. The Cauchy–Schwarz inequality

Theorem 2.1. *Let A be a C^*-algebra. Let $(H, [\,.\,,\,.\,])$ be a semi-inner product A-module. If $f, g \in H$ and $|f| \in Z(A)$, then*

$$|[f,g]|^2 \leq |f|^2 |g|^2. \tag{2.1}$$

Proof. Without loss of generality we may assume that A is unital with identity e. For all $f, g \in H$ and $a \in A$ we have

$$0 \leq [fa-g, fa-g] = a^*|f|^2 a - [g,f]a - a^*[f,g] + |g|^2$$

which implies

$$a^*[f,g] + [g,f]a \leq a^*|f|^2 a + |g|^2. \tag{2.2}$$

Let $\varepsilon > 0$ be an arbitrary real number. Let $x \in A$ be such that $\varepsilon e \leq x$. Since $x^* = x$, use of spectral theory yields $x \leq \|x\| e$ (see e.g. [4, Theorem 2.2.5]), which implies $\frac{x}{\|x\|} \leq e$. Thus we have

$$0 \leq e - \frac{x}{\|x\|} \leq e - \frac{\varepsilon}{\|x\|} e,$$

which, according to [4, Theorem 2.2.5.(3)], implies

$$\left\| e - \frac{x}{\|x\|} \right\| \leq 1 - \frac{\varepsilon}{\|x\|} < 1.$$

Hence, $\frac{x}{\|x\|}$ is invertible, thus x is also invertible. In particular, $|f| + \varepsilon e$ is invertible and it commutes with every element in A. Thus its inverse also commutes with every element in A.

If we put

$$a = [f,g](|f| + \varepsilon e)^{-2}$$

in (2.2), we get

$$2(|f| + \varepsilon e)^{-2} |[f,g]|^2 \leq (|f| + \varepsilon e)^{-4} |f|^2 |[f,g]|^2 + |g|^2.$$

Multiplying this by $(|f| + \varepsilon e)^2$, using the fact that $|f| + \varepsilon e \in Z(A)$ and applying [4, Theorem 2.2.5.(2)], we get

$$2|[f,g]|^2 \leq (|f| + \varepsilon e)^{-2} |f|^2 |[f,g]|^2 + (|f| + \varepsilon e)^2 |g|^2,$$

that is

$$|[f,g]|^2 - (|f| + \varepsilon e)^2 |g|^2 \leq (|f| + \varepsilon e)^{-2} |f|^2 |[f,g]|^2 - |[f,g]|^2. \tag{2.3}$$

Multiplying the inequality $|f|^2 \le (|f| + \varepsilon e)^2$ by $(|f| + \varepsilon e)^{-1}[g, f]$ from the left, and by $(|f| + \varepsilon e)^{-1}[f, g]$ from the right, and applying [4, Theorem 2.2.5.(2)], we get

$$(|f| + \varepsilon e)^{-2}|f|^2|[f, g]|^2 \le |[f, g]|^2.$$

Taking this into account in (2.3), we conclude

$$|[f, g]|^2 - |f|^2|g|^2 \le \varepsilon(2|f| + \varepsilon e)|g|^2.$$

Letting $\varepsilon \to 0$, we finally get $|[f, g]|^2 \le |f|^2|g|^2$. □

Remark 2.2. It is known that $a^*b^*ba \le \|b\|^2 a^*a$ for all $a, b \in A$ (see [4, Theorem 2.2.5]). In particular, if $|f| \in Z(A)$, then $|f|^2|g|^2 = |g||f|^2|g| \le \|\,|f|\,\|^2|g|^2$, so (2.1) sharpens (1.2). If A is unital with identity e and $Z(A) = \mathbb{C}e$, then the inequality (2.1) coincides with the Cauchy–Schwarz inequality (1.2) for all $f, g \in H$ such that $|f| \in Z(A)$.

The following corollary is of independent interest.

