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ON SOME GEOMETRIC PROPERTIES OF GENERALIZED MUSIELAK-ORLICZ SEQUENCE SPACE AND CORRESPONDING OPERATOR IDEALS

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ABSTRACT. Let $\Phi = (\phi_n)$ be a Musielak-Orlicz function, X be a real Banach space and A be any infinite matrix. In this paper, a generalized vector-valued Musielak-Orlicz sequence space $l_{\Phi}^A(X)$ is introduced. It is shown that the space is a complete normed linear space under certain conditions on the matrix A . It is also shown that $l_{\Phi}^A(X)$ is a σ -Dedekind complete whenever X is so. We have discussed some geometric properties, namely, uniformly monotone, uniform Opial property for this space. Using the sequence of s -number (in the sense of Pietsch), the operators of s -type l_{Φ}^A and operator ideals under certain conditions on the matrix A are discussed.

1. INTRODUCTION

The theory of sequence spaces has several important applications in many branches of mathematical analysis. The classical sequence space l_2 is extended to l_p , $1 < p < \infty$ ($p \neq 2$) by Reisz [23], and further its generalizations to Lorentz sequence space $l_{p,q}$, for $0 < p, q < \infty$ due to Hardy and Littlewood [8]. W. Orlicz [18] has generalized the sequence space l_p to Orlicz sequence space l_{ϕ} with the help of Orlicz function ϕ while Woo [26] has generalized the Orlicz sequence space to modular sequence space. In recent years, many mathematicians are interested to study the theory of sequence spaces generated by Cesàro mean, Orlicz function, Musielak-Orlicz function or using the combination of these. The Cesàro-Orlicz

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sequence space is introduced by Lim and Lee [15]. Later on, Cui et al. [3] have studied and discussed its basic topological properties as well as geometric properties. In 1990, Kamińska introduced Orlicz-Lorentz space [11] and after that Foralewski et al. (2008, [4]) have introduced generalized Orlicz-Lorentz sequence space. Srivastava and Ghosh [25] also studied vector valued sequence spaces using Orlicz function.

With the help of s -numbers, many authors such as Pietsch [21], Carl [1] have introduced and studied operator ideal over the sequence spaces. The theory of operator ideal is important in spectral theory, the geometry of Banach spaces, theory of eigenvalue distributions. The s -numbers, in particular, approximation numbers of a bounded linear operator plays an important role in the study of compactness, eigenvalue problem etc. Pietsch [19] studied the class of l_p , $0 < p < \infty$ type operators and $l_{p,q}$, $0 < p, q < \infty$ type operators. In 2012, Gupta and Acharya [6] have introduced the class of l_ϕ type operators. Recently, Gupta and Bhar [7] studied generalized Orlicz-Lorentz sequence space and developed operator ideals with the help of s -number. Maji and Srivastava [17] also introduced and studied the class of s -type $|A, p|$ operators using s -number and $|A, p|$ space.

The natural question is: can we unify the sequence spaces and the various type of operators described in the above paragraph? The present work is an attempt in this direction. We have introduced a vector-valued new sequence space with the help of sequence of Orlicz functions and an infinite matrix. It is shown that sequence spaces such as Musielak-Orlicz, Cesàro-Orlicz, Orlicz-Lorentz. are the particular cases with the suitable choice of an infinite matrix. We have proved that the space is a complete normed linear space under certain conditions on the matrix. We have also discussed some geometric properties, namely, uniformly monotone, uniform Opial property for this space. With the help of s -number, we have also developed operator ideals under certain conditions on the matrix.

2. PRELIMINARIES

Throughout the paper, we consider $(X, \|\cdot\|)$ as a real Banach space and $S(X)$ be the unit sphere in X . Let $l_\infty(X) = \{\bar{x} = (x_n) : x_n \in X, \sup_{n \geq 1} \|x_n\| < \infty\}$. Let w ,

\mathbb{R} , \mathbb{R}^+ and \mathbb{N} stand for the set of all real sequences, the set of real numbers, the set of all non-negative real numbers and the set of all natural numbers respectively. A sequence space $(\lambda, \|\cdot\|_\lambda)$ is called a BK space if it is a Banach space with continuous coordinates $p_n : \lambda \rightarrow \mathbb{R}$, i.e., $p_n(\alpha) = \alpha_n$ for all $\alpha \in \lambda$ and every $n \in \mathbb{N}$, where $\alpha = (\alpha_1, \alpha_2, \dots)$. A BK-space $(\lambda, \|\cdot\|_\lambda)$ is called an AK-space if $\alpha^{[n]} \rightarrow \alpha$, where $\alpha^{[n]} = (\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, \dots)$, the n th section of α . An infinite matrix $A = (a_{nk})$ is called a triangle if $a_{nn} \neq 0$ and $a_{nk} = 0$ for all $k > n$. The following notations are used:

$x|_i$ stand for $(x_1, x_2, \dots, x_i, 0, 0, \dots)$ and $x|_{\mathbb{N}-i}$ stand for $(0, 0, \dots, 0, x_{(i+1)}, x_{(i+2)}, \dots)$, where $x \in w$ and $i \in \mathbb{N}$.

Definition 2.1. [16] *An Orlicz function $\phi : [0, \infty) \rightarrow [0, \infty)$ is a convex, nondecreasing, continuous function on $[0, \infty)$ such that $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ and $\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$.*

Definition 2.2. [16] An Orlicz function ϕ is said to satisfy Δ_2 -condition at zero ($\phi \in \Delta_2(0)$ for short) if there exist $K > 0$ and $t_0 > 0$ such that $\phi(2t) \leq K\phi(t)$ for all $t \in [0, t_0]$.

Definition 2.3. [16] A sequence $\Phi = (\phi_n)$, where each ϕ_n is an Orlicz function, is said to be a Musielak-Orlicz function.

The Musielak-Orlicz sequence space l_Φ (see [13], [16]) generated by $\Phi = (\phi_n)$ is defined as

$$l_\Phi = \left\{ x = (x_n) \in w : \sum_{n=1}^{\infty} \phi_n\left(\frac{|x_n|}{\sigma}\right) < \infty \text{ for some } \sigma > 0 \right\}$$

and convex modular ρ_Φ is defined as $\rho_\Phi(x) = \sum_{n=1}^{\infty} \phi_n(|x_n|)$.

A Musielak-Orlicz function $\Phi = (\phi_n)$ is said to satisfy the condition δ_2 ($\Phi \in \delta_2$ for short) if there exist $K > 0$, $\delta > 0$ and a nonnegative scalar sequence $(c_n) \in l_1$ such that for each $n \in \mathbb{N}$ and all $x \geq 0$,

$$\phi_n(2x) \leq K\phi_n(x) + c_n$$

whenever $\phi_n(x) \leq \delta$.

It is easy to check that $\Phi = (\phi_n) \in \delta_2$ if and only if for all $\beta > 1$ there exist $K > 0$, $\delta > 0$ and a nonnegative scalar sequence $(c_n) \in l_1$ such that for all $n \in \mathbb{N}$ and $x \geq 0$

$$\phi_n(\beta x) \leq K\phi_n(x) + c_n$$

holds if $\phi_n(x) \leq \delta$. For details on Musielak-Orlicz function, one can see [13], [16].

A Musielak-Orlicz function $\Phi = (\phi_n)$ is said to satisfy the condition $(*)$ (see [10]) if for any $\epsilon \in (0, 1)$ there exists $\delta > 0$ such that

$$\phi_n(u) < 1 - \epsilon \text{ implies } \phi_n((1 + \delta)u) \leq 1 \text{ for all } n \in \mathbb{N} \text{ and } u \geq 0. \quad (2.1)$$

Definition 2.4. [12] A real Banach space X , endowed with partial order \leq satisfying the properties $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in X$ and $0 \leq tx$ for $0 \leq x$ and $t \in \mathbb{R}^+$, is said to be a Banach lattice if $\|x\| \leq \|y\|$ whenever $|x| < |y|$ for $x, y \in X$, where $|x| = \sup\{x, -x\}$, the absolute value of x .

Let $X = (X, \leq, \|\cdot\|)$ be a Banach lattice with a lattice norm $\|\cdot\|$. The positive cone X_+ of X is defined as $X_+ = \{x \in X : 0 \leq x\}$. A Banach lattice X is said to be σ -Dedekind complete (σ -DC for short) if any nonnegative order bounded sequence (x_n) in X has supremum in X .

