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KOLMOGOROV TYPE INEQUALITIES FOR HYPERSINGULAR INTEGRALS WITH HOMOGENEOUS CHARACTERISTIC

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by G. V. Milovanović

ABSTRACT. New sharp Kolmogorov type inequalities for hypersingular integrals with homogeneous characteristic of multivariate functions from Hölder spaces are obtained. Proved inequalities are used to solve the Stechkin's problem on the best approximation of unbounded hypersingular integral operator by bounded ones on functional classes which are defined by a majorant of the modulus of continuity.

1. INTRODUCTION AND PRELIMINARIES

Kolmogorov type inequalities for univariate and multivariate functions, especially with sharp constants, that estimate the norm of intermediate derivatives in terms of the norms of the function itself and its derivative of the higher order, are of great importance for many branches of mathematics. After the inequality of A.N. Kolmogorov [15] there were obtained a lot of results in this direction for univariate functions; the surveys of known sharp inequalities in the case of derivatives of integer order and further references can be found, for example, in [2], [3], [4], [9], [10], [11]. The case of the fractional order derivatives has been studied much less (for known results see [14], [1], [13], [18], [6], [7]). In addition observe that in the case of functions of two and more variables very few sharp

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Kolmogorov type inequalities are known (see the papers [16], [13], [24], [8], [5], [12]).

In this paper we obtain sharp Kolmogorov type inequalities for certain natural generalizations of the fractional differentiation for the case of multivariate functions and give applications of obtained results in approximation theory.

We shall describe in this section univariate results which we shall generalize in this paper. One of the natural and useful definitions of the fractional derivative for univariate functions is the following definition of Marchaud fractional derivative [19] (see, also, [21, p. 95–97]):

$$(\mathbb{D}^\alpha_\pm f)(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^\infty \frac{f(x) - f(x \mp t)}{t^{1+\alpha}} dt.$$

Let $C(\mathbb{R})$ be the space of all bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$ endowed with the norm

$$\|f\|_C := \sup\{|f(x)| \ : \ x \in \mathbb{R}\}.$$

Let $\omega(t)$ be a certain modulus of continuity, that is a continuous nondecreasing semi-additive function defined on the real half-line and such that $\omega(0) = 0$. We shall consider the space $H^\omega(\mathbb{R})$ of functions $f \in C(\mathbb{R})$ for which the quantity

$$\|f\|_{H^\omega} := \sup_{\substack{x,y\in\mathbb{R} \\ x\neq y}} \frac{|f(x) - f(y)|}{\omega(|x-y|)}$$

is finite. If $\omega(t) = t^\beta$, $\beta \in (0, 1]$, then instead of H^ω we write H^β.

If $\omega(t)$ is a modulus of continuity such that

$$\int\limits_0^1 \frac{\omega(t)}{t^{\alpha+1}} dt < \infty, \tag{1.1}$$

then for any $h > 0$ the quantity

$$I_{\omega,\alpha}(h) := \int\limits_0^h \frac{\omega(t)}{t^{\alpha+1}} dt$$

is defined.

In [6] the following additive Kolmogorov type inequality for derivatives in Marchaud sense of order $\alpha \in (0, 1)$ for univariate functions was proved:

$$\left\|\mathbb{D}^\alpha_\pm f\right\|_C \le \frac{1}{\Gamma(1-\alpha)} \left(\alpha \|f\|_{H^\omega} I_{\omega,\alpha}(h) + \frac{2\|f\|_C}{h^\alpha}\right) \tag{1.2}$$

with any $h > 0$ and $\omega(t)$ such that (1.1) holds. Moreover, this inequality becomes an equality for the function $f_h(x)$ that is defined as:

$$f_h(x) = \begin{cases} \omega(|x|) - \frac{\omega(h)}{2}, & |x| \le h; \\ \frac{\omega(h)}{2}, & |x| \ge h. \end{cases}$$

If $\omega(t) = t^\beta$ and $\alpha < \beta \le 1$, then inequality (1.2) can be represented in the following multiplicative form:

$$\|\mathbb{D}^\alpha_\pm f\|_C \le \frac{1}{\Gamma(1-\alpha)} \frac{2^{1-\frac{\alpha}{\beta}}}{1-\frac{\alpha}{\beta}} \|f\|_C^{1-\frac{\alpha}{\beta}} \|f\|_{H^\beta}^{\frac{\alpha}{\beta}}.$$

2. Main results

Among the best known generalizations of fractional derivatives for multivariate functions are fractional derivative with respect to a direction [21, p. 348] and the Riesz derivative [21, p. 367–370].