Corollary 2.3. *Let A be a C^*-algebra. If $a \in A$ is such that $|a| \in Z(A)$, then $|a^*| \le |a|$. In particular, if both $|a|$ and $|a^*|$ are in $Z(A)$, then $|a| = |a^*|$.*

Proof. Applying Theorem 2.1 for $H = A$, we get $|a^*b|^2 \le |a|^2|b|^2$, that is $b^*(|a|^2 - |a^*|^2)b \ge 0$, for all $b \in A$. If b runs through the generalized sequence $\{e_\alpha\}$ such that (1.1) holds, we conclude $|a^*|^2 \le |a|^2$. This yields $|a^*| \le |a|$ (e.g. [4, Theorem 2.2.6]). □

Remark 2.4. The involution $*$ is said to be positive when $a^*a = 0$ if and only if $a = 0$. The involution of a C^*-algebra is an example of a positive involution. In any $*$-ring A with positive involution, if both a^*a and aa^* are in $Z(A)$, then $a^*a = aa^*$. Namely,

$$\begin{aligned} & (a^*a - aa^*)(a^*a - aa^*) \\ = \; & a^*(aa^*)a - (aa^*)(a^*a) - (a^*a)(aa^*) + a(a^*a)a^* \\ = \; & (aa^*)a^*a - (aa^*)(a^*a) - (a^*a)(aa^*) + (a^*a)aa^* = 0, \end{aligned}$$

so $a^*a - aa^* = 0$.

3. A reverse of the Cauchy–Schwarz inequality

Let us prove an additive reverse of the Cauchy–Schwarz inequality (2.1).

Theorem 3.1. *Let A be a C^*-algebra. Let $(H, [\,.\,,\,.\,])$ be a semi-inner product A-module. If $a, b \in A$ and $f, g \in H$ are such that $|f| \in Z(A)$ and that the assumption*

$$\left| f\left(\frac{a+b}{2}\right) - g \right|^2 \le \frac{1}{4}|f|^2|a - b|^2 \tag{3.1}$$

holds, then we have the inequality

$$\begin{aligned} 0 \le |f|^2|g|^2 - |[f, g]|^2 &\le \frac{1}{4}|f|^4|a - b|^2 - \left| |f|^2\left(\frac{a+b}{2}\right) - [f, g] \right|^2 \\ &\le \frac{1}{4}|f|^4|a - b|^2. \end{aligned} \tag{3.2}$$

Proof. First note that (3.1) is equivalent to

$$|g|^2 \leq \frac{1}{4}|f|^2|a-b|^2 + \frac{1}{2}[g,f](a+b) + \frac{1}{2}(a^*+b^*)[f,g] - \frac{1}{4}|f|^2|a+b|^2.$$

This implies, using $|f| \in Z(A)$ and [4, Theorem 2.2.5.(2)],

$$\begin{aligned}
& |f|^2|g|^2 - |[f,g]|^2 \\
\leq\ & \frac{1}{4}|f|^4|a-b|^2 + \frac{1}{2}|f|^2[g,f](a+b) + \frac{1}{2}(a^*+b^*)|f|^2[f,g] \\
& -\frac{1}{4}|f|^4|a+b|^2 - |[f,g]|^2 \\
=\ & \frac{1}{4}|f|^4|a-b|^2 - \Big(\frac{1}{4}|f|^4|a+b|^2 - \frac{1}{2}|f|^2[g,f](a+b) \\
& -\frac{1}{2}(a^*+b^*)|f|^2[f,g] + |[f,g]|^2\Big) \\
=\ & \frac{1}{4}|f|^4|a-b|^2 - \Big(\frac{1}{2}|f|^2(a+b) - [f,g]\Big)^* \Big(\frac{1}{2}|f|^2(a+b) - [f,g]\Big) \\
=\ & \frac{1}{4}|f|^4|a-b|^2 - \Big||f|^2\Big(\frac{a+b}{2}\Big) - [f,g]\Big|^2 \\
\leq\ & \frac{1}{4}|f|^4|a-b|^2.
\end{aligned}$$