The norm $\|\cdot\|$ in a Banach lattice X is said to be strictly monotone if $x, y \in X_+$ with $y \leq x$ and $y \neq x$, there holds $\|y\| < \|x\|$ and the space X has strictly monotone property. The norm $\|\cdot\|$ is said to be uniformly monotone (UM for short) if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\|x + y\| \geq 1 + \delta(\epsilon),$$

for each $x, y \in X_+$ with $\|x\| = 1$ and $\|y\| \geq \epsilon$. Then the space X has UM property. A Banach lattice X is said to be an AL-space if $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$. An element $x \in X$ is said to be order continuous if for any

sequence (x_n) in X such that $0 \leq x_n \leq |x|$ for all $n \in \mathbb{N}$, $x_1 \geq x_2 \geq \dots$ and $\inf x_n = 0$, there holds $\|x_n\| \rightarrow 0$. The space X is said to be order continuous if every element of X is order continuous. For details on Banach lattice, refer to [9], [12].

A Banach space X is said to have Opial property if for every weakly null sequence (x_n) and for every $x \neq 0$ in X , we have

$$\liminf_{n \rightarrow \infty} \|x_n\| < \liminf_{n \rightarrow \infty} \|x_n + x\|.$$

A Banach space X is said to have the uniform Opial property if for every $\epsilon > 0$ there exists $\mu > 0$ such that

$$1 + \mu \leq \liminf_{n \rightarrow \infty} \|x_n + x\|$$

for any weakly null sequence $(x_n) \in S(X)$ and $x \in X$ with $\|x\| \geq \epsilon$.

Let \mathcal{L} be the class of all bounded linear operators between two arbitrary Banach spaces and $\mathcal{L}(E, F)$ be the space of all bounded linear operators from a Banach space E to a Banach space F . We denote E' as the dual of E and x' as the continuous linear functional on E . Define $x' \otimes y : E \rightarrow F$ by $x' \otimes y(x) = x'(x)y$ for all $x \in E$ and $y \in F$.

Definition 2.5. (*s-numbers of a bounded linear operator*) ([1], [21]) A map $s = (s_n) : \mathcal{L} \rightarrow \mathbb{R}^{\mathbb{N}}$ assigning to every operator $T \in \mathcal{L}$ a non-negative scalar sequence $(s_n(T))$ is called an *s-number sequence* if the following conditions are satisfied:

- (S1): *monotonicity:* $\|T\| = s_1(T) \geq s_2(T) \geq \dots \geq 0$, for $T \in \mathcal{L}(E, F)$
- (S2): *additivity:* $s_{m+n-1}(S + T) \leq s_m(S) + s_n(T)$, for $S, T \in \mathcal{L}(E, F)$, $m, n \in \mathbb{N}$
- (S3): *ideal property:* $s_n(RST) \leq \|R\|s_n(S)\|T\|$, for some $R \in \mathcal{L}(F, F_0)$, $S \in \mathcal{L}(E, F)$ and $T \in \mathcal{L}(E_0, E)$, where E_0, F_0 are arbitrary Banach spaces
- (S4): *rank property:* if $\text{rank}(T) \leq n$ then $s_n(T) = 0$
- (S5): *norming property:* $s_n(I : l_2^n \rightarrow l_2^n) = 1$, where I denotes the identity operator on the n -dimensional Hilbert space l_2^n .

We call $s_n(T)$ the n -th *s-number* of the operator T . In particular the approximation number is an example of an *s-number* and the n -th approximation number, denoted by $a_n(T)$, is defined as

$$a_n(T) = \inf \left\{ \|T - L\| : L \in \mathcal{L}(E, F), \text{rank}(L) < n \right\}.$$

Definition 2.6. ([21], p.90) An *s-number sequence* $s = (s_n)$ is called *injective* if, given any metric injection $J \in \mathcal{L}(F, F_0)$, $s_n(T) = s_n(JT)$ for all $T \in \mathcal{L}(E, F)$.

Definition 2.7. ([21], p.95) An *s-number sequence* $s = (s_n)$ is called *surjective* if, given any metric surjection $Q \in \mathcal{L}(E_0, E)$, $s_n(T) = s_n(TQ)$ for all $T \in \mathcal{L}(E, F)$.

Definition 2.8. (*Operator ideals*) ([20], [22]) A sub collection \mathcal{M} of \mathcal{L} is said to be an *operator ideal* if each component $\mathcal{M}(E, F) = \mathcal{M} \cap \mathcal{L}(E, F)$ satisfies the following conditions:

- (OI1) if $x' \in E'$, $y \in F$ then $x' \otimes y \in \mathcal{M}(E, F)$
(OI2) if $S, T \in \mathcal{M}(E, F)$ then $S + T \in \mathcal{M}(E, F)$
(OI3) if $S \in \mathcal{M}(E, F)$, $T \in \mathcal{L}(E_0, E)$ and $R \in \mathcal{L}(F, F_0)$ then $RST \in \mathcal{M}(E_0, F_0)$.

Definition 2.9. ([20], [22]) A function $\alpha : \mathcal{M} \rightarrow \mathbb{R}^+$ is said to be a quasi-norm on the ideal \mathcal{M} if the following conditions hold:

- (QN1) if $x' \in E'$, $y \in F$ then $\alpha(x' \otimes y) = \|x'\| \|y\|$
(QN2) if $S, T \in \mathcal{M}(E, F)$ then there exists a constant $C \geq 1$ such that $\alpha(S+T) \leq C[\alpha(S) + \alpha(T)]$
(QN3) if $S \in \mathcal{M}(E, F)$, $T \in \mathcal{L}(E_0, E)$ and $R \in \mathcal{L}(F, F_0)$, then $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$.

In particular if $C = 1$ then α becomes a norm on the operator ideal \mathcal{M} .

An ideal \mathcal{M} with a quasi-norm α , denoted by $[\mathcal{M}, \alpha]$ is said to be a quasi-Banach operator ideal if each component $\mathcal{M}(E, F)$ is complete under the quasi-norm α .

3. MAIN RESULTS

Let $A = (a_{nk})$ be an infinite real matrix, $\Phi = (\phi_n)$ be a Musielak-Orlicz function and X be a real Banach space. Now we introduce generalized Musielak-Orlicz sequence space using an infinite matrix $A = (a_{nk})$ as

$$l_{\Phi}^A(X) = \left\{ \bar{x} = (x_k) \in l_{\infty}(X) : \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\sigma} \right) < \infty \text{ for some } \sigma > 0 \right\}$$

and

$$h_{\Phi}^A(X) = \left\{ \bar{x} = (x_k) \in l_{\infty}(X) : \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\sigma} \right) < \infty \text{ for all } \sigma > 0 \right\}.$$

It can be proved that the space $l_{\Phi}^A(X)$ is a linear space and the space $h_{\Phi}^A(X)$ is a linear subspace of $l_{\Phi}^A(X)$. If $X = \mathbb{R}$ then we write l_{Φ}^A and h_{Φ}^A instead of $l_{\Phi}^A(\mathbb{R})$ and $h_{\Phi}^A(\mathbb{R})$ respectively. Let us define a function ϱ_{Φ}^A on $l_{\Phi}^A(X)$ by

$$\varrho_{\Phi}^A(\bar{x}) = \sum_{n=1}^{\infty} \phi_n \left(\sum_{k=1}^{\infty} \|a_{nk}x_k\| \right).$$

Let \mathcal{A} be a class of infinite matrices $A = (a_{nk})$ such that each column of a matrix is non zero, i.e., for each k there exists at least one n_0 such that $a_{n_0k} \neq 0$. If $A = (a_{nk}) \in \mathcal{A}$ then it can be proved that the function ϱ_{Φ}^A is a convex modular on $l_{\Phi}^A(X)$.

Proposition 3.1. *The following conditions are equivalent.*

- (1). $l_{\Phi}^A(X) \neq \{0\}$.
- (2). For some $k \in \mathbb{N}$, $(|a_{nk}|)_{n=1}^{\infty} \in l_{\Phi}$.

Proof. (1) \Rightarrow (2).