Let $C(\mathbb{R}^n)$ be the space of all bounded continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ endowed with the norm

$$\|f\|_C := \sup\{|f(x)| \,:\, x \in \mathbb{R}^n\}.$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ $|x| := \sqrt{\sum\limits_{i=1}^n x_i^2}$ denotes the usual Euclidean norm in $\mathbb{R}^n$. For a given modulus of continuity $\omega(t)$ we shall consider the space $H^\omega(\mathbb{R}^n)$ of functions $f \in C(\mathbb{R}^n)$ for which the quantity

$$\|f\|_{H^\omega} := \sup_{\substack{x,y\in\mathbb{R}^n \\ x\ne y}} \frac{|f(x) - f(y)|}{\omega(|x-y|)}$$

is finite. If $\omega(t) = t^\beta$, $\beta \in (0,1]$, then instead of H^ω we write H^β.

Fractional derivative of order $\alpha \in (0,1)$ with respect to the direction $\theta \in \mathbb{R}^n$, $|\theta| = 1$, of the function $f : \mathbb{R}^n \to \mathbb{R}$ in Marchaud sense is defined by

$$(D^\alpha_\theta f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^\infty \frac{f(x) - f(x-\xi\theta)}{\xi^{1+\alpha}} d\xi.$$

The Riesz fractional derivative of order $\alpha \in (0,1)$ of the function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(D^\alpha f)(x) = \frac{1}{d_{n,1}(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(x) - f(x-\xi)}{|\xi|^{n+\alpha}} d\xi,$$

where

$$d_{n,1}(\alpha) = \frac{\pi^{1+n/2}}{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{n+\alpha}{2}\right) \sin\frac{\alpha\pi}{2}}.$$

Note that the Riesz derivative represents a fractional power of the Laplas operator:

$$D^\alpha f = (-\Delta)^{\alpha/2} f,$$

where

$$\Delta f := \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}.$$

We shall consider still more general notion that is the notion of hypersingular integral with homogeneous characteristic (for the references see [21, p. 425]).

For $\alpha \in (0,1)$ the hypersingular integral is defined in the following way (for more details see, for example, [21, p. 381]):

$$\left(D_\Omega^\alpha f\right)(x)=\frac{1}{d_{n,1}\left(\alpha\right)}\int\limits_{\mathbb{R}^n}\frac{f(x)-f(x-\xi)}{|\xi|^{n+\alpha}}\Omega\left(\frac{\xi}{|\xi|}\right)d\xi,$$

where the function Ω (which is called characteristic) is homogeneous of degree 0 with respect to ξ.

The notion of such hypersingular integrals allows in particular to consider problems connected with fractional derivative with respect to a direction and the Riesz derivative from the general point of view. Let $S^{n-1}\subset\mathbb{R}^n$ be the unit sphere with induced Lebesgue measure. Let also $B_{S^{n-1}}(\theta,r)$ be the ball of radius r (in the sense of spherical distance, for more details see, for example, [20]) on the sphere S^{n-1}, centered at the point of intersection of the direction θ and the sphere S^{n-1}. Denote by $|B_{S^{n-1}}(\theta,r)|$ the measure of this ball on the sphere S^{n-1}, and by

$$\chi_{B_{S^{n-1}}(\theta,r)}(x)=\begin{cases}1, & x\in B_{S^{n-1}}(\theta,r);\\ 0, & x\in S^{n-1}\setminus B_{S^{n-1}}(\theta,r)\end{cases}$$

denote its indicator function. Then choosing the function

$$\Omega(x)=\frac{\chi_{B_{S^{n-1}}(\theta,r)}(x)}{|B_{S^{n-1}}(\theta,r)|},$$

as a characteristic and letting $r\to 0$ we obtain the fractional derivative with respect to the direction θ; if $r\to\pi$ we obtain the Riesz derivative.

Further results are formulated for the case of hypersingular integrals with homogeneous characteristic.