□

Remark 3.2. Let $f \in H$ be such that $|f| \in Z(A)$ and let $a, b \in A$ be such that $a \neq b$. Assume that there exists $h \in H$ satisfying $|h| = |f|$ and $[f,h] = 0$. Let

$$g = f\Big(\frac{a+b}{2}\Big) + h\Big(\frac{a-b}{2}\Big).$$

Then

$$\begin{aligned}
|g|^2 &= \Big(\frac{a+b}{2}\Big)^*|f|^2\Big(\frac{a+b}{2}\Big) + \Big(\frac{a-b}{2}\Big)^*|h|^2\Big(\frac{a-b}{2}\Big) \\
&= \frac{1}{4}|f|^2|a+b|^2 + \frac{1}{4}|f|^2|a-b|^2
\end{aligned}$$

and

$$|[f,g]|^2 = \frac{1}{4}|f|^4|a+b|^2.$$

Thus

$$|f|^2|g|^2 - |[f,g]|^2 = \frac{1}{4}|f|^4|a-b|^2.$$

Hence, the constant $\frac{1}{4}$ in (3.2) is the best possible.

Remark 3.3. A complex semi-inner product space is a semi-inner product C^*-module over the C^*-algebra of complex numbers. Thus, if $(\mathcal{H}, \langle\,.\,,\,.\,\rangle)$ is a complex semi-inner product space, then the absolute value in $\mathcal{H}$ coincides with the norm in $\mathcal{H}$, i.e. $|x| = \|x\|$ for all $x \in \mathcal{H}$, and if $\dim \mathcal{H} > 1$ then the assumption of Remark 3.2 holds. In this particular case, Theorem 3.1 yields Theorem 1.1. However, as we can see, the proof of Theorem 3.1 is simpler than the proof of Theorem 1.1 given in [1].

4. Remarks on semi-inner product H^*-modules

Another generalization of a complex semi-inner product space is obtained when the complex field is replaced not by a C^*-algebra, but by a proper H^*-algebra.

A proper H^*-algebra is a complex Banach $*$-algebra $(A, \|\,.\,\|)$ whose underlying Banach space is a Hilbert space with respect to the inner product $\langle\,.\,,\,.\,\rangle$ satisfying $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ba, c\rangle = \langle b, ca^*\rangle$ for all $a, b, c \in A$. It represents an abstract characterization of Hilbert-Schmidt operators.

An element $a \in A$ is called positive (we write $a \geq 0$) if $\langle ax, x\rangle \geq 0$ for all $x \in A$. If $a \geq 0$, then $a^* = a$. Analogously as in C^*-algebras, the relation '$\leq$' on A is given by $a \leq b$ if and only if $b - a \geq 0$. For every $a \in A$ there is a unique positive $b \in A$ such that $a^*a = b^2$; such b is denoted by $|a|$.

There is a norm $\|\,.\,\|_\infty$ on a proper H^*-algebra A (but not complete) such that $\|ax\| \leq \|a\|_\infty \|x\|$ for all $a, x \in A$ ([7, Lemma 3.10]).

The concept of a semi-inner product H^*-module can be defined analogously as the concept of a semi-inner product C^*-module. These structures have many common properties.

The proof of [3, Proposition 1.1] can be adjusted to the case when A is a proper H^*-algebra, noting that $a^*b^*ba \leq \|b\|_\infty^2\, a^*a$ for all $a, b \in A$. Namely, we have

$$\begin{aligned} &\langle (a^*b^*ba)x, x\rangle = \|b(ax)\|^2 \leq \|b\|_\infty^2 \|ax\|^2 \\ = \;& \|b\|_\infty^2 \langle a^*ax, x\rangle = \langle (\|b\|_\infty^2 a^*a)x, x\rangle \end{aligned}$$

for all $x \in A$. Hence, if H is a semi-inner product A-module over a proper H^*-algebra A, then as in the proof of [3, Proposition 1.1] we arrive at

$$|[f, g]|^2 \leq \|\,|f|\,\|_\infty^2 \cdot |g|^2 \qquad (f, g \in H). \tag{4.1}$$

In the proof of Theorem 2.1, we used the fact that any C^*-algebra can be embedded into a unital C^*-algebra. In distinction from C^*-algebras, proper H^*-algebras are nonunital in general, and the identity cannot be added to them within their category.

The technique of the proof of Theorem 3.1 is applicable to semi-inner product H^*-modules as well.

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