Let $\bar{0} \neq \bar{x} \in l_{\Phi}^A(X)$. Then there exists some $l \in \mathbb{N}$ such that $x_l \in X$ and $x_l \neq 0$. Clearly $\bar{y} = (0, 0, \dots, 0, x_l, 0, 0, \dots) \in l_{\Phi}^A(X)$. Thus for some $t > 0$, we have $\varrho_{\Phi}^A(t\bar{y}) < \infty$. Now

$$\varrho_{\Phi}^A(t\bar{y}) = \sum_{n=1}^{\infty} \phi_n \left(t \sum_{k=1}^{\infty} \|a_{nk}y_k\| \right) = \sum_{n=1}^{\infty} \phi_n \left(t|a_{nl}||x_l| \right).$$

Choose $t_0 = t|x_l| > 0$. Then $\varrho_{\Phi}^A(t\bar{y}) = \sum_{n=1}^{\infty} \phi_n \left(t_0|a_{nl}| \right) = \varrho_{\Phi} \left(t_0(|a_{nl}|)_{n=1}^{\infty} \right)$. Hence $(|a_{nl}|)_{n=1}^{\infty} \in l_{\Phi}$.

(2) \Rightarrow (1).

Let for some $k \in \mathbb{N}$, $(|a_{nk}|)_{n=1}^{\infty} \in l_{\Phi}$. Define $\bar{x} = (0, 0, \dots, 0, x_l, 0, 0, \dots)$. Then $\bar{x} \in l_{\infty}(X)$. Now for $k = l$, $t = \|x_l\|$, $\varrho_{\Phi} \left(t(|a_{nl}|)_{n=1}^{\infty} \right) = \sum_{n=1}^{\infty} \phi_n \left(\|x_l\| |a_{nl}| \right) < \infty$.

Again

$$\varrho_{\Phi}^A(1.\bar{x}) = \sum_{n=1}^{\infty} \phi_n \left(\sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) = \sum_{n=1}^{\infty} \phi_n \left(\|x_l\| |a_{nl}| \right) = \varrho_{\Phi} \left(t(|a_{nl}|)_{n=1}^{\infty} \right) < \infty.$$

This implies $\bar{x} \in l_{\Phi}^A(X)$. This completes the proof. \square

Applications:

The linear space $l_{\Phi}^A(X)$ contains many known sequence spaces as particular cases with the suitable choice of the matrix $A = (a_{nk})$. For example:

1. Choose the matrix $A = (a_{nk})$ as an identity matrix and $X = \mathbb{R}$, then $l_{\Phi}^A(X)$ reduces to Musielak-Orlicz sequence space [13]. In addition, if for all $n \in \mathbb{N}$, $\phi_n = \phi$, an Orlicz function then the space gives Orlicz sequence space.
2. Choose the matrix $A = (a_{nk})$ as a Cesàro matrix of order 1 and $X = \mathbb{R}$, then $l_{\Phi}^A(X)$ reduces to Cesàro-Orlicz sequence space [14].
3. Choose the matrix $A = (a_{nk})$ such that $a_{nk} = n^{\frac{1}{p} - \frac{1}{q}}$ for $n = k$ and 0 otherwise, where $p, q > 0$ and suppose the non-increasing rearrangement of $\bar{x} = (x_n) \in l_{\infty}(X)$, denoted by (x_n^*) is given as

$$x_n^* = \inf \{ c \geq 0 : |\{i \in \mathbb{N} : \|x_i\| > c\}| < n \},$$

where the vertical bars indicate number of elements in the enclosed set, then $l_{\Phi}^A(X)$ reduces to Orlicz-Lorentz space [5]. In addition, if $\phi(x) = x^q$, then the space reduces to Lorentz space $l_{p,q}$ [21].

4. Choose $\phi_n = x^p$, $0 < p < \infty$ for all n and $X = \mathbb{R}$, then $l_{\Phi}^A(X)$ reduces to $A - p$ spaces denoted by $|A, p|$ studied by Rhoades [24].

Throughout the paper, we assume that $A = (a_{nk}) \in \mathcal{A}$. Let $\bar{x} \in l_{\Phi}^A(X)$ and define

$$\|\bar{x}\|_{\Phi}^A = \inf \left\{ \sigma > 0 : \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\sigma} \right) \leq 1 \right\}. \quad (3.1)$$

Theorem 3.2. *The space $l_{\Phi}^A(X)$ is a complete normed linear space with the norm $\|\cdot\|_{\Phi}^A$ as defined by (3.1).*

Proof. First we prove that $\|\cdot\|_{\Phi}^A$ is a norm on the space $l_{\Phi}^A(X)$.

Let $\|\bar{x}\|_{\Phi}^A = 0$. Then by definition given by (3.1), for all $\sigma > 0$ there exists $C > 0$ such that for all n

$$\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\sigma} \leq C \quad \Rightarrow \quad |a_{nk}|\|x_k\| \leq C\sigma$$

holds for all n and any arbitrary $\sigma > 0$. Thus $\|x_k\| = 0$, hence $x_k = 0$ for all k . Therefore $\bar{x} = \bar{0}$. Clearly if $\bar{x} = \bar{0}$, then $\|\bar{x}\|_{\Phi}^A = 0$.

Let $\bar{x} = (x_k), \bar{y} = (y_k) \in l_{\Phi}^A(X)$. Let $\epsilon > 0$ be any number. Then there exist $\sigma_1 > 0, \sigma_2 > 0$ such that

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\sigma_1} \right) \leq 1, \sigma_1 \leq \|\bar{x}\|_{\Phi}^A + \frac{\epsilon}{2} \text{ and}$$

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}y_k\|}{\sigma_2} \right) \leq 1, \sigma_2 \leq \|\bar{y}\|_{\Phi}^A + \frac{\epsilon}{2}.$$

Using the convexity property of each ϕ_n , we get

$$\begin{aligned} \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}(x_k + y_k)\|}{\sigma_1 + \sigma_2} \right) &\leq \frac{\sigma_1}{\sigma_1 + \sigma_2} \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\sigma_1} \right) \\ &\quad + \frac{\sigma_2}{\sigma_1 + \sigma_2} \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}y_k\|}{\sigma_2} \right) \\ &\leq 1. \end{aligned}$$

Thus

$$\|\bar{x} + \bar{y}\|_{\Phi}^A \leq \sigma_1 + \sigma_2 \leq \|\bar{x}\|_{\Phi}^A + \|\bar{y}\|_{\Phi}^A + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\|\bar{x} + \bar{y}\|_{\Phi}^A \leq \|\bar{x}\|_{\Phi}^A + \|\bar{y}\|_{\Phi}^A$. It is easy to show that $\|\alpha\bar{x}\|_{\Phi}^A = |\alpha|\|\bar{x}\|_{\Phi}^A$ for any scalar α . Hence $\|\cdot\|_{\Phi}^A$ is a norm on the space $l_{\Phi}^A(X)$.

To show $l_{\Phi}^A(X)$ is complete, let $(\bar{x}^{(m)})$ be a Cauchy sequence in $l_{\Phi}^A(X)$. Then for each $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $\|\bar{x}^{(m)} - \bar{x}^{(l)}\|_{\Phi}^A < \epsilon$ for all $m, l \geq m_0$, i.e.,

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}(x_k^{(m)} - x_k^{(l)})\|}{\epsilon} \right) \leq 1. \quad (3.2)$$

Thus the sequence $\left\{ \frac{\|x_k^{(m)} - x_k^{(l)}\|}{\epsilon} \right\}$ is bounded for all $m, l \geq m_0$. Therefore there exists $C > 0$ such that

$$\frac{\|x_k^{(m)} - x_k^{(l)}\|}{\epsilon} \leq C \text{ for all } m, l \geq m_0.$$

Hence $(x_k^{(m)})$ is a Cauchy sequence in X for each k . Since X is a Banach space, so $(x_k^{(m)})$ is convergent in X . Let $x_k = \lim_{m \rightarrow \infty} x_k^{(m)}$ for each k . Since each ϕ_k is continuous so taking $m \rightarrow \infty$, we get from (3.2),

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}(x_k - x_k^{(l)})\|}{\epsilon} \right) \leq 1$$

for all $l \geq m_0$. Thus $\|\bar{x} - \bar{x}^{(l)}\|_{\Phi}^A < \epsilon$ for all $l \geq m_0$. Thus $(\bar{x}^{(l)})$ converges to \bar{x} in $l_{\Phi}^A(X)$. Since $\bar{x} - \bar{x}^{(m_0)} \in l_{\Phi}^A(X)$, so $\bar{x} = \bar{x}^{(m_0)} + (\bar{x} - \bar{x}^{(m_0)}) \in l_{\Phi}^A(X)$. This completes the proof. \square