Let $B_h^n\subset\mathbb{R}^n$ be the ball of radius h centered at the origin. We introduce the operator $D_{\Omega,h}^\alpha : C\left(\mathbb{R}^n\right)\to C\left(\mathbb{R}^n\right)$ defined as

$$\left(D_{\Omega,h}^\alpha f\right)(x)=\frac{1}{d_{n,1}\left(\alpha\right)}\int\limits_{\mathbb{R}^n\setminus B_h^n}\frac{f(x)-f(x-\xi)}{|\xi|^{n+\alpha}}\Omega\left(\frac{\xi}{|\xi|}\right)d\xi.$$

Theorem 2.1. *Let $\Omega(x)$ be a non-negative homogeneous of degree 0 with respect to x function, integrable on the unit sphere $S^{n-1}\subset\mathbb{R}^n$. Let also $\alpha\in(0;1)$ and $\omega(t)$ be such a modulus of continuity that (1.1) holds. Then for any function $f\in H^\omega\left(\mathbb{R}^n\right)$ and any $h>0$ the sharp inequality*

$$\|D_\Omega^\alpha f\|_C\le\frac{1}{d_{n,1}(\alpha)}\int\limits_{S^{n-1}}\Omega\left(\xi'\right)d\xi'\left(\|f\|_{H^\omega}I_{\omega,\alpha}(h)+\frac{2\|f\|_C}{\alpha h^\alpha}\right)\tag{2.1}$$

holds with the extremal function

$$f_h(x)=\begin{cases}\omega\left(|x|\right)-\frac{\omega(h)}{2}, & x\in B_h^n;\\ \frac{\omega(h)}{2}, & x\in\mathbb{R}^n\setminus B_h^n.\end{cases}\tag{2.2}$$

Proof. For any $f\in H^\omega\left(\mathbb{R}^n\right)$ and any $h>0$, we obviously have

$$\|D_\Omega^\alpha f\|_C\le\left\|D_\Omega^\alpha f-D_{\Omega,h}^\alpha f\right\|_C+\left\|D_{\Omega,h}^\alpha f\right\|_C.$$

To estimate the first term we use the fact that the function f belongs to the space $H^{\omega}(\mathbb{R}^n)$ and switch to the polar coordinates in the integral:

$$\begin{aligned}
\left\|D^{\alpha}_{\Omega}f - D^{\alpha}_{\Omega,h}f\right\|_C &= \frac{1}{d_{n,1}(\alpha)}\left\|\int\limits_{B^n_h}\frac{f(\cdot)-f(\cdot+\xi)}{|\xi|^{n+\alpha}}\Omega\left(\frac{\xi}{|\xi|}\right)d\xi\right\|_C \\
&\le \frac{1}{d_{n,1}(\alpha)}\|f\|_{H^{\omega}}\int\limits_{B^n_h}\frac{\omega(|\xi|)}{|\xi|^{n+\alpha}}\Omega\left(\frac{\xi}{|\xi|}\right)d\xi \qquad (2.3)\\
&= \frac{1}{d_{n,1}(\alpha)}\int\limits_{S^{n-1}}\Omega\left(\xi'\right)d\xi'\|f\|_{H^{\omega}}I_{\omega,\alpha}(h).
\end{aligned}$$

To estimate the second term we use the fact that the function f is bounded and again switch to the polar coordinates in the integral:

$$\begin{aligned}
\left\|D^{\alpha}_{\Omega,h}f\right\|_C &= \frac{1}{d_{n,1}(\alpha)}\left\|\int\limits_{\mathbb{R}^n\setminus B^n_h}\frac{f(\cdot)-f(\cdot+\xi)}{|\xi|^{n+\alpha}}\Omega\left(\frac{\xi}{|\xi|}\right)d\xi\right\|_C \\
&\le \frac{2\|f\|_C}{d_{n,1}(\alpha)}\int\limits_{\mathbb{R}^n\setminus B^n_h}\frac{1}{|\xi|^{n+\alpha}}\Omega\left(\frac{\xi}{|\xi|}\right)d\xi \qquad (2.4)\\
&= \frac{1}{d_{n,1}(\alpha)}\int\limits_{S^{n-1}}\Omega\left(\xi'\right)d\xi'\frac{2\|f\|_C}{\alpha h^{\alpha}}.
\end{aligned}$$

It follows from the obtained estimates that inequality (2.1) holds true.

Let us show that this inequality is sharp. It is easily seen that for the function (2.2) we have

$$\|f_h\|_C = \frac{\omega(h)}{2}.$$

Let $x', x'' \in \mathbb{R}^n$. There are three possible cases:

1) $x',\ x'' \in B^n_h$;
2) $x' \in B^n_h,\ x'' \notin B^n_h$;
3) $x',\ x'' \notin B^n_h$.