Theorem 3.3. *The space $h_{\Phi}^A(X)$ is an AK-BK space.*

Proof. Clearly $h_{\Phi}^A(X)$ is a linear subspace of $l_{\Phi}^A(X)$. To prove $h_{\Phi}^A(X)$ is a BK space, it is enough to prove that the space $h_{\Phi}^A(X)$ is a closed subspace of $l_{\Phi}^A(X)$. Let \bar{x} belongs to closure of $h_{\Phi}^A(X)$ in the norm topology of $l_{\Phi}^A(X)$. Then there exists a sequence $(\bar{y}^{(m)})$ in $h_{\Phi}^A(X)$, where $\bar{y}^{(m)} = (y_k^{(m)})_{k \geq 1}$ such that $\|\bar{y}^{(m)} - \bar{x}\|_{\Phi}^A \rightarrow 0$ as $m \rightarrow \infty$. So for any $\epsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}(x_k - y_k^{(m)})\|}{\epsilon} \right) \leq 1 \text{ for all } m \geq m_0.$$

Now for any $\epsilon > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{2\epsilon} \right) &\leq \sum_{n=1}^{\infty} \phi_n \left(\frac{1}{2} \frac{\sum_{k=1}^{\infty} \|a_{nk}(x_k - y_k^{(m_0)})\|}{\epsilon} + \frac{1}{2} \frac{\sum_{k=1}^{\infty} \|a_{nk}y_k^{(m_0)}\|}{\epsilon} \right) \\ &\leq \frac{1}{2} \left[\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}(x_k - y_k^{(m_0)})\|}{\epsilon} \right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}y_k^{(m_0)}\|}{\epsilon} \right) \right] < \infty. \end{aligned}$$

Thus $\bar{x} \in h_{\Phi}^A(X)$.

To show $h_{\Phi}^A(X)$ is an AK-space, let $\bar{x} \in h_{\Phi}^A(X)$. Now for each $\epsilon, 0 < \epsilon < 1$ there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\epsilon} \right) \leq 1$. Now for $m \geq n_0$,

$$\begin{aligned} \|\bar{x} - \bar{x}^{[m]}\|_{\Phi}^A &= \inf \left\{ \sigma > 0 : \sum_{n=m+1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\sigma} \right) \leq 1 \right\} \\ &\leq \inf \left\{ \sigma > 0 : \sum_{n=m}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} \|a_{nk}x_k\|}{\sigma} \right) \leq 1 \right\} \leq \epsilon. \end{aligned}$$

This shows that $x^{[m]} \rightarrow x$ as $m \rightarrow \infty$ under the norm $\|\cdot\|_{\Phi}^A$. Hence $h_{\Phi}^A(X)$ is an AK space. This completes the proof. \square

Theorem 3.4. *If a Musielak-Orlicz function $\Phi = (\phi_n) \in \delta_2$, then $l_{\Phi}^A(X) = h_{\Phi}^A(X)$.*

Proof. It suffices to show that $l_{\Phi}^A(X) \subseteq h_{\Phi}^A(X)$. Let $\bar{x} \in l_{\Phi}^A(X)$. Then for some $t > 0$, we have

$$\varrho_{\Phi}^A(t\bar{x}) = \sum_{n=1}^{\infty} \phi_n \left(t \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) < \infty.$$

Therefore for $\delta_0 > 0$ there exists $n_0 \in \mathbb{N}$ such that $\phi_n \left(t \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) < \delta_0$ for all $n \geq n_0$. We have to show that for any $r > 0$, $\varrho_{\Phi}^A(r\bar{x}) < \infty$. If r be any number such that $r \leq t$, then by the non-decreasing property of ϕ_n for all n , we have $\varrho_{\Phi}^A(r\bar{x}) \leq \varrho_{\Phi}^A(t\bar{x}) < \infty$.

If $r > t$, choose $\beta > 1$ such that $r < \beta t$. Thus

$$\phi_n \left(r \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) \leq \phi_n \left(\beta t \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right).$$

Since $\Phi = (\phi_n) \in \delta_2$, so there exist $K > 0$, $\delta_0 > 0$ and a non-negative sequence $(c_n) \in l_1$ such that for $n \geq n_0$

$$\phi_n \left(\beta t \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) \leq K \phi_n \left(t \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) + c_n,$$

whenever $\phi_n \left(t \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) < \delta_0$. Therefore

$$\sum_{n=n_0}^{\infty} \phi_n \left(\beta t \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) \leq K \sum_{n=n_0}^{\infty} \phi_n \left(t \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) + \sum_{n=n_0}^{\infty} c_n.$$

Since $\varrho_{\Phi}^A(t\bar{x}) < \infty$, so we have $\sum_{n=n_0}^{\infty} \phi_n \left(r \sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) < \infty$. Thus for any $r > 0$, we have $\varrho_{\Phi}^A(r\bar{x}) < \infty$. Hence $\bar{x} \in h_{\Phi}^A(X)$. This completes the proof. \square

Example 3.5. Let $\phi_n(u) = u^{p_n} \left(\log(1+u) + 1 \right)$ for $u \geq 0$, $1 \leq p_n < \infty$ with $\sup_{n \geq 1} p_n < \infty$. Then $\Phi = (\phi_n)$ is a Musielak-Orlicz function and satisfies δ_2 condition. Hence $l_{\Phi}^A(X) = h_{\Phi}^A(X)$. In addition if we take A as an identity matrix then $l_{\Phi}(X) \subseteq l(\{p_n\}, X)$, where $l(\{p_n\}, X)$ denotes vector-valued Nakano sequence space.

Now we state some known lemmas which will be needed in the sequel. Proofs of the corresponding lemmas run parallel lines as in the references (see [2], [3], [10]).

Lemma 3.6. Let $\bar{x} \in h_{\Phi}^A(X)$ be an arbitrary element. Then $\|\bar{x}\|_{\Phi}^A = 1$ if and only if $\varrho_{\Phi}^A(\bar{x}) = 1$.

Lemma 3.7. Let $\Phi \in \delta_2$. Then for any $\bar{x} \in l_{\Phi}^A(X)$,
 $\|\bar{x}\|_{\Phi}^A = 1$ if and only if $\varrho_{\Phi}^A(\bar{x}) = 1$.

Lemma 3.8. Let $\Phi \in \delta_2$. Then for any sequence $(\bar{x}^{(l)})$ in $l_{\Phi}^A(X)$, $\|\bar{x}^{(l)}\|_{\Phi}^A \rightarrow 0$ if and only if $\varrho_{\Phi}^A(\bar{x}^{(l)}) \rightarrow 0$.

Lemma 3.9. Let $\Phi \in \delta_2$ and satisfies the condition (*) given by (2.1). Then for any $\bar{x} \in l_{\Phi}^A(X)$ and every $\epsilon \in (0, 1)$ there exists $\delta(\epsilon) \in (0, 1)$ such that $\varrho_{\Phi}^A(\bar{x}) \leq 1 - \epsilon$ implies $\|\bar{x}\|_{\Phi}^A \leq 1 - \delta$.

Lemma 3.10. Let $(X, \|\cdot\|)$ be a normed space. If $f : X \rightarrow \mathbb{R}$ is a convex function in the set $K = \{x \in X : \|x\| \leq 1\}$ and $|f(x)| \leq M$ for all $x \in K$ and some $M > 0$ then f is almost uniformly continuous in K i.e., for all $d \in (0, 1)$ and $\epsilon > 0$ there exists $\delta > 0$ such that $\|y\| \leq d$ and $\|x - y\| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in K$.

Lemma 3.11. Let $\Phi \in \delta_2$ and satisfies the condition (*) given by (2.1). Then for each $d \in (0, 1)$ and $\epsilon > 0$ there exists $\delta = \delta(d, \epsilon) > 0$ such that $\varrho_{\Phi}^A(\bar{x}) \leq d$, $\varrho_{\Phi}^A(\bar{y}) \leq \delta$ imply

$$|\varrho_{\Phi}^A(\bar{x} + \bar{y}) - \varrho_{\Phi}^A(\bar{x})| < \epsilon \text{ for any } \bar{x}, \bar{y} \in l_{\Phi}^A(X). \quad (3.3)$$

Proof. Let $\epsilon > 0$ be any number. Assume that for each $d \in (0, 1)$ and $\epsilon > 0$ there exists $\delta = \delta(d, \epsilon) > 0$ such that $\varrho_{\Phi}^A(\bar{x}) \leq d$ and $\varrho_{\Phi}^A(\bar{y}) \leq \delta$ for $\bar{x}, \bar{y} \in l_{\Phi}^A(X)$. Since $\Phi \in \delta_2$ and satisfies condition (*), so by Lemma 3.9, there exists $d_1 \in (0, 1)$ such that $\|\bar{x}\|_{\Phi}^A \leq d_1$. Also by using Lemma 3.8, for $\delta > 0$ we have $\delta_1 > 0$ such that $\|\bar{y}\|_{\Phi}^A \leq \delta_1$ whenever $\varrho_{\Phi}^A(\bar{y}) \leq \delta$ and $\bar{y} \in l_{\Phi}^A(X)$. Since ϱ_{Φ}^A is a convex and bounded on $l_{\Phi}^A(X)$, by Lemma 3.10, we have $|\varrho_{\Phi}^A(\bar{x} + \bar{y}) - \varrho_{\Phi}^A(\bar{x})| < \epsilon$. \square

Lemma 3.12. Let $\Phi \in \delta_2$. Then for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\varrho_{\Phi}^A(\bar{x}) > \delta$ whenever $\|\bar{x}\|_{\Phi}^A \geq \epsilon$.