In the case 1) we have using the properties of modulus of continuity

$$\begin{aligned}
|f_h(x') - f_h(x'')| &= |\omega(|x'|) - \omega(|x''|)| \\
&\le \omega(||x'| - |x''||) \\
&\le \omega(|x' - x''|).
\end{aligned}$$

In the case 2) there exists a point $y'' \in B^n_h$ such that

$$|x' - y''| \le |x' - x''|$$

and

$$|f_h(x') - f_h(x'')| = |\omega(|x'|) - \omega(|y''|)|.$$

Using the previous estimation we obtain

$$\begin{aligned} |f_h(x') - f_h(x'')| &= |\omega(|x'|) - \omega(|y''|)| \\ &\leq \omega(|x' - y''|) \\ &\leq \omega(|x' - x''|). \end{aligned}$$

Finally, in the case 3) we have

$$|f_h(x') - f_h(x'')| = 0 \leq \omega(|x' - x''|).$$

Therefore the function f_h belongs to the space $H^\omega(\mathbb{R}^n)$ and

$$\|f_h\|_{H^\omega} \leq 1.$$

We obtain from inequality (2.1) that

$$\|D^\alpha_\Omega f_h\|_C \leq \frac{1}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi' \left(I_{\omega,\alpha}(h) + \frac{\omega(h)}{\alpha h^\alpha} \right).$$

On the other hand

$$\begin{aligned} \|D^\alpha_\Omega f_h\|_C &\geq |(D^\alpha_\Omega f_h)(0)| \\ &= \frac{1}{d_{n,1}(\alpha)} \left| \int\limits_{\mathbb{R}^n} \frac{f_h(0) - f_h(\xi)}{|\xi|^{n+\alpha}} \Omega\left(\frac{\xi}{|\xi|}\right) d\xi \right| \\ &= \frac{1}{d_{n,1}(\alpha)} \left| \int\limits_{B^n_h} \frac{f_h(0) - f_h(\xi)}{|\xi|^{n+\alpha}} \Omega\left(\frac{\xi}{|\xi|}\right) d\xi \right. \\ &\quad \left. + \int\limits_{\mathbb{R}^n \setminus B^n_h} \frac{f_h(0) - f_h(\xi)}{|\xi|^{n+\alpha}} \Omega\left(\frac{\xi}{|\xi|}\right) d\xi \right| \\ &= \frac{1}{d_{n,1}(\alpha)} \left(\int\limits_{B^n_h} \frac{\omega(|\xi|)}{|\xi|^{n+\alpha}} \Omega\left(\frac{\xi}{|\xi|}\right) d\xi + \int\limits_{\mathbb{R}^n \setminus B^n_h} \frac{\omega(h)}{|\xi|^{n+\alpha}} \Omega\left(\frac{\xi}{|\xi|}\right) d\xi \right) \\ &= \frac{1}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi' \left(I_{\omega,\alpha}(h) + \frac{\omega(h)}{\alpha h^\alpha} \right). \end{aligned}$$

Comparing the obtained estimates for the norm $\|D^\alpha_\Omega f_h\|_C$ we arrive to the identity which proves the sharpness of inequality (2.1). The proof is complete. □

If $\omega(t) = t^\beta$, $0 < \beta \leq 1$, then as a consequence from Theorem 1 we obtain

Theorem 2.2. *Let $0 < \alpha < \beta \leq 1$, $\Omega(x)$ be such as in Theorem 1. For any function $f \in H^\beta(\mathbb{R}^n)$ sharp inequalities hold:*

$$\|D^\alpha_\Omega f\|_C \leq \frac{h^{-\alpha}}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi' \left(\|f\|_{H^\beta} \frac{h^\beta}{\beta - \alpha} + \frac{2\|f\|_C}{\alpha} \right) \qquad (2.5)$$

for any $h > 0$, and

$$\|D_{\Omega}^{\alpha} f\|_C \le \frac{2^{1-\frac{\alpha}{\beta}}}{\alpha d_{n,1}(\alpha)(1-\frac{\alpha}{\beta})} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi' \|f\|_C^{1-\frac{\alpha}{\beta}} \|f\|_{H^\beta}^{\frac{\alpha}{\beta}}. \tag{2.6}$$

The extremal function for both these inequalities is

$$f_h(x) = \begin{cases} |x|^\beta - \frac{h^\beta}{2}, & x \in B_h^n; \\ \frac{h^\beta}{2}, & x \in \mathbb{R}^n \setminus B_h^n, \end{cases} \qquad h > 0. \tag{2.7}$$

Proof. Inequality (2.5) and its sharpness for any $h > 0$ follow immediately from Theorem 1, since in this case

$$I_{\omega,\alpha}(h) = \frac{h^{\beta-\alpha}}{\beta-\alpha}.$$

Substituting the quantity

$$h_* = 2^{\frac{1}{\beta}} \|f\|_C^{\frac{1}{\beta}} \|f\|_{H^\beta}^{-\frac{1}{\beta}}$$

to the right hand side of additive inequality we obtain

$$\begin{aligned} \|D_{\Omega}^{\alpha} f\|_C &\le \frac{1}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi' \left(\frac{\|f\|_{H^\beta}}{\beta-\alpha} \left(h_*\right)^{\beta-\alpha} + \frac{2\|f\|_C}{\alpha} \left(h_*\right)^{-\alpha} \right) \\ &= \frac{2^{1-\frac{\alpha}{\beta}}}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi' \|f\|_C^{1-\frac{\alpha}{\beta}} \|f\|_{H^\beta}^{\frac{\alpha}{\beta}} \left[\frac{1}{\beta-\alpha} + \frac{1}{\alpha} \right] \\ &= \frac{2^{1-\frac{\alpha}{\beta}} \cdot \beta}{\alpha(\beta-\alpha) d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi' \|f\|_C^{1-\frac{\alpha}{\beta}} \|f\|_{H^\beta}^{\frac{\alpha}{\beta}}. \end{aligned}$$

Thus, we obtain multiplicative inequality (2.6):

$$\|D_{\Omega}^{\alpha} f\|_C \le \frac{2^{1-\frac{\alpha}{\beta}}}{\alpha d_{n,1}(\alpha)(1-\frac{\alpha}{\beta})} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi' \|f\|_C^{1-\frac{\alpha}{\beta}} \|f\|_{H^\beta}^{\frac{\alpha}{\beta}}.$$

Now we shall prove the sharpness of this inequality. For the function of the form (2.7) with any $h > 0$ we have

$$\|f_h\|_C = \frac{h^\beta}{2}, \quad \|f_h\|_{H^\beta} \le 1,$$

i.e. $f_h \in H^\beta\left(\mathbb{R}^n\right)$, therefore inequality (2.6) holds for this function:

$$\|D_{\Omega}^{\alpha} f_h\|_C \le \frac{2^{1-\frac{\alpha}{\beta}}}{\alpha d_{n,1}(\alpha)(1-\frac{\alpha}{\beta})} \left(\frac{h^\beta}{2}\right)^{1-\frac{\alpha}{\beta}} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi'.$$

On the other hand

$$\begin{aligned} \|D_{\Omega}^{\alpha} f_h\|_C &\ge \left|\left(D_{\Omega}^{\alpha} f_h\right)(0)\right| \\ &= \frac{h^{\beta-\alpha}}{\alpha d_{n,1}(\alpha)(1-\frac{\alpha}{\beta})} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi'. \end{aligned}$$

Comparing the obtained estimates for the norm $\|D_\Omega^\alpha f_h\|_C$, we arrive to the identity, which proves the sharpness of multiplicative inequality (2.6). The proof is complete. □

Using Stein's method [23] (see, also, [17, §1.8], [11, p. 84]) we obtain analogue of inequality (2.6) in the space $L_p(\mathbb{R}^n)$, $1 \le p < \infty$, of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|_p := \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

For $\beta \in (0,1]$ denote by $H_p^\beta(\mathbb{R}^n)$ the set of functions $f \in L_p(\mathbb{R}^n)$ such that

$$\|f\|_{H_p^\beta} := \sup_{\substack{t \in \mathbb{R} \\ t \neq 0}} \frac{\|f(\cdot) - f(\cdot + t)\|_p}{|t|^\beta} < \infty.$$

Corollary 2.3. *Let $0 < \alpha < \beta \le 1$, $1 \le p < \infty$, $\Omega(x)$ be such as in Theorem 1. For any function $f \in H_p^\beta(\mathbb{R}^n)$ inequality*

$$\|D_\Omega^\alpha f\|_p \le \frac{2^{1-\frac{\alpha}{\beta}}}{\alpha d_{n,1}(\alpha)(1-\frac{\alpha}{\beta})} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi' \|f\|_p^{1-\frac{\alpha}{\beta}} \|f\|_{H_p^\beta}^{\frac{\alpha}{\beta}}$$

holds.