Proof. The proof follows from Lemma 3.8. \square

Lemma 3.13. *Let $\Phi \in \delta_2$ and satisfies the condition (*). Then for any $\bar{x} \in l_{\Phi}^A(X)$ and any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\varrho_{\Phi}^A(\bar{x}) \geq 1 + \epsilon$ implies $\|\bar{x}\|_{\Phi}^A \geq 1 + \delta$.*

Now we will prove the main result in this section.

Theorem 3.14. *Let $A = (a_{nk})$ be a triangle matrix. If a Musielak-Orlicz function $\Phi = (\phi_n) \in \delta_2$ and satisfies condition (*) given by (2.1), then $l_{\Phi}^A(X)$ has the uniform Opial property.*

Proof. Let $S(l_{\Phi}^A(X))$ be the unit sphere of $l_{\Phi}^A(X)$ and $(\bar{x}^{(l)}) \subset S(l_{\Phi}^A(X))$ be any weakly null sequence. We will show that for any $\epsilon > 0$ there exists $\mu > 0$ such that

$$\liminf_{l \rightarrow \infty} \|\bar{x}^{(l)} + \bar{x}\|_{\Phi}^A \geq 1 + \mu,$$

where $\bar{x} \in l_{\Phi}^A(X)$ satisfying $\|\bar{x}\|_{\Phi}^A \geq \epsilon$. Since $\Phi \in \delta_2$ and $\|\bar{x}\|_{\Phi}^A \geq \epsilon$, so by using Lemma 3.12, for each $\epsilon > 0$ there exists $\delta \in (0, 1)$ such that $\varrho_{\Phi}^A(\bar{x}) \geq \delta$, and also by Lemma 3.11 there exists $\delta_1 \in (0, \delta)$ such that $\varrho_{\Phi}^A(\bar{u}) \leq 1$, $\varrho_{\Phi}^A(\bar{v}) \leq \delta_1$ implies

$$|\varrho_{\Phi}^A(\bar{u} + \bar{v}) - \varrho_{\Phi}^A(\bar{u})| < \frac{\delta}{6} \text{ for any } \bar{u}, \bar{v} \in l_{\Phi}^A(X). \quad (3.4)$$

Since $\varrho_{\Phi}^A(\bar{x}) < \infty$, so there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \phi_n \left(\sum_{k=1}^n \|a_{nk}x_k\| \right) \leq \frac{\delta_1}{6}. \quad (3.5)$$

From equation (3.5), it follows that

$$\delta \leq \sum_{n=1}^{n_0} \phi_n \left(\sum_{k=1}^n \|a_{nk}x_k\| \right) + \sum_{n=n_0+1}^{\infty} \phi_n \left(\sum_{k=1}^n \|a_{nk}x_k\| \right) \leq \sum_{n=1}^{n_0} \phi_n \left(\sum_{k=1}^n \|a_{nk}x_k\| \right) + \frac{\delta_1}{6},$$

which implies $\sum_{n=1}^{n_0} \phi_n \left(\sum_{k=1}^n \|a_{nk}x_k\| \right) \geq \delta - \frac{\delta_1}{6} > \delta - \frac{\delta}{6} = \frac{5\delta}{6}$. Since $\bar{x}^{(l)} \rightarrow 0$

weakly, so $\bar{x}_k^{(l)} \rightarrow 0$ for each k . Hence there exists l_0 such that for all $l \geq l_0$ the last inequality yields

$$\sum_{n=1}^{n_0} \phi_n \left(\sum_{k=1}^n \|a_{nk}(x_k^{(l)} + x_k)\| \right) \geq \frac{5\delta}{6}. \quad (3.6)$$

Again $\bar{x}^{(l)} \rightarrow 0$ weakly, so we can choose $n_0 \in \mathbb{N}$ such that $\varrho_{\Phi}^A(\bar{x}^{(l)}|_{n_0}) \rightarrow 0$ as $l \rightarrow \infty$. Therefore there exists $l_1 > l_0$ such that $\varrho_{\Phi}^A(\bar{x}^{(l)}|_{n_0}) \leq \delta_1$ for all $l \geq l_1$. Since $(\bar{x}^{(l)}) \subset S(l_{\Phi}^A(X))$, so by Lemma 3.7, we have $\varrho_{\Phi}^A(\bar{x}^{(l)}) = 1$, and hence for $n_0 \in \mathbb{N}$, we have $\varrho_{\Phi}^A(\bar{x}^{(l)}|_{\mathbb{N}-n_0}) \leq 1$. Let us set $\bar{u} = \bar{x}^{(l)}|_{\mathbb{N}-n_0}$ and $\bar{v} = \bar{x}^{(l)}|_{n_0}$. Then $\bar{u}, \bar{v} \in l_{\Phi}^A(X)$ and $\varrho_{\Phi}^A(\bar{u}) \leq 1$, $\varrho_{\Phi}^A(\bar{v}) \leq \delta_1$. Using (3.4), we have

$$|\varrho_{\Phi}^A(\bar{x}^{(l)}|_{\mathbb{N}-n_0} + \bar{x}^{(l)}|_{n_0}) - \varrho_{\Phi}^A(\bar{x}^{(l)}|_{\mathbb{N}-n_0})| < \frac{\delta}{6},$$

for all $l \geq l_1$. Thus $\varrho_{\Phi}^A(\bar{x}^{(l)}) - \frac{\delta}{6} < \varrho_{\Phi}^A(\bar{x}^{(l)}|_{\mathbb{N}-n_0})$ for all $l \geq l_1$, i.e.,

$$\sum_{n=n_0+1}^{\infty} \phi_n \left(\sum_{k=1}^n \|a_{nk}x_k^{(l)}\| \right) > 1 - \frac{\delta}{6} \text{ for all } l \geq l_1.$$

Again, since $\varrho_{\Phi}^A(\bar{x}^{(l)}|_{\mathbb{N}-n_0}) \leq 1$ and $\varrho_{\Phi}^A(\bar{x}|_{\mathbb{N}-n_0}) \leq \frac{\delta_1}{6} < \delta_1$, so from the equations (3.4), we have

$$\left| \varrho_{\Phi}^A(\bar{x}^{(l)} + \bar{x}|_{\mathbb{N}-n_0}) - \varrho_{\Phi}^A(\bar{x}|_{\mathbb{N}-n_0}) \right| < \frac{\delta}{6}.$$

For $l \geq l_1$, we have

$$\begin{aligned} \varrho_{\Phi}^A(\bar{x}^{(l)} + \bar{x}) &= \sum_{n=1}^{n_0} \phi_n \left(\sum_{k=1}^n \|a_{nk}(x_k^{(l)} + x_k)\| \right) + \sum_{n=n_0+1}^{\infty} \phi_n \left(\sum_{k=1}^n \|a_{nk}(x_k^{(l)} + x_k)\| \right) \\ &> \sum_{n=1}^{n_0} \phi_n \left(\sum_{k=1}^n \|a_{nk}(x_k^{(l)} + x_k)\| \right) + \sum_{n=n_0+1}^{\infty} \phi_n \left(\sum_{k=1}^n \|a_{nk}x_k^{(l)}\| \right) - \frac{\delta}{6} \\ &> \frac{5\delta}{6} + \left(1 - \frac{\delta}{6}\right) - \frac{\delta}{6} = 1 + \frac{\delta}{2}. \end{aligned}$$

Since $\Phi \in \delta_2$ and satisfies the condition (*), so by Lemma 3.13 there exists $\mu > 0$ depending only on δ such that $\|\bar{x}^{(l)} + \bar{x}\|_{\Phi}^A > 1 + \mu$ for $l \geq l_1$. Hence $\liminf_{l \rightarrow \infty} \|\bar{x}^{(l)} + \bar{x}\|_{\Phi}^A \geq 1 + \mu$. This completes the proof. \square

Remark 3.15. Let A be an identity matrix, $X = \mathbb{R}$ and $\phi_n(u) = u^{p_n}$ for all $u \geq 0$, $1 \leq p_n < \infty$ such that $\sup_{n \geq 1} p_n < \infty$. Then $\Phi = (\phi_n) \in \delta_2$ and satisfies the condition (*) given by (2.1). The space l_{Φ} has the uniform Opial property studied by Cui and Hudzik [2]. It would be interesting to find the necessary condition for the space $l_{\Phi}^A(X)$ to have the uniform Opial property.