3. Some applications

The general statement of the Stechkin's problem on approximation an unbounded operator by bounded ones is as follows [22] (see, also, [11, p. 391]).

Let X, Y be Banach spaces and let $A : X \to Y$ be a certain operator (not necessarily linear) with the domain $D(A) \subset X$. Let $\mathcal{L}(N) = \mathcal{L}(N; X, Y)$ be a set of linear bounded operators $T : X \to Y$ such that the norms $\|T\| = \|T\|_{X \to Y}$ do not exceed number $N > 0$. Let $Q \subset D(A)$ be a certain class of elements. The quantity

$$U(T) = \sup\{\|Ax - Tx\|_Y : \ x \in Q\}$$

is called a deviation of operator $T \in \mathcal{L}(N)$ from operator A on the class Q, and the quantity

$$\begin{aligned} E(N) &= E(N; A, Q) \\ &:= \inf\{U(T) : \ T \in \mathcal{L}(N)\} \end{aligned} \tag{3.1}$$

is called the best approximation of operator A by the set of bounded operators $\mathcal{L}(N)$ on the class Q.

The problem is to compute (analyze) the quantity $E(N)$ and to find (study questions of existence, uniqueness, characterization) the extremal operator, i.e. the operator which achieves the lower bound in the right part of (3.1).

Let $0 < \alpha < 1$, $\Omega(x)$ be a non-negative homogeneous of degree 0 with respect to x function which is integrable on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and let $\omega(t)$ be a certain modulus of continuity. We consider the problem of best approximation of the operator $D_\Omega^\alpha : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ by a set of linear bounded operators

$T : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ for which $\|T\| \le N$, $N > 0$, on the class $WH^\omega(\mathbb{R}^n) = \{f \in H^\omega(\mathbb{R}^n) \, : \, \|f\|_{H^\omega} \le 1\}$.

Theorem 3.1. *Let $\Omega(x)$ be such as in Theorem 1. Let also $\alpha \in (0;1)$ and $\omega(t)$ be such a modulus of continuity that (1.1) holds. Then for the best approximation $E(N)$ of operator D^α_Ω on the class $WH^\omega(\mathbb{R}^n)$ equality*

$$\begin{aligned} E(N) &= U\left(D^\alpha_{\Omega,h_N}\right) \\ &= \frac{1}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi' \cdot I_{\omega,\alpha}(h_N), \end{aligned}$$

holds, where

$$h_N = \left(\frac{2}{\alpha N d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi' \right)^{\frac{1}{\alpha}}.$$

Proof. It follows from (2.4) that

$$\begin{aligned} \left\| D^\alpha_{\Omega,h} \right\| &= \sup_{\|f\|_C \le 1} \left\| D^\alpha_{\Omega,h} f \right\|_C \\ &\le \frac{2}{\alpha d_{n,1}(\alpha) h^\alpha} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi'. \end{aligned}$$

On the other hand, for the function f_h defined in Theorem 1 we have

$$\begin{aligned} \left\| D^\alpha_{\Omega,h} \right\| &\ge \left\| D^\alpha_{\Omega,h} \frac{f_h}{\|f_h\|_C} \right\|_C \\ &\ge \frac{2}{\omega(h)} \left| \left(D^\alpha_{\Omega,h} f_h \right)(0) \right| \\ &= \frac{2}{\omega(h) d_{n,1}(\alpha)} \left| \int\limits_{\mathbb{R}^n \setminus B^n_h} \frac{-\omega(h)}{|\xi|^{n+\alpha}} \Omega\left(\frac{\xi}{|\xi|} \right) d\xi \right| \\ &= \frac{2}{\alpha d_{n,1}(\alpha) h^\alpha} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi'. \end{aligned}$$

Therefore,

$$\left\| D^\alpha_{\Omega,h} \right\| = \frac{2}{\alpha d_{n,1}(\alpha) h^\alpha} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi'.$$