Theorem 3.16. Let X be a σ -DC Banach lattice. Then the space $l_{\Phi}^A(X)$ is a σ -DC Banach lattice.

Proof. We have proved that $l_{\Phi}^A(X)$ is a Banach space with the norm $\|\cdot\|_{\Phi}^A$. To prove $l_{\Phi}^A(X)$ is a Banach lattice, let $\bar{x} = (x_k), \bar{y} = (y_k) \in l_{\Phi}^A(X)$ such that $|x_k| \leq |y_k|$ for all k . Since by hypothesis X is a Banach lattice and $x_k, y_k \in X$ for all k . Therefore $|x_k| \leq |y_k| \Rightarrow \|x_k\| \leq \|y_k\|$ for all k .

Let $\sigma > 0$ be any number. Now using the non-decreasing property of each ϕ_n , we get

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}| \|x_k\|}{\sigma} \right) \leq \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}| \|y_k\|}{\sigma} \right) \Rightarrow \|\bar{x}\|_{\Phi}^A \leq \|\bar{y}\|_{\Phi}^A.$$

Hence $l_{\Phi}^A(X)$ is a Banach lattice.

Now we shall show that the space $l_{\Phi}^A(X)$ is σ -DC. Let $(\bar{x}^{(n)})$ be a non-negative order bounded sequence and bounded above by $\bar{y} \in l_{\Phi}^A(X)$, i.e., for each $k \in \mathbb{N}$, $x_k^{(n)} \leq y_k$ for all $n \geq 1$.

Since X is a σ -DC, there exists $(x_k) \subset X$ such that $\sup_n x_k^{(n)} = x_k$ for all k .

Therefore $x_k \leq y_k$ for all k . Again $\bar{y} \in l_{\Phi}^A(X)$, so $\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}| \|x_k\|}{\sigma} \right) < \infty$ for some $\sigma > 0$, and hence $\bar{x} \in l_{\Phi}^A(X)$. This completes the proof. \square

Corollary 3.17. *If X is a Banach lattice, then $h_{\Phi}^A(X)$ is a Banach lattice.*

Theorem 3.18. *Let a Banach lattice X be an AL-space. If $\Phi = (\phi_n) \in \delta_2$ and satisfies the condition $(*)$ given by (2.1), then the Banach lattice $l_{\Phi}^A(X)$ is uniformly monotone.*

Proof. Let $\epsilon > 0$ be any number. Let $\bar{x} = (x_k), \bar{y} = (y_k) \in l_{\Phi}^A(X)$ such that $\bar{0} \leq \bar{x} \leq \bar{y}$ with $\|\bar{x}\|_{\Phi}^A = 1$ and $\|\bar{y}\|_{\Phi}^A \geq \epsilon$. Since $\Phi = (\phi_n) \in \delta_2$, from Lemma 3.7, we get $\varrho_{\Phi}^A(\bar{x}) = 1$ as $\|\bar{x}\|_{\Phi}^A = 1$ and from Lemma 3.12, there exists $\delta(\epsilon) > 0$ such that $\varrho_{\Phi}^A(\bar{y}) > \delta(\epsilon)$ as $\|\bar{y}\|_{\Phi}^A \geq \epsilon$.

Now if $u, v \geq 0$ and ϕ is an Orlicz function, then we will prove that $\phi(u+v) \geq \phi(u) + \phi(v)$. With out loss of generality, we assume that $u > v > 0$. Clearly $u < u+v$ and $v < u+v$. Then $\phi(u) = \phi(v \cdot \frac{u}{v}) \geq \frac{u}{v} \phi(v)$ as ϕ is an Orlicz function. Thus for $u > v$, we have $\frac{\phi(u)}{u} \geq \frac{\phi(v)}{v}$. Since $u+v > u$ and $u+v > v$, we get

$$\frac{\phi(u+v)}{u+v} \geq \frac{\phi(u)}{u} \quad \text{and} \quad \frac{\phi(u+v)}{u+v} \geq \frac{\phi(v)}{v}.$$

Therefore $\phi(u+v) \geq \phi(u) + \phi(v)$ for $u, v \geq 0$.

Now $\varrho_{\Phi}^A(\bar{x} + \bar{y}) = \sum_{n=1}^{\infty} \phi_n \left(\sum_{k=1}^{\infty} \|a_{nk}(x_k + y_k)\| \right)$. Since X is an AL-space, so $\sum_{k=1}^{\infty} \|a_{nk}(x_k + y_k)\| = \sum_{k=1}^{\infty} \|a_{nk}x_k\| + \sum_{k=1}^{\infty} \|a_{nk}y_k\|$. Take $\sum_{k=1}^{\infty} \|a_{nk}x_k\| = u$ and $\sum_{k=1}^{\infty} \|a_{nk}y_k\| = v$. Therefore for each Orlicz function ϕ_n , we have

$$\phi_n \left(\sum_{k=1}^{\infty} \|a_{nk}x_k\| + \sum_{k=1}^{\infty} \|a_{nk}y_k\| \right) \geq \phi_n \left(\sum_{k=1}^{\infty} \|a_{nk}x_k\| \right) + \phi_n \left(\sum_{k=1}^{\infty} \|a_{nk}y_k\| \right).$$

Hence

$$\varrho_{\Phi}^A(\bar{x} + \bar{y}) \geq \varrho_{\Phi}^A(\bar{x}) + \varrho_{\Phi}^A(\bar{y}).$$

Therefore $\varrho_{\Phi}^A(\bar{x} + \bar{y}) \geq 1 + \delta(\epsilon)$. Since $\Phi = (\phi_n) \in \delta_2$ and satisfies the condition $(*)$, by using Lemma 3.13, there exists $\mu > 0$ independent of \bar{x}, \bar{y} such that $\|\bar{x} + \bar{y}\|_{\Phi}^A \geq 1 + \mu$. This finishes the proof. \square

Corollary 3.19. *If a Banach lattice X is an AL-space and $\Phi = (\phi_n)$ satisfies the condition $(*)$, then $h_{\Phi}^A(X)$ is strictly monotone.*

Theorem 3.20. *Let A be a triangle. Then $h_{\Phi}^A(X)$ is the subspace of all order continuous elements of $l_{\Phi}^A(X)$.*

Proof. Let $\epsilon > 0$ be any number and $\bar{x} = (x_n) \in h_{\Phi}^A(X)$. We will show that \bar{x} is order continuous. Since $\bar{x} \in h_{\Phi}^A(X)$, so there is $t > 0$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \phi_n \left(t \sum_{k=1}^n \|a_{nk}x_k\| \right) < \frac{\epsilon}{2}.$$

Suppose $(\bar{x}^{(m)})$ is a sequence in $h_{\Phi}^A(X)$ such that $\bar{x}^{(m)} \rightarrow \bar{0}$ coordinate wise and $\bar{0} \leq \bar{x}^{(m)} \leq |\bar{x}|$ for all $m \in \mathbb{N}$. Let us denote for all $n \in \mathbb{N}$,

$$\phi_n \left(t \sum_{k=1}^n \|a_{nk}x_k\| \right) = \beta(n) \text{ and } \phi_n \left(t \sum_{k=1}^n \|a_{nk}x_k^{(m)}\| \right) = \beta^{(m)}(n).$$

Since $x_k^{(m)} \rightarrow \bar{0}$ and each ϕ_n is continuous, we get $\beta^{(m)}(n) \rightarrow 0$ as $m \rightarrow \infty$ for all $n \in \mathbb{N}$. Therefore we can choose $n_1 \in \mathbb{N}$ such that $\sum_{n=1}^{n_0} \beta^{(m)}(n) < \frac{\epsilon}{2}$ for all $m \geq n_1$. Since $\bar{x}^{(m)} \leq |\bar{x}|$ for all $m \in \mathbb{N}$, we have

$$\sum_{n=n_0+1}^{\infty} \beta^{(m)}(n) \leq \sum_{n=n_0+1}^{\infty} \beta(n) < \frac{\epsilon}{2}.$$

Thus for all $m \geq n_1$ and $t > 0$,

$$\begin{aligned} \varrho_{\Phi}^A(t\bar{x}^{(m)}) &= \sum_{n=1}^{n_0} \phi_n \left(t \sum_{k=1}^n \|a_{nk}x_k^{(m)}\| \right) + \sum_{n=n_0+1}^{\infty} \phi_n \left(t \sum_{k=1}^n \|a_{nk}x_k^{(m)}\| \right) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore any arbitrary $t > 0$, we get $\varrho_{\Phi}^A(t\bar{x}^{(m)}) \rightarrow 0$ as $m \rightarrow \infty$, and hence $\|\bar{x}^{(m)}\| \rightarrow 0$. Thus \bar{x} is an order continuous in $h_{\Phi}^A(X)$. Arbitrariness of \bar{x} implies that the space $h_{\Phi}^A(X)$ is order continuous. \square

3.1. Operators of s -type l_{Φ}^A . Let E, F be two Banach spaces. Then

$$\mathcal{L}_{\Phi}^A(E, F) = \left\{ T \in \mathcal{L}(E, F) : (s_n(T)) \in l_{\Phi}^A \right\}.$$

For $T \in \mathcal{L}_{\Phi}^A(E, F)$, we define

$$\|T\|_{\Phi}^A = \inf \left\{ \sigma > 0 : \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(T)|}{\sigma} \right) \leq 1 \right\}.$$

We denote \mathcal{L}_{Φ}^A as the class of s -type l_{Φ}^A operators between any two arbitrary Banach spaces. We also define

$$\mathcal{H}_{\Phi}^A(E, F) = \left\{ T \in \mathcal{L}(E, F) : (s_n(T)) \in h_{\Phi}^A \right\}.$$

Let $A = (a_{nk})$ be a matrix from the collection \mathcal{A} satisfying the condition

$$|a_{n,2k-1}| + |a_{n,2k}| \leq M|a_{nk}| \quad \text{for each } k \text{ and } n, \quad (3.7)$$

where M is a constant independent of n and k .

Example 3.21. It is easy to give example of matrices which satisfy condition (3.7). For example,

1. Nörlund matrix $A = (a_{nk})$, where a_{nk} is defined as

$$a_{nk} = \begin{cases} \frac{a_{n+1-k}}{A_n} & : 1 \leq k \leq n \\ 0 & : k > n \end{cases}$$

where a_n is non negative for each n and $A_n = \sum_{k=1}^n a_k > 0$.

2. Hilbert matrix $A = (a_{nk})$, where

$$a_{nk} = \left\{ \frac{1}{n+k-1} : 1 \leq n, k < \infty. \right.$$

Proposition 3.22. Let $A = (a_{nk})$ be an infinite matrix such that $A = (a_{nk}) \in \mathcal{A}$ and satisfies the condition (3.7). Then $(\mathcal{L}_{\Phi}^A(E, F), \|\cdot\|_{\Phi}^A)$ is a quasi-Banach space. Moreover, the inclusion map from $(\mathcal{L}_{\Phi}^A(E, F), \|\cdot\|_{\Phi}^A)$ to $(\mathcal{L}(E, F), \|\cdot\|)$ is continuous.

Proof. Let $S, T \in \mathcal{L}_{\Phi}^A(E, F)$. Then there exist $\sigma_1 > 0, \sigma_2 > 0$ such that

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(S)|}{\sigma_1} \right) < \infty \text{ and } \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\sigma_2} \right) < \infty.$$

Now using the non-increasing property of s -number and the condition (3.7) on the matrix $A = (a_{nk})$, we get

$$\sum_{k=1}^{\infty} |a_{nk} s_k(S+T)| \leq M \sum_{k=1}^{\infty} |a_{n,k}| (s_k(S) + s_k(T)) \quad (3.8)$$

Using (3.8) and convexity property of each ϕ_n , we have

$$\begin{aligned} \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(S+T)|}{M(\sigma_1 + \sigma_2)} \right) &\leq \frac{\sigma_1}{\sigma_1 + \sigma_2} \left[\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(S)|}{\sigma_1} \right) \right] \\ &\quad + \frac{\sigma_2}{\sigma_1 + \sigma_2} \left[\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\sigma_2} \right) \right] \\ &< \infty. \end{aligned}$$

This shows that $S+T \in \mathcal{L}_{\Phi}^A(E, F)$.

Let $\alpha \in \mathbb{R}$ and $T \in \mathcal{L}_{\Phi}^A(E, F)$. If $\alpha = 0$, then it is trivial. Suppose $\alpha \neq 0$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(\alpha T)|}{\sigma} \right) &\leq \sum_{n=1}^{\infty} \phi_n \left(|\alpha| \frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\sigma} \right) = \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\frac{\sigma}{|\alpha|}} \right) \\ &< \infty \end{aligned}$$

as $T \in \mathcal{L}_{\Phi}^A(E, F)$.

This shows that $\alpha T \in \mathcal{L}_{\Phi}^A(E, F)$. Hence $\mathcal{L}_{\Phi}^A(E, F)$ is a linear space.

To show $\|\cdot\|_{\Phi}^A$ is a quasi-norm on the space $\mathcal{L}_{\Phi}^A(E, F)$, let $T \in \mathcal{L}_{\Phi}^A(E, F)$ such that $\|T\|_{\Phi}^A = 0$. Then for all $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\epsilon} \right) \leq 1.$$

Therefore the sequence $\left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\epsilon} \right)$ is bounded, so there exists $C > 0$ such

that $\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\epsilon} \leq C$ for all n . Since $(a_{nk}) \in \mathcal{A}$, there exists $n_0 \in \mathbb{N}$ such that $a_{n_0 1} \neq 0$ and hence

$$|a_{n_0 1} s_1(T)| \leq C\epsilon \text{ for all } n \quad (3.9)$$

which is true for any arbitrary $\epsilon > 0$. Thus $\|T\| = s_1(T) = 0$, and hence $T = 0$.

Let $S, T \in \mathcal{L}_{\Phi}^A(E, F)$ and $\epsilon > 0$ be any positive number. Choose $\sigma_1 > 0, \sigma_2 > 0$ such that

$$\begin{aligned} \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(S)|}{\sigma_1} \right) &\leq 1, \quad \sigma_1 \leq \|S\|_{\Phi}^A + \frac{\epsilon}{2} \text{ and} \\ \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\sigma_2} \right) &\leq 1, \quad \sigma_2 \leq \|T\|_{\Phi}^A + \frac{\epsilon}{2} \text{ hold.} \end{aligned}$$

With out loss of generality, we can choose $M > 1$. Now from the above, we have

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(S+T)|}{M(\sigma_1 + \sigma_2)} \right) \leq 1$$

which implies

$$\|S+T\|_{\Phi}^A \leq M[\sigma_1 + \sigma_2] \leq M[\|S\|_{\Phi}^A + \|T\|_{\Phi}^A + \epsilon].$$

Since $\epsilon > 0$ is arbitrary, so we have

$$\|S+T\|_{\Phi}^A \leq M[\|S\|_{\Phi}^A + \|T\|_{\Phi}^A].$$

Hence $\|\cdot\|_{\Phi}^A$ is a quasi-norm on the space $\mathcal{L}_{\Phi}^A(E, F)$.

To prove completeness, let $(T^{(m)})$ be a Cauchy sequence in $\mathcal{L}_{\Phi}^A(E, F)$. Then for each $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T^{(l)} - T^{(m)})|}{\epsilon} \right) \leq 1 \text{ for all } m, l \geq m_0. \quad (3.10)$$

Therefore the sequence $\left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(T^{(l)} - T^{(m)})|}{\epsilon}\right)$ is bounded. Using the same argument as above, we have $\|T^{(l)} - T^{(m)}\| \rightarrow 0$ as $l, m \rightarrow \infty$.

Thus $(T^{(m)})$ is a Cauchy sequence in $\mathcal{L}(E, F)$, and hence converges. Let $T = \lim_{m \rightarrow \infty} T^{(m)}$. Also $s_k(T^{(l)} - T^{(m)}) \rightarrow s_k(T^{(l)} - T)$ as $m \rightarrow \infty$ for each $k \in \mathbb{N}$. Since each ϕ_n is continuous so taking $m \rightarrow \infty$, we get from (3.10),

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(T^{(l)} - T)|}{\epsilon} \right) \leq 1 \text{ for all } l \geq m_0.$$

Thus $(T^{(l)})$ converges to T in $\mathcal{L}_{\Phi}^A(E, F)$. In particular $T - T^{(m_0)} \in \mathcal{L}_{\Phi}^A(E, F)$, so $T = T^{(m_0)} + (T - T^{(m_0)}) \in \mathcal{L}_{\Phi}^A(E, F)$.