Moreover, using (2.3) we obtain the upper bound for the deviation of the operator $D^\alpha_{\Omega,h}$ from the operator D^α_Ω on the class $WH^\omega(\mathbb{R}^n)$:

$$\begin{aligned} U\left(D^\alpha_{\Omega,h}\right) &= \sup_{f \in WH^\omega(\mathbb{R}^n)} \left\| D^\alpha_\Omega f - D^\alpha_{\Omega,h} f \right\|_C \\ &\le \frac{1}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega(\xi')\, d\xi' I_{\omega,\alpha}(h). \end{aligned}$$

On the other hand, for the function f_h defined in Theorem 1 we have (since $\|f_h\|_{H^\omega} \leq 1$):

$$\begin{aligned} U\left(D^\alpha_{\Omega,h}\right) &\geq \left\|D^\alpha_\Omega f_h - D^\alpha_{\Omega,h} f_h\right\|_C \\ &\geq \left|\left(D^\alpha_\Omega f_h\right)(0) - \left(D^\alpha_{\Omega,h} f_h\right)(0)\right| \\ &= \frac{1}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi' \cdot I_{\omega,\alpha}(h). \end{aligned}$$

Therefore,

$$U\left(D^\alpha_{\Omega,h}\right) = \frac{1}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi' I_{\omega,\alpha}(h).$$

For given $N > 0$ we find the corresponding h_N under the condition

$$\left\|D^\alpha_{\Omega,h_N}\right\| = N,$$

that is equivalent to

$$h_N = \left(\frac{2}{\alpha N d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi'\right)^{\frac{1}{\alpha}}.$$

Then

$$E\left(N\right) \leq U\left(D^\alpha_{\Omega,h_N}\right).$$

Now we shall show that

$$E\left(N\right) = U\left(D^\alpha_{\Omega,h_N}\right).$$

For this we note that additive inequality (2.1) obtained in Theorem 1 for the functions from the space $H^\omega\left(\mathbb{R}^n\right)$ can be rewritten as

$$\|D^\alpha_\Omega f\|_C \leq U\left(D^\alpha_{\Omega,h}\right)\|f\|_{H^\omega} + \left\|D^\alpha_{\Omega,h}\right\| \cdot \|f\|_C\,,$$

and for an arbitrary $h > 0$ there exist the function $f_h \in WH^\omega\left(\mathbb{R}^n\right)$ (f_h is extremal function in inequality (2.1)) and the linear bounded operator $D^\alpha_{\Omega,h}$ such that

$$\|D^\alpha_\Omega f_h\|_C = U\left(D^\alpha_{\Omega,h}\right) + \left\|D^\alpha_{\Omega,h}\right\| \cdot \|f_h\|_C\,.$$

Then for any operator T with $\|T\| \leq N$ we have

$$\begin{aligned} U\left(T\right) &= \sup_{f \in WH^\omega(\mathbb{R}^n)} \|D^\alpha_\Omega f - Tf\|_C \\ &\geq \sup_{f \in WH^\omega(\mathbb{R}^n)} \left(\|D^\alpha_\Omega f\|_C - N\|f\|_C\right) \\ &\geq \|D^\alpha_\Omega f_{h_N}\|_C - N\|f_{h_N}\|_C = U\left(D^\alpha_{\Omega,h_N}\right), \end{aligned}$$

therefore,

$$E\left(N\right) = U\left(D^\alpha_{\Omega,h_N}\right),$$

or

$$E\left(N\right) = \frac{1}{d_{n,1}(\alpha)} \int\limits_{S^{n-1}} \Omega\left(\xi'\right) d\xi' I_{\omega,\alpha}(h_N),$$

and the operator D^{α}_{Ω,h_N} is the operator of best approximation. The proof is complete. $\square$

In particular, if $\omega(t) = t^{\beta}$, $0 < \beta \le 1$, we obtain the following

Corollary 3.2. *Let $0 < \alpha < \beta \le 1$, $\Omega(x)$ be such as in Theorem 1. Then for the best approximation $E(N)$ of operator D^{α}_{Ω} on the class $WH^{\beta}(\mathbb{R}^n)$ the following equality holds*

$$\begin{aligned} E(N) &= U\left(D^{\alpha}_{\Omega,h_N}\right) \\ &= \left(\frac{2}{\alpha N}\right)^{\frac{\beta}{\alpha}-1} \cdot \frac{1}{\beta-\alpha}\left(\frac{1}{d_{n,1}(\alpha)}\int\limits_{S^{n-1}} \Omega(\xi')\,d\xi'\right)^{\frac{\beta}{\alpha}}. \end{aligned}$$

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