To prove the second part, we have from (3.9)

$$\|T\| \leq \frac{C}{|a_{n_0 1}|} \inf \left\{ \epsilon > 0 : \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(T)|}{\epsilon} \right) \leq 1 \right\}$$

i.e., $\|T\| \leq \frac{C}{|a_{n_0 1}|} \|T\|_{\Phi}^A.$

Hence the inclusion map from $(\mathcal{L}_{\Phi}^A(E, F), \|\cdot\|_{\Phi}^A)$ to $(\mathcal{L}(E, F), \|\cdot\|)$ is continuous. This completes the proof. \square

Remark 3.23. 1. If we take the matrix $A = (a_{nk})$ as an identity matrix and s -number as approximation number then the s -type l_{Φ}^A operators become l_p type [19] and l_{ϕ} type [7] operators when $\phi_n = x^p$ for $0 < p < \infty$ and $\phi_n = \phi$, an Orlicz function respectively.

2. If we take the matrix $A = (a_{nk})$ such that the matrix satisfies the condition (3.7), then the class of s -type l_{Φ}^A operators becomes s -type $|A, p|$ operators introduced by Maji and Srivastava [17].

Theorem 3.24. Let $A = (a_{nk})$ be an infinite matrix such that $A = (a_{nk}) \in \mathcal{A}$. If $A = (a_{nk})$ satisfies the condition (3.7) and $(|a_{n1}|) \in l_{\Phi}$, then the class \mathcal{L}_{Φ}^A is a quasi-Banach operator ideal.

Proof. Let E, F be any two Banach spaces and $\mathcal{L}_{\Phi}^A(E, F)$ be any one of the component of \mathcal{L}_{Φ}^A . To prove \mathcal{L}_{Φ}^A is a quasi-Banach operator ideal, it is enough to prove (OI1) and (OI3) in Definition 2.8. Let $x' \in E'$ and $y \in F$. Then $x' \otimes y : E \rightarrow F$ is a rank one operator, and hence $s_k(x' \otimes y) = 0$ for $k \geq 2$. Then for some $\sigma > 0$

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(x' \otimes y)|}{\sigma} \right) = \|x' \otimes y\| \sum_{n=1}^{\infty} \phi_n \left(\frac{|a_{n1}|}{\sigma} \right) < \infty$$

as $(|a_{n1}|) \in l_{\Phi}$. Thus $x' \otimes y \in \mathcal{L}_{\Phi}^A(E, F)$.

Let $T \in \mathcal{L}(E_0, E)$, $R \in \mathcal{L}(F, F_0)$ and $S \in \mathcal{L}_{\Phi}^A(E, F)$. It is required to prove $RST \in \mathcal{L}_{\Phi}^A(E_0, F_0)$.

Using the property (S3) in Definition 2.5, we have

$$s_n(RST) \leq \|R\|s_n(S)\|T\| \text{ for all } n \in \mathbb{N}.$$

Since $S \in \mathcal{L}_{\Phi}^A(E, F)$, there exists some $\sigma > 0$ such that

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(S)|}{\sigma} \right) < \infty.$$

Therefore

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(RST)|}{\|R\|\|T\|\sigma} \right) < \infty.$$

Thus $RST \in \mathcal{L}_{\Phi}^A(E_0, F_0)$ and therefore (OI3) is proved. Hence \mathcal{L}_{Φ}^A is an operator ideal.

To prove $\|\cdot\|_{\Phi}^A$ is an ideal-norm on \mathcal{L}_{Φ}^A , let $S \in \mathcal{L}_{\Phi}^A(E, F)$. Then for given $\epsilon > 0$, there exists some σ_0 such that $\sigma_0 < \|S\|_{\Phi}^A + \epsilon$ with

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(S)|}{\sigma_0} \right) \leq 1.$$

Thus for $T \in \mathcal{L}(E_0, E)$, $R \in \mathcal{L}(F, F_0)$, we have

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk}s_k(RST)|}{\|R\|\|T\|\sigma_0} \right) \leq 1.$$

Hence $\|RST\|_{\Phi}^A \leq \|R\|\|T\|\sigma_0 < \|R\|\|T\|(\|S\|_{\Phi}^A + \epsilon)$. As $\epsilon > 0$ is arbitrary, we have $\|RST\|_{\Phi}^A \leq \|R\|\|S\|_{\Phi}^A\|T\|$. Thus \mathcal{L}_{Φ}^A is a quasi-Banach operator ideal. \square

Remark 3.25. In particular if we choose $\phi_n = \phi$, an Orlicz function for all n and $A = (a_{nk})$ as a diagonal matrix $a_{nn} = n^{\frac{1}{p} - \frac{1}{q}}$ for $0 < p, q < \infty$, the operator ideal \mathcal{L}_{Φ}^A becomes $\mathcal{L}_{p,q,\Phi}$ studied by Gupta and Bhar [7]. This example also shows that the condition (3.7) on the matrix A is only sufficient condition to form operator ideal.

Proposition 3.26. The space $\mathcal{H}_{\Phi}^A(E, F)$ is a closed subspace of $\mathcal{L}_{\Phi}^A(E, F)$.

Proof. Clearly $\mathcal{H}_{\Phi}^A(E, F)$ is a subspace of $\mathcal{L}_{\Phi}^A(E, F)$. To show $\mathcal{H}_{\Phi}^A(E, F)$ is a closed subspace of $\mathcal{L}_{\Phi}^A(E, F)$, let T belongs to the closure of the space $\mathcal{H}_{\Phi}^A(E, F)$ in the norm topology of $\mathcal{L}_{\Phi}^A(E, F)$. Then there exists a sequence $(T^{(m)})$ in $\mathcal{H}_{\Phi}^A(E, F)$ such that $\lim_{n \rightarrow \infty} \|T^{(m)} - T\|_{\Phi}^A = 0$. Thus for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|T^{(m)} - T\|_{\Phi}^A < \frac{\epsilon}{2} \text{ for all } m \geq n_0.$$

Now using the condition (3.7) on the matrix $A = (a_{nk})$ and (3.8), we get

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{M\epsilon} \right) \leq \frac{1}{2} \left[\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{n,k} s_k(T - T^{(n_0)})|}{\frac{\epsilon}{2}} \right) + \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{n,k} s_k(T^{(n_0)})|}{\frac{\epsilon}{2}} \right) \right] < \infty,$$

as $T^{(n_0)} \in \mathcal{H}_{\Phi}^A(E, F)$ and $\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T - T^{(n_0)})|}{\|T - T^{(n_0)}\|_{\Phi}^A} \right) < \infty$. Thus $T \in \mathcal{H}_{\Phi}^A(E, F)$ and hence the proof is complete. \square

Proposition 3.27. *If the s -number sequence is injective, then the quasi-Banach operator ideal $[\mathcal{L}_{\Phi}^A, \|\cdot\|_{\Phi}^A]$ is injective.*

Proof. Let $T \in \mathcal{L}(E, F)$ and $J \in \mathcal{L}(F, F_0)$ be any metric injection. Suppose that $JT \in \mathcal{L}_{\Phi}^A(E, F_0)$. Then for some $\sigma_0 > 0$, we have

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(JT)|}{\sigma_0} \right) < \infty.$$

Since the s -number sequence $s = (s_n)$ is injective, we have $s_n(T) = s_n(JT)$ for all $T \in \mathcal{L}(E, F)$, $n \in \mathbb{N}$. Hence

$$\sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(T)|}{\sigma_0} \right) = \sum_{n=1}^{\infty} \phi_n \left(\frac{\sum_{k=1}^{\infty} |a_{nk} s_k(JT)|}{\sigma_0} \right) < \infty.$$

Thus $T \in \mathcal{L}_{\Phi}^A(E, F_0)$ and clearly $\|JT\|_{\Phi}^A = \|T\|_{\Phi}^A$ holds. Hence the operator ideal \mathcal{L}_{Φ}^A is injective. This completes the proof. \square

Proposition 3.28. *If the s -number sequence is surjective, then the quasi-Banach operator ideal $[\mathcal{L}_{\Phi}^A, \|\cdot\|_{\Phi}^A]$ is surjective.*

Proof. We omit the proof as it follows in similar lines from the preceding proposition. \square